SOLUTIONS

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2019
Instructor: Prof. A. R. Reibman

Exam 1

Fall 2019, T/Th 3:00-4:15pm
(September 23, 2019)

This is a closed book exam with 8 multi-part problems. Neither calculators nor help sheets are allowed.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name or initials on every page of the exam and turn in everything when time is up.

Write your answers in the boxes provided. We will be scanning the exams, so **DO NOT WRITE ON THE BACK of the pages!**.

Name: ______________________ PUID: ______________________

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: ____

Problem 1. (True/False: 4 points each, total 24 points)

For each of the following statements, determine which is valid.
If you show your reasoning you might get partial credit.
Finding a counter-example might be helpful if the answer is FALSE.
(Note: if a statement is not always true, then it is FALSE.)

Clearly label each statement T or F in the box to the left of the problem.

$\boxed{T}$ $P(X > x) = 1 - P(\{X > x\}^c) = 1 - P(X \le x)$

(a) If X is a RV and $F_X(x)$ is its CDF, then $P(X > x) = 1 - F_X(x)$ for all x.

$= 1 - F_X(x)$ for any RV

$\boxed{F}$

(b) If X is a RV and $f_x(x)$ is its PDF, then $f_X(x) \le 1$ for all x.

ex: $f_x(x) = \begin{cases} 2 & 0 < x < 1/2 \\ 0 & \text{else} \end{cases}$ is a valid PDF

$\boxed{T}$

(c) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ for any two sets A and B.

Add $P(A \cup B)$ and subtract $P(A \cap B)$ from both sides to get a corollary

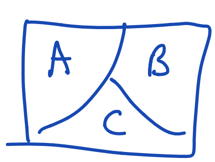
$\boxed{F}$

(d) $E[(X-1)^3] = [E(X) - 1]^3$

$E(X^3 - 3X^2 + X - 1)$
$= E(X^3) - 3E(X)^2 + E(X) - 1$
$\neq E(X)^3 - 3E(X)^2 + E(X) - 1$

$\boxed{T}$

(e) If A, B, and C form a partition, then A^c, B^c, and C^c are collectively exhaustive.



$A^c \cup B^c = S$, so surely $A^c \cup B^c \cup C^c = S$ too

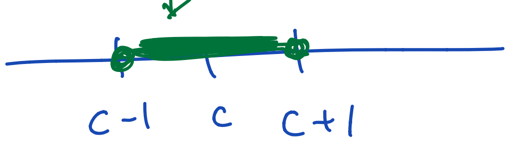
$\boxed{T}$

(f) A correct mathematical expression for "the probability that X is no more than 1 away from a constant c is at least 95%" is

$$P(|X - c| \le 1) \ge 0.95$$

The event of interest

underlined words are the event of interest



Problem 2. (12 POINTS (6 POINTS FOR (A); 4 POINTS FOR (B)))
One Blue and one Green die are rolled. Both dice are fair. Let A be the event the number on the Green die is even, and let C be the event that the sum of the two rolls is equal to 6.

(a) What is $P(C|A)$?

(b) Are events A and C independent? Explain your reasoning.
(No points without a proper reason.)

$S = \{(g, b) : g = \text{green die}, b = \text{blue die}, 1 \le g, b \le 6\}$

$|S| = 36$

$A = \{\text{green die is even}\}$ $\qquad P(A) = 1/2$

$C = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$ $\qquad P(C) = 5/36$

$A \cap C = \{(2,4), (4,2)\}$ $\qquad P(A \cap C) = 2/36$

a) $P(C|A) = \dfrac{P(A \cap C)}{P(A)} = \dfrac{2/36}{1/2} = \dfrac{4}{36} = \dfrac{1}{9}$

b) Need $P(A \cap C) = P(A)P(C)$, or equivalently, $P(C|A) = P(C)$

(a) $1/9$

(b) No. $P(A \cap C) = \dfrac{2}{36} \neq \dfrac{1}{2} \dfrac{5}{36} = P(A)P(C)$

and $P(C|A) = \dfrac{1}{9} \neq P(C) = 5/36$

Problem 3. (12 POINTS (6 POINTS EACH PART))
An engineer is designing a system to detect whales in the ocean. Let the event W indicate a whale is present, and let W^c indicate there is no whale. Let the event D indicate the system *decides* a whale is present, and let D^c indicate the system decides there is no whale.

(a) If the system decides wrong 5% of the time there actually is a whale, and 5% of the time there is actually no whale, and the prior probability of there being a whale is $P(W) = 0.2$, what is the probability that the system decides D?

(b) Let there be a cost for two possible outcomes: that of missing the whale, and that of saying there's a whale when there's no whale. Let the cost of the first case be 1000 for each occurrence, and the cost of the second be 10. Assume there is no cost to a correct decision. Using the result from part (a), what is the expected cost of this system each time it is used?

a) decides wrong 5% of time there is a whale

$\Rightarrow P(D^c | W) = 0.05$

decides wrong 5% of time there is no whale

$\Rightarrow P(D | W^c) = 0.05$

Thm total prob: $P(D) = P(D|W)P(W) + P(D|W^c)P(W^c)$

$= (1-0.05)(0.2) + (0.05)(1-0.2)$

$= 0.19 + 0.04 = 0.23$

b) $C = \text{cost} = \begin{cases} 1000 & \text{if } D^c \cap W \\ 10 & \text{if } D \cap W^c \\ 0 & \text{else} \end{cases}$

$E(C) = 1000\, P(D \cap W^c) + 10\, P(D^c \cap W)$

(a) 0.23

(b) 10.4

$E(C) = 1000\, P(D^c | W)P(W) + 10\, P(D | W^c)P(W^c)$

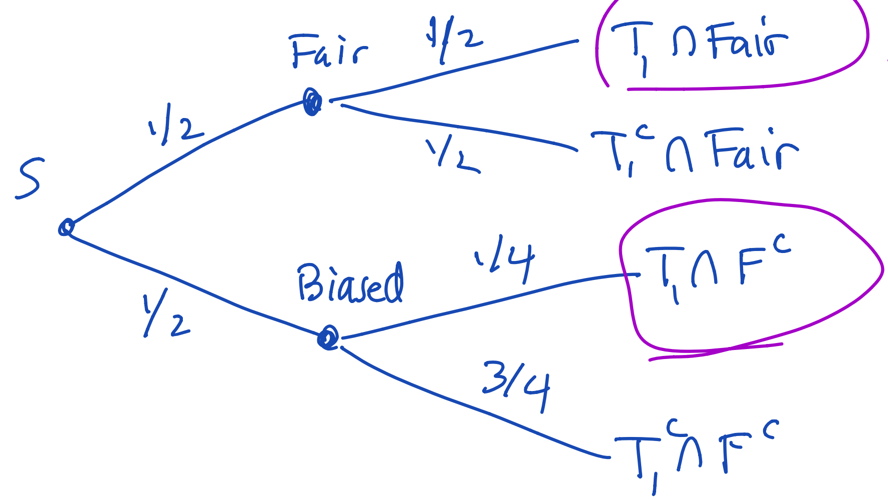
$= 1000(.05)(.2) + 10(0.05)(.8) = 10 + 0.4$

Problem 4. (12 POINTS (6 POINTS EACH PART))

A wallet contains two coins. One is a fair coin; the other has a probability of tails of 0.25. You pick a coin at random and give the other coin to someone for safe keeping.

(a) You flip the coin you picked; what is the probability of getting a tails?

(b) You flip the coin you picked a second time. What is the probability of getting a tail on the second flip, given that you got a tail on the first flip?



the two events yielding 1st Tails

a)

$$P(T_1) = P(T_1 | Fair)P(Fair) + P(T_1 | Fair^c)P(Fair^c)$$
$$= \frac{1}{2}\frac{1}{2} + \frac{1}{4}\frac{1}{2} = 3/8$$

b).

$$P(T_2 | T_1) = \frac{P(T_1 \cap T_2)}{P(T_1)} = \frac{P(T_1 \cap T_2 | Fair)P(Fair) + P(T_1 \cap T_2 | Fair^c)P(Fair^c)}{P(T_1)}$$

(a) 3/8

(b) 5/12

Note: you are using the same coin.
You can also compute $P(Fair | T_1)$

$$= \frac{P(T_1 | Fair)P(Fair)}{P(T_1)} = \frac{1/2 \cdot 1/2}{3/8} = \frac{2}{3}$$

The implication here, so, if Fair, more likely to get a second head

$$= \frac{\frac{1}{2}\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{2}}{3/8} = \frac{\frac{4}{32} + \frac{1}{32}}{3/8} = \frac{5}{32}\frac{8}{3} = \frac{5}{12}$$

Problem 5. (12 POINTS (6 POINTS EACH PART))

Suppose the PDF of X is given by

$$f_X(x) = \begin{cases} cx^2 & \text{if } \; 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P(X \le 1)$.

a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_0^2 cx^2\,dx = \frac{cx^3}{3}\Big|_0^2$

$= c\,\frac{8}{3} = 1 \Rightarrow c = 3/8$

b) $P(X \le 1) = \int_0^1 f_X(x)\,dx = \frac{cx^3}{3}\Big|_0^1$

$= c\,\frac{1}{3} = 1/8$

(a) $3/8$
(b) $1/8$

recall sifting property

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-a)\,dx = g(a)$$

Problem 6. (12 POINTS (6 POINTS EACH PART))

Suppose the PDF of X is given by

$$f_X(x) = \begin{cases} x + \delta(x-1/2)/2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the expected value of X: $E(X)$.

(b) Find the variance of X: $\mathrm{VAR}(X)$

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_0^1 x\left(x + \tfrac{1}{2}\delta(x-\tfrac{1}{2})\right)dx$$

$$= \int_0^1 x^2\,dx + \tfrac{1}{2}\int_0^1 x\,\delta(x-\tfrac{1}{2})\,dx = \frac{x^3}{3}\Big|_0^1 + \frac{1}{2}\left(\frac{1}{2}\right)$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \boxed{7/12}$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^3\,dx + \frac{1}{2}\int_0^1 x^2\,\delta(x-\tfrac{1}{2})\,dx$$

$$= \frac{x^4}{4}\Big|_0^1 + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{8} = 3/8$$

(a)	$7/12$
(b)	$\mathrm{Var}(X) = \frac{3}{8} - \left(\frac{7}{12}\right)^2 = \frac{3\cdot 18}{8\cdot 18} - \frac{49}{144} = \frac{54}{144} - \frac{49}{144}$

$8\overline{)144}$ 18

$$\begin{array}{r} 18 \\ \times\ 3 \\ \hline 54 \end{array}$$

$$= \boxed{5/144}$$

Problem 7. (12 POINTS (6 POINTS EACH PART))

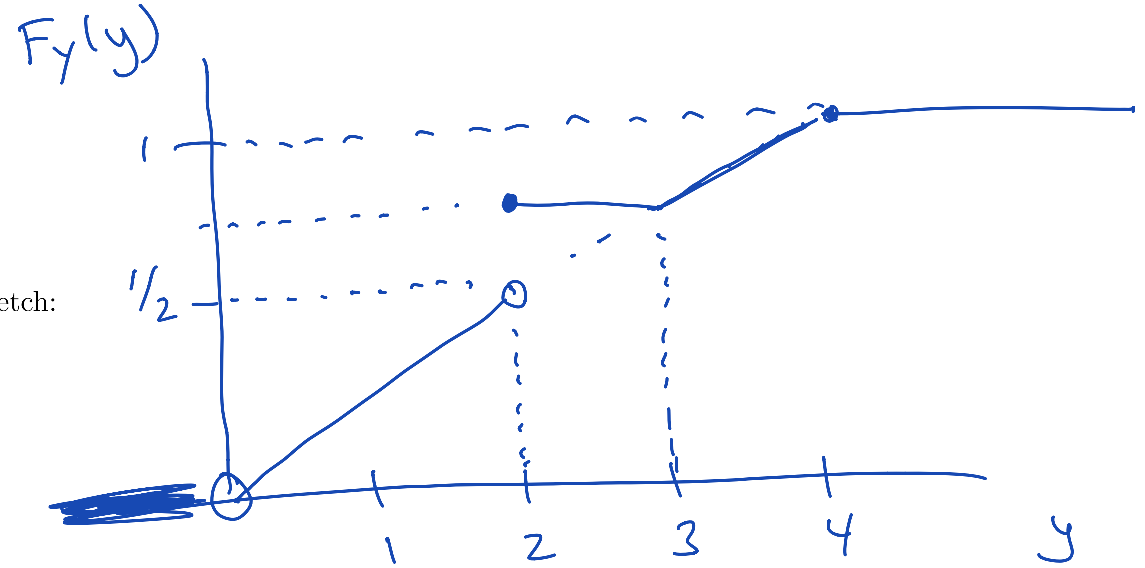
Suppose the CDF of the random variable Y is given by

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y/4 & \text{for } 0 \leq y < 2 \\ 3/4 & \text{for } 2 \leq y < 3 \\ y/4 & \text{for } 3 \leq y < 4 \\ 1 & \text{for } y \geq 4 \end{cases}$$

(a) Sketch the CDF. Is Y a continuous, discrete, or mixed random variable?

(b) Find $P(1 \leq Y \leq 2)$.

Sketch:



Continuous, discrete, or mixed: mixed, There is at least one jump, and at least one interval of continuity

(b) $P(1 \leq Y \leq 2) = P(1 < Y \leq 2) + P(Y=1)$

$= F_X(2) - F_X(1) + 0$

$= 3/4 - 1/4 = \boxed{1/2}$

Problem 8. (6 POINTS)

Let X be a random variable with mean μ and variance σ^2, and let $Y = 2X^2 + 4X - 1$. Express $E(Y)$ in terms of μ and σ.

$$E(Y) = E(2X^2 + 4X - 1)$$

$$= 2E(X^2) + 4E(X) - E(1)$$

$$= 2E(X^2) + 4\mu - 1$$

$$Var(X) = E(X^2) - E(X)^2$$

so $E(X^2) = Var(X) + E(X)^2$

$$= \sigma^2 + \mu^2$$

So $E(Y) = 2\sigma^2 + 2\mu^2 + 4\mu - 1$

$E(Y) =$ $2\sigma^2 + 2\mu^2 + 4\mu - 1$