

Exam 2

Fall 2019, T/Th 3:00-4:15pm
(November 5, 2019)



This is a closed book exam with 9 multi-part problems. Neither calculators nor help sheets are allowed.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name or initials on every page of the exam and turn in everything when time is up.

Write your answers in the boxes provided. We will be scanning the exams, so **DO NOT WRITE ON THE BACK** of the pages!.

Name: _____

PUID: _____

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: _____

Problem 1. (TRUE/FALSE: 4 POINTS EACH, TOTAL 20 POINTS)

For each of the following statements, determine which is valid.

If you show your reasoning you might get partial credit.

Finding a counter-example might be helpful if the answer is FALSE.

(Note: if a statement is not always true, then it is FALSE.)

Clearly label each statement T or F in the box to the left of the problem.

(a) If X and Y have $COV(X, Y) = c_{XY}$,
then $COV(X - E(X), Y - E(Y)) = c_{XY} - E(X)E(Y)$.

(b) $E(X) = \sum_{i=1}^n E(X|A_i)$ if A_1, A_2, \dots, A_n form a partition.

(c) If $E(XY) \neq E(X)E(Y)$, then X and Y cannot be independent RVs.

(d) The function $F_{XY}(x, y)$ below is a valid CDF for two random variables X and Y .

$$F_{XY}(x, y) = \begin{cases} 1 - \exp(-x - y) & \text{for } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{else} \end{cases}$$

(e) $COV(3X + 4, -2Y - 1) = 36COV(X, Y)$.

Problem 2. (MULTIPLE CHOICE: 6 POINTS)

6 cars compete in a race that takes 200 laps to complete. Suppose the probability that each car breaks down in one lap is $p = 0.1$. Let X be a random variable indicating the number of cars that complete the first lap without breaking down. Assume the probabilities of break-down for any car in any lap are independent.

The random variable X is well-modeled by which of the following distributions?

If you show your reasoning you might get partial credit.

Answer

- (a) Geometric with parameter $p = 0.1$ and sample space starting with 0.
- (b) Geometric with parameter $p = 0.9$ and sample space starting with 0.
- (c) Geometric with parameter $p = 0.1$ and sample space starting with 1.
- (d) Geometric with parameter $p = 0.9$ and sample space starting with 1.
- (e) Binomial with parameters $n = 6$ and $p = 0.1$.
- (f) Binomial with parameters $n = 200$ and $p = 0.1$.
- (g) Binomial with parameters $n = 6$ and $p = 0.9$.
- (h) Binomial with parameters $n = 200$ and $p = 0.9$.
- (i) Bernoulli with parameter $p = 0.9$
- (j) Bernoulli with parameter $p = 0.1$
- (k) Insufficient information to determine.
- (l) None of the above

Problem 3. (MULTIPLE CHOICE: 6 POINTS)

Suppose X is a Gaussian Random Variable with mean $\mu = 2$ and variance $\sigma^2 = 4$. Compute the probability of the event $\{X^2 \leq 4\}$.

You may express your answer in terms of the Φ -function.

If you show your reasoning you might get partial credit.

Answer

- (a) $\Phi(0) - \Phi(-2)$
- (b) $\Phi(2) - \Phi(0)$
- (c) $2\Phi(-1)$
- (d) $\Phi(0)$
- (e) $\Phi(0) - \Phi(-1)$
- (f) None of the above.

Problem 4. (6 POINTS)

A lawyer is working on a task, where the time it takes to complete the task is a random variable X that is uniformly distributed between 30 minutes and 2 hours. Given the lawyer has already worked on the task for 1 hour, what is the probability it will take no more than 30 more minutes to finish the task?

Answer:

Problem 5. (18 POINTS TOTAL (6 POINTS FOR EACH PART))

Suppose buses arrive completely at random at a certain bus stop, and the number of buses that arrive in a time period of length β is a random variable with mean $\beta/5$.

(a) What is the PMF of N , $p_N(n)$, the number of buses that arrive in β minutes?

NOTE: Even if you cannot solve answer (a), you can set up equations to express answers to (b) and (c) in terms of $p_N(n)$.

(b) What is the probability that no buses arrive in a 10-minute interval?

(c) How much time should you allow so that with probability 0.99 at least one bus arrives?

ALSO: Leave your answers in terms of e (Euler's number) and $\ln()$, the natural logarithm.

Problem 6. (20 POINTS (7 POINTS FOR PART (A) AND FOR PART (B) AND 6 POINTS FOR PART (C)))

Given the joint Probability Density Function

$$f_{XY}(x, y) = \begin{cases} cx & \text{when } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ -cy & \text{when } -1 \leq x \leq 0 \text{ and } -1 \leq y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Sketch the Region of Support and find c .
- (b) Find $f_X(x)$, the marginal PDF of X .
- (c) Are X and Y independent? Why or why not?

Problem 7. (6 POINTS)

A runner is coming back from a long break of not running. Each exercise period, she interleaves running and walking for a total of 30 minutes. Her running speed is uniformly distributed between 5.5 and 6.5 miles per hour, and her walking speed is uniformly distributed between 3.8 and 4.2 miles per hour.

In one exercise period, she runs for 10 minutes and in a later exercise period, she runs for 20 minutes. How much further does she travel in the second exercise period than the first?

Answer:

Problem 8. (12 POINTS (6 POINTS EACH PART))

Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} 1/6 & \text{for } -3 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find **and sketch** the PDF of the random variable $Y = 2X - 3$.

- (b) For the same $f_X(x)$ in part (a), find **and sketch** the PDF of the random variable $Y = g(X)$, where

$$g(x) = \begin{cases} (x+1) & \text{for } x \leq -1 \\ 0 & \text{for } x > -1 \end{cases}$$

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Problem 9. (6 POINTS))

Hong and Yao decide to meet next to the clock tower at noon. Hong is often late, and Yao is often impatient.

Suppose Yao arrives at the clock tower at a random time (uniformly distributed) between 11:50 and 12:10, and will leave if Hong doesn't show up within 15 minutes of the time Yao arrives. Suppose Hong arrives at the clock tower at a random time (uniformly distributed) between 12:05 and 12:25, and will wait until Yao arrives (if necessary). Their arrival times are independent of each other.

What is the probability that Hong and Yao meet?

(Assume neither have cell phones or any way to contact the other, and assume they cannot see each other arriving or departing.)

Answer:

Extra space to solve – label problem clearly so I can give you credit!
**ALSO: Write at the bottom of original problem that you're solving here, so I know
to look here!**

Discrete Random Variables

- Bernoulli Random Variable, parameter p
 $S = \{0, 1\}$
 $p_0 = 1 - p, p_1 = p; 0 \leq p \leq 1$
 $E(X) = p; \text{VAR}(X) = p(1 - p)$
- Binomial Random Variable, parameters (n, p)
 $S = \{0, 1, \dots, n\}$
 $p_k = \binom{n}{k} p^k (1 - p)^{n-k}; k = 0, 1, \dots, n; 0 \leq p \leq 1$
 $E(X) = np; \text{VAR}(X) = np(1 - p)$
- Geometric Random Variable, parameter p
 $S = \{0, 1, \dots\}$
 $p_k = p(1 - p)^k; k = 0, 1, \dots; 0 \leq p \leq 1$
 $E(X) = (1 - p)/p; \text{VAR}(X) = (1 - p)/p^2$
- Poisson Random Variable, parameter α
 $S = \{0, 1, \dots\}$
 $p_k = \alpha^k e^{-\alpha} / k! \quad k = 0, 1, \dots,$
 $E(X) = \alpha; \text{VAR}(X) = \alpha$
- Uniform Random Variable
 $S = \{0, 1, \dots, L\}$
 $p_k = 1/L \quad k = 0, 1, \dots, L$
 $E(X) = (L + 1)/2; \text{VAR}(X) = (L^2 - 1)/12$

Continuous Random Variables

- Uniform Random Variable
Equally likely outcomes
 $S = [a, b]$
 $f_X(x) = 1/(b - a), \quad a \leq x \leq b$
 $E(X) = (a + b)/2; \quad \text{VAR}(X) = (b - a)^2/12$
- Exponential Random Variable, parameter λ
 $S = [0, \infty)$
 $f_X(x) = \lambda \exp(-\lambda x), \quad x \geq 0, \lambda > 0$
 $E(X) = 1/\lambda; \quad \text{VAR}(X) = 1/\lambda^2$
- One Gaussian Random Variable, parameters μ, σ^2
 $S = (-\infty, \infty)$
 $f_X(x) = \exp(-(x - \mu)^2 / (2\sigma^2)) / \sqrt{2\pi\sigma^2}$
 $E(X) = \mu; \quad \text{VAR}(X) = \sigma^2$