SOLUTIONS

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2019
Instructor: Prof. A. R. Reibman

Exam 2

Fall 2019, T/Th 3:00-4:15pm
(November 5, 2019)

This is a closed book exam with 9 multi-part problems. Neither calculators nor help sheets are allowed.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name or initials on every page of the exam and turn in everything when time is up.

Write your answers in the boxes provided. We will be scanning the exams, so **DO NOT WRITE ON THE BACK of the pages!**.

Name: ______________________ PUID: ______________________

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: ____

Problem 1. (TRUE/FALSE: 4 POINTS EACH, TOTAL 20 POINTS)

For each of the following statements, determine which is valid.
If you show your reasoning you might get partial credit.
Finding a counter-example might be helpful if the answer is FALSE.
(Note: if a statement is not always true, then it is FALSE.)

Clearly label each statement T or F in the box to the left of the problem.

$\boxed{F}$

(a) If X and Y have $COV(X,Y) = c_{XY}$,
then $COV(X - E(X), Y - E(Y)) = c_{XY} - E(X)E(Y)$.

$cov(X - E(X), Y - E(Y))$
$= cov(X, Y) = c_{XY}$

$\boxed{F}$

missing $P(A_i)$ factor

(b) $E(X) = \sum_{i=1}^{n} E(X|A_i)$ if $A_1, A_2, \ldots, A_n$ form a partition.

correct: $E(X) = \sum_{i=1}^{n} E(X|A_i)P(A_i)$

$\boxed{T}$

(c) If $E(XY) \neq E(X)E(Y)$, then X and Y cannot be independent RVs.

since independence $\rightarrow$ $E(XY) = E(X)E(Y)$
then NOT $\{E(XY) = E(X)E(Y)$ $\rightarrow$ NOT indep.

$\boxed{F}$

(d) The function $F_{XY}(x, y)$ below is a valid CDF for two random variables X and Y.

$$F_{XY}(x, y) = \begin{cases} 1 - \exp(-x - y) & \text{for } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y) = -\exp(-x-y)$$

for $x \geq 0$ and $y \geq 0$

<u>BUT</u> this is always negative
$\Rightarrow$ not allowed.

$\boxed{F}$

(e) $COV(3X + 4, -2Y - 1) = 36COV(X, Y)$.

$cov(3X + 4, -2Y - 1)$
$= cov(3X, -2Y)$
$= -6\, cov(X, Y)$

Problem 2. (MULTIPLE CHOICE: 6 POINTS)
6 cars compete in a race that takes 200 laps to complete. Suppose the probability that each car breaks down in one lap is $p = 0.1$. Let X be a random variable indicating the number of cars that complete the first lap without breaking down. Assume the probabilities of break-down for any car in any lap are independent.

The random variable X is well-modeled by which of the following distributions?
If you show your reasoning you might get partial credit.

$\boxed{g}$ Answer

X = # cars that do NOT break down in 1^{st} lap.

(a) Geometric with parameter $p = 0.1$ and sample space starting with 0.

(b) Geometric with parameter $p = 0.9$ and sample space starting with 0.

(c) Geometric with parameter $p = 0.1$ and sample space starting with 1.

(d) Geometric with parameter $p = 0.9$ and sample space starting with 1.

(e) Binomial with parameters $n = 6$ and $p = 0.1$.

(f) Binomial with parameters $n = 200$ and $p = 0.1$.

(g) Binomial with parameters $n = 6$ and $p = 0.9$.

(h) Binomial with parameters $n = 200$ and $p = 0.9$.

(i) Bernoulli with parameter $p = 0.9$

(j) Bernoulli with parameter $p = 0.1$

(k) Insufficient information to determine.

(l) None of the above

$\Rightarrow$ there are 6 cars

$A_i = \{ \text{car}_i \text{ doesn't break down} \}$

$i = 1, \dots, 6$

$P(A_i) = 0.9$

Each car can either break down or not $\Rightarrow$ binomial

Problem 3. (Multiple choice: 6 points)

Suppose X is a Gaussian Random Variable with mean $\mu = 2$ and variance $\sigma^2 = 4$. Compute the probability of the event $\{X^2 \leq 4\}$.
You may express your answer in terms of the Φ-function.
If you show your reasoning you might get partial credit.

$\boxed{A}$ (or B)

Answer

$$P(X^2 \leq 4)$$

$$= P(-2 \leq X \leq 2)$$

$$= P\left(\frac{-2-2}{\sqrt{4}} \leq \frac{X-2}{\sqrt{4}} \leq \frac{2-2}{\sqrt{4}}\right)$$

$$= P(-2 \leq Z \leq 0)$$

$$= \Phi(0) - \Phi(-2)$$

(a) $\Phi(0) - \Phi(-2)$

(b) $\Phi(2) - \Phi(0)$

(c) $2\Phi(-1)$

(d) $\Phi(0)$

(e) $\Phi(0) - \Phi(-1)$

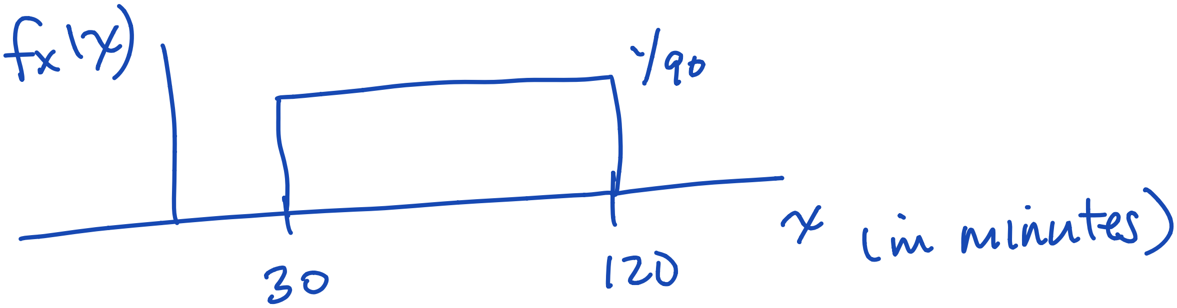
(f) None of the above.

Note: since $\Phi(a) = 1 - \Phi(-a)$ and $\Phi(0) = 1/2$, then the answer can also be written as

$$= \Phi(0) - (1 - \Phi(2))$$

$$= \Phi(2) - 1/2 = \Phi(2) - \Phi(0)$$

Problem 4. (6 POINTS)
A lawyer is working on a task, where the time it takes to complete the task is a random variable X that is uniformly distributed between 30 minutes and 2 hours. Given the lawyer has already worked on the task for 1 hour, what is the probability it will take no more than 30 more minutes to finish the task?



Find $P(X \le 90 \mid X \ge 60) = \dfrac{P(X \le 90 \cap X \ge 60)}{P(X \ge 60)}$

$$= \frac{P(60 \le X \le 90)}{P(X \ge 60)} = \frac{(30)\left(\frac{1}{90}\right)}{(60)\left(\frac{1}{90}\right)} = \frac{1}{2}$$

Answer: $1/2$

Problem 5. (18 POINTS TOTAL (6 POINTS FOR EACH PART))
Suppose buses arrive completely at random at a certain bus stop, and the number of buses that arrive in a time period of length β is a random variable with mean $\beta/5$.

(a) What is the PMF of N, $p_N(n)$, the number of buses that arrive in β minutes?

NOTE: Even if you cannot solve answer (a), you can set up equations to express answers to (b) and (c) in terms of $p_N(n)$.

(b) What is the probability that no buses arrive in a 10-minute interval?

(c) How much time should you allow so that with probability 0.99 at least one bus arrives?

ALSO: Leave your answers in terms of e (Euler's number) and ln(), the natural logarithm.

a) "buses arrive at random" $\Rightarrow$ Poisson

$$E(N) = \alpha = \beta/5$$

so $$p_N(n) = e^{-\beta/5} \frac{(\beta/5)^n}{n!} \quad \text{for } n=0,1,\ldots$$

b) $P(N=0)$ when $\beta = 10$.

$$p_N(0) = e^{-2} \frac{2^0}{0!} = e^{-2}$$

c) set $P(N \geq 1) = 0.99 \xrightarrow{\text{and}}$ solve for β.

$1 - P(N=0) = 0.99$ so

$$P(N=0) = 0.01 = e^{-\beta/5} \frac{(\beta/5)^0}{0!}$$

$$0.01 = e^{-\beta/5} \Rightarrow \beta = -5 \ln 0.01$$
$$= 5 \ln 100$$

Problem 6. (20 POINTS (7 POINTS FOR PART (A) AND FOR PART (B) AND 6 POINTS FOR PART (C)))

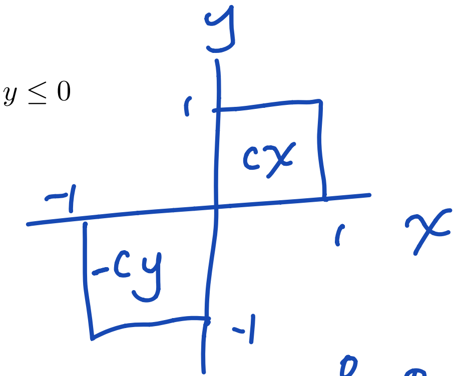
Given the joint Probability Density Function

$$f_{XY}(x,y) = \begin{cases} cx & \text{when } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ -cy & \text{when } -1 \le x \le 0 \text{ and } -1 \le y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the Region of Support and find c.

(b) Find $f_X(x)$, the marginal PDF of X.

(c) Are X and Y independent? Why or why not?



$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = \int_0^1\int_0^1 cx\,dx\,dy + \int_{-1}^{0}\int_{-1}^{0} -cy\,dx\,dy$$

$$= \int_{y=0}^{1} \frac{cx^2}{2}\Big|_0^1 dy + \int_{y=-1}^{0} -cyx\Big|_0^1 dy = \int_0^1 \frac{c}{2}\,dy + \int_{-1}^{0} -cy\,dy$$

$$= \frac{c}{2}y\Big|_0^1 + -\frac{cy^2}{2}\Big|_0^1 = \frac{c}{2} + \frac{c}{2} = \boxed{c = 1}$$

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \begin{cases} \int_0^1 cx\,dy & 0 \le x \le 1 \\ -\int_{-1}^{0} cy\,dy & -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$\boxed{f_x(x) = \begin{cases} x & 0 \le x \le 1 \\ \frac{1}{2} & -1 \le x \le 0 \\ 0 & \text{else} \end{cases}}$$

c). Not independent.

ROS not in product form.

Problem 7. (6 POINTS)

A runner is coming back from a long break of not running. Each exercise period, she interleaves running and walking for a total of 30 minutes. Her running speed is uniformly distributed between 5.5 and 6.5 miles per hour, and her walking speed is uniformly distributed between 3.8 and 4.2 miles per hour.

In one exercise period, she runs for 10 minutes and in a later exercise period, she runs for 20 minutes. How much further does she travel in the second exercise period than the first?

D = distance traveled

$E(D|\text{walk}) = 4 \frac{\text{miles}}{\text{hour}}$ $\quad E(D|\text{run}) = 6 \frac{\text{miles}}{\text{hour}}$

$E(D) = E(D|\text{walk})P(\text{walk}) + E(D|\text{run})P(\text{run})$,

Exercise period 1: $P(\text{walk}) = 2/3$ $\quad P(\text{run}) = 1/3$

$E(D) = 4 \cdot \frac{2}{3} + 6 \cdot 1/3 = \frac{14}{3} \cancel{\frac{\text{miles}}{\text{hour}}}$

Exercise period 2: $P(\text{walk}) = 1/3$ $\quad P(\text{run}) = 2/3$

$E(D) = 4 \cdot 1/3 + 6 \cdot 2/3 = 16/3$ miles/hour

But she only exercises for half an hour.

Diff. in distance is $\left(\frac{16}{3} - \frac{14}{3}\right)\left(\frac{\text{miles}}{\text{hour}}\right)\left(\frac{1}{2} \text{ hour}\right)$

Answer: $1/3$ miles

Problem 8. (12 POINTS (6 POINTS EACH PART))
Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} 1/6 & \text{for } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find **and sketch** the PDF of the random variable $Y = 2X - 3$.

Method 1: apply linear short cut

$Y = aX + b$ where $a = 2$ $b = -3$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{2} f_X\left(\frac{y+3}{2}\right)$$

$$= \begin{cases} \frac{1}{12} & \text{when } -3 \le \frac{y+3}{2} \le 3 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1/12 & \text{when } -9 \le y \le 3 \\ 0 & \text{else} \end{cases}$$

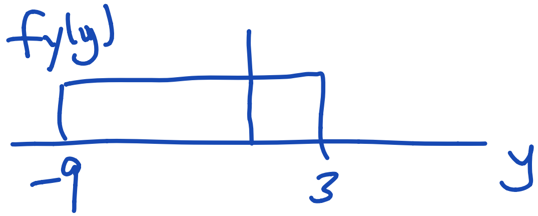
Method 2: 2-step process

$$F_Y(y) = P(Y \le y) = P(2X - 3 \le y) = P\left(X \le \frac{y+3}{2}\right)$$

$$= F_X\left(\frac{y+3}{2}\right)$$

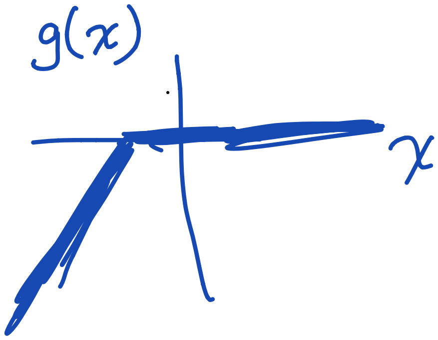
$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X\left(\frac{y+3}{2}\right) \cdot \frac{1}{2}$$

which, unsurprisingly, matches interim solution above and the final answer can be found the same way



(b) For the same $f_X(x)$ in part (a), find **and sketch** the PDF of the random variable $Y = g(X)$, where

$$g(x) = \begin{cases} (x+1) & \text{for } x \le -1 \\ 0 & \text{for } x > -1 \end{cases}$$



Note: based on sketch of $y = g(x)$ we can see $y > 0$ is not possible. Break y range into 3 regions.

region $y > 0$: $f_Y(y) = 0$, or $F_Y(y) = P(Y \le y) = 1$

region $y < 0$: $g(x)$ linear in this region $\Rightarrow$ $a = 1$, $b = 1$

$$f_Y(y) = \frac{1}{1} f_X(y-1)$$

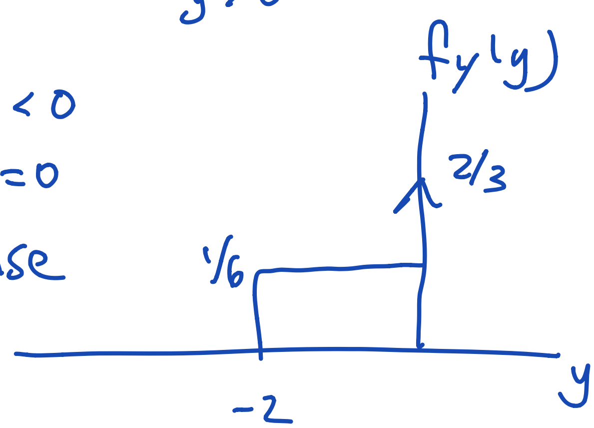
substituting, $f_Y(y) = \frac{1}{6}$ for $-2 < y < 0$

region $y = 0$: a flat region $\Rightarrow$ a jump in $F_Y(y)$ @ $y = 0$

$P(Y \le 0) = 1$ and $P(Y = 0) = P(X \ge -1) = 2/3$

Combining: $$F_Y(y) = \begin{cases} y/6 + 1/3 & -2 < y < 0 \\ 1 & y = 0 \\ 1 & y > 0 \end{cases}$$

$$f_Y(y) = \begin{cases} 1/6 & -2 < y < 0 \\ 2/3\, \delta(y) & y = 0 \\ 0 & \text{else} \end{cases}$$



Problem 9. (6 POINTS))

Hong and Yao decide to meet next to the clock tower at noon. Hong is often late, and Yao is often impatient.

Suppose Yao arrives at the clock tower at a random time (uniformly distributed) between 11:50 and 12:10, and will leave if Hong doesn't show up within 15 minutes of the time Yao arrives. Suppose Hong arrives at the clock tower at a random time (uniformly distributed) between 12:05 and 12:25, and will wait until Yao arrives (if necessary). Their arrival times are independent of each other.

What is the probability that Hong and Yao meet?

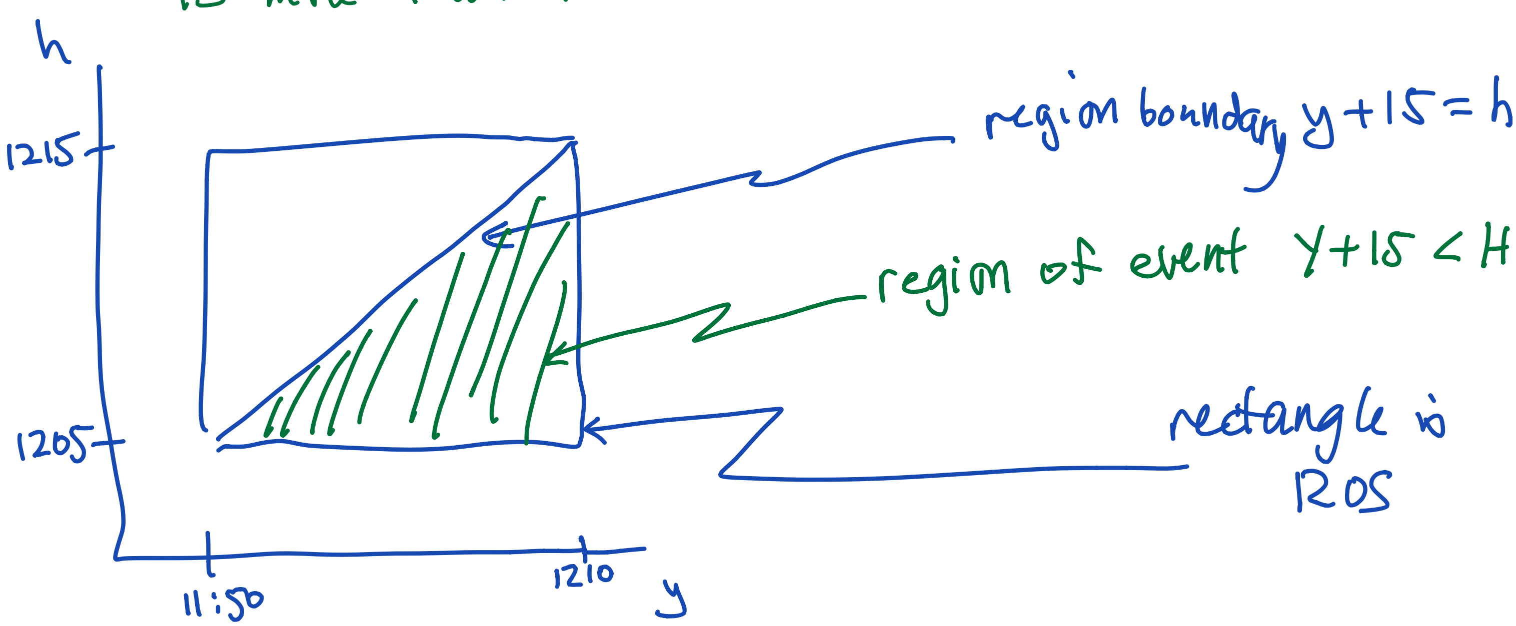
(Assume neither have cell phones or any way to contact the other, and assume they cannot see each other arriving or departing.)

H and Y are both RVs (times of arrival) so analyze in 2D:

$$f_{YH}(y,h) = f_Y(y) f_H(h) = \begin{cases} \frac{1}{20}\,\frac{1}{20} & \text{in ROS} \\ 0 & \text{else} \end{cases}$$

Event of interest: $Y + 15 < H$

(Hong's arrival time must be sooner than 15 more than Yao's arrival time)



Answer: $1/2$

Joint PDF is uniform ⇒ can use area argument

$$P(Y+15<H) = \frac{\text{area}(Y+15<H)}{\text{area (ROS)}} = \frac{\frac{1}{2}(20)(20)}{(20)(20)} = \frac{1}{2}$$