PURDUE UNIVERSITY

Part 2:

#10-12,
14-19,
33-37

Past Exam Questions
(Fall 2015, Spring 2016, Fall 2016, Fall 2017)
Chapter 5 and beyond

Reibman

(November 2018)

SOLUTIONS

These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapter 5, 7, 9, and 10 in our textbook.

I will post the last pages of the final – with the formulas that you'll have available to you – separately.

Problem 10. (15 POINTS)
Given the Joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 0.50 & \text{for } 0 < x \leq 0.5 \text{ and } 0 < y \leq 0.5 \\ 1.25 & \text{for } 0.5 < x \leq 1 \text{ and } 0 < y \leq 0.5 \\ 1.50 & \text{for } 0 < x \leq 0.5 \text{ and } 0.5 < y \leq 1 \\ 0.75 & \text{for } 0.5 < x \leq 1 \text{ and } 0.5 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$y = 3/4$ is in this piece

(a) Find and sketch the conditional PDF of X given that $Y = 3/4$.

(b) What is $E(X|Y = 3/4)$?

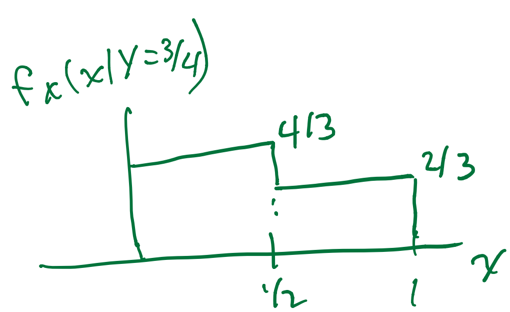
a) We know $Y = 3/4$, so look just @ that particular value of y.

$$f_X(x|Y=3/4) = \begin{cases} b\ 1.5 & 0 < x \leq 0.5 \\ b\ 0.75 & 0.5 < x \leq 1 \end{cases}$$

find b: $\quad 1 = \int_0^{1/2} b\ 3/2\, dx + \int_{1/2}^{1} b\ 3/4\, dx$

$$= \frac{b}{2}\left(\frac{3}{2} + \frac{3}{4}\right) = \frac{b}{8}(6+3) = \frac{9}{8} b$$

$$\Rightarrow b = 8/9$$



$$f_X(x|y=3/4) = \begin{cases} 4/3 & 0 < x \leq 0.5 \\ 2/3 & 0.5 < x \leq 1 \end{cases}$$

b) $E(X|Y=3/4) = \int_0^{1/2} x\, 4/3\, dx + \int_{1/2}^{1} x\, 2/3\, dx = \frac{4}{3}\frac{x^2}{2}\Big|_0^{1/2} + \frac{2}{3}\frac{x^2}{2}\Big|_{1/2}^{1}$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(1 - \left(\frac{1}{2}\right)^2\right) = \frac{1}{3}\left(\frac{2}{4} + \frac{3}{4}\right) = \boxed{\frac{5}{12}}$$

Problem 11. (5 POINTS)
Let X and Y be two discrete RV's with joint PMF given by

$$p_{X,Y}(x,y) = \begin{cases} 0.800 & \text{for } x=0 \text{ and } y=0 \\ 0.050 & \text{for } x=1 \text{ and } y=0 \\ 0.025 & \text{for } x=0 \text{ and } y=1 \\ 0.125 & \text{for } x=1 \text{ and } y=1 \\ 0 & \text{otherwise} \end{cases}$$

only

What is $Var(X|Y=1)$?

first, find the conditional pmf of X given Y=1.

Narrow the sample space + renormalize
≡ chop and scale

$$p_X(x|Y=1) = \begin{cases} b\ 0.025 & \text{for } x=0 \\ b\ 0.125 & \text{for } x=1 \end{cases}$$

find b: $b(0.025+0.125) = 1 = b(0.15)$

$b = 1/0.15$

so $p_X(x|Y=1) = \begin{cases} 1/6 & \text{for } x=0 \\ 5/6 & \text{for } x=1 \end{cases}$

$$E(X|Y=1) = 0(1/6) + 1(5/6) = 5/6$$

$$E(X^2|Y=1) = 0^2(1/6) + 1^2(5/6) = 5/6$$

So $Var(X|Y=1) = E(X^2|Y=1) - E(X|Y=1)^2$

$$= 5/6 - (5/6)^2 = \frac{5\cdot 6}{6\cdot 6} - \frac{5\cdot 5}{6\cdot 6}$$

$$= \boxed{5/36}$$

Problem 12.
X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (4x+2y)/3 & \text{for } 0 \le x \le 1; 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) For which values of y is $f_X(x|y)$ defined? $0 \le y \le 1$

(b) Find $f_X(x|y)$

(c) For which values of x is $f_Y(y|x)$ defined? $0 \le x \le 1$

(d) Find $f_Y(y|x)$

b) $f_x(x|y) = \dfrac{f_{xy}(x,y)}{f_y(y)}$ (when $f_y(y) > 0$) so need to find $f_y(y)$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \frac{1}{3}\int_0^1 (4x+2y)\,dx = \frac{1}{3}\left(\frac{4x^2}{2} + 2xy\right)\Big|_0^1$$

$$= \frac{1}{3}(2+2y) - 0 = \frac{2}{3}(1+y) \text{ when } 0 \le y \le 1$$

and $\boxed{f_x(x|y) = \dfrac{4x+2y}{2(1+y)} \text{ when } 0 \le x \le 1 \text{ and } 0 \le y \le 1}$

d) need $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \frac{1}{3}\int_0^1 (4x+2y)\,dy$

$$= \frac{1}{3}\left(4xy + \frac{2y^2}{2}\right)\Big|_0^1 = \frac{1}{3}(4x+1) \quad \text{when } 0 \le x \le 1$$

so $f_Y(y|x) = \dfrac{f_{xy}(x,y)}{f_x(x)} = \boxed{\dfrac{4x+2y}{4x+1} \text{ when } 0 \le x \le 1 \text{ and } 0 \le y \le 1}$

Problem 13.
X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} k & \text{for } 0 < y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

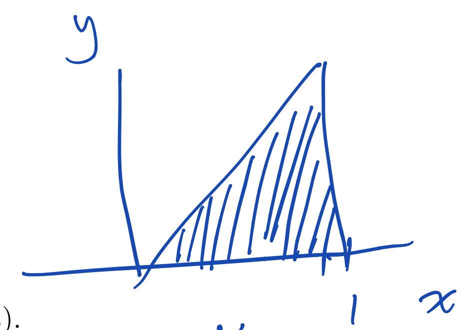
(a) Find the marginal pdf's of X and Y (you don't have to find k yet).

(b) Find k.

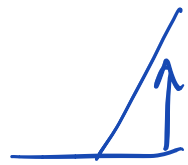
(c) Find $P(0 < X < 1/2; 0 < Y < 1/2)$.

(d) Find the conditional pdf's $f_Y(y|x)$ and $f_X(x|y)$.

(e) Compute the conditional means $E(Y|x)$ and $E(X|y)$.



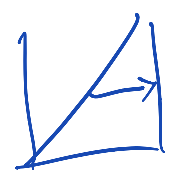
area 1/2
$\Rightarrow K = 2.$



integrate from 0 to x

a) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$= \int_0^x k\,dy = \begin{cases} kx & \text{when } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$

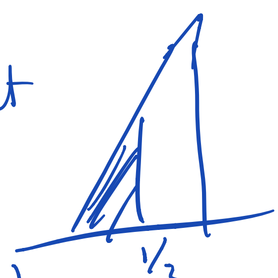


integrate from y to 1

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_y^1 k\,dx = \begin{cases} k(1-y) & \text{when } 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$

b) $K = 2$ by inspection above

c) Solve using areas, since $f_{XY}(x,y)$ is flat



area of shaded triangle is $\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

area of full region of support is $\frac{1}{2}(1)(1)$

$\Rightarrow P(X < 1/2, Y < 1/2) = \frac{1/8}{1/2} = 1/4$

d) $f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{K}{Kx} = \begin{cases} \frac{1}{x} & \text{when } 0 < y \leq x \leq 1 \\ 0 & \text{else} \end{cases}$

$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-y} & \text{when } 0 < y \leq x < 1 \\ 0 & \text{else} \end{cases}$

#13 e) $E(y|x) = \int_{-\infty}^{\infty} y \, f_y(y|x) \, dy$

$= \int_0^1 \frac{y}{x} \, dy = \frac{y^2}{2x} \Big|_0^1 = \frac{1}{2x}$

for $0 < x < 1$

$E(X|y) = \int_{-\infty}^{\infty} x \, f_x(x|y) \, dx$

$= \int_0^1 \frac{x}{1-y} \, dx = \frac{x^2}{2(1-y)} \Big|_0^1$

$= \frac{1}{2(1-y)}$

for $0 < y < 1$

Problem 14. (15 POINTS)

Let X be the number of bugs in the first draft of a piece of software, and let Y be the number of bugs remaining in the second draft of the same software. Their joint PMF is given by the following table:

$p_{xy}(x,y)$

		Y=		
		0	1	2
	0	1/6	0	0
X=	1	1/3	1/6	0
	2	1/12	1/12	1/6

(a) What is the conditional PMF of Y given $X = 2$?

(b) What is the marginal PMF of X?

(c) Find the Variance of X.

a) PMF of Y given X=2 ⟶ grab the row of X=2 and scale it to make it sum to 1.

current sum $= \frac{1}{12} + \frac{1}{12} + \frac{1}{6} = \frac{2}{6}$ ⟹ multiply by 3.

$$p_y(y \mid x=2) = \begin{cases} 3/12 & \text{if } y=0 \\ 3/12 & \text{if } y=1 \\ 6/12 & \text{if } y=2 \end{cases}$$

b) marginal of X

$$p_x(x) = \sum_{y=0}^{2} p_{xy}(x,y) = \begin{cases} 1/6 & x=0 \\ 3/6 & x=1 \\ 2/6 & x=2 \end{cases}$$

c) $\text{var}(X) = E(X^2) - E(X)^2$

$E(X) = 0(1/6) + 1(3/6) + 2(2/6) = 7/6$

$E(X^2) = 0^2(1/6) + 1^2(3/6) + 2^2(2/6) = 3/6 + 8/6 = 11/6$

$$\text{Var}(X) = \frac{11}{6} - \left(\frac{7}{6}\right)^2 = \frac{66}{36} - \frac{49}{36} = \boxed{\frac{17}{36}}$$

Problem 15. (20 POINTS)
Given X with PDF

$$f_X(x) = \begin{cases} 1/2 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Then Y is uniformly distributed between x and $2x$.

(a) Sketch the region of support. That is, indicate where $f_{XY}(x, y)$ is nonzero.

(b) Find the joint PDF $f_{XY}(x, y)$.

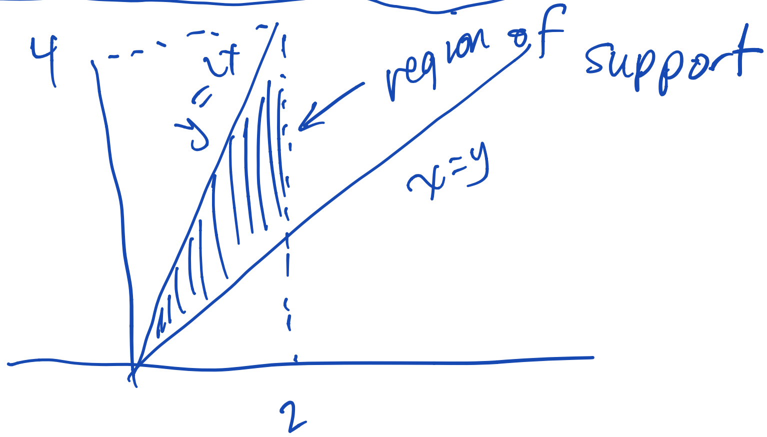
(c) What is $P(Y < 1)$?

(d) What is $E(Y)$? (Hint: you may use the law of iterated expectations.)

"y is uniform between x and 2x" means that the pdf of Y is defined conditionally based on the value of X. So

$f_Y(y|x) = \frac{1}{x}$ $x \le y \le 2x$ because it's uniform

a)



region of support is $0 \le x \le 2$ and $x \le y \le 2x$

b) $f_{XY}(x,y) = f_Y(y|x)\, f_X(x) = \begin{cases} \frac{1}{2x} & 0 \le x \le 2 \text{ and } x \le y \le 2x \\ 0 & \text{else} \end{cases}$

c) $P(Y<1)$ requires a double integration. Note that since $f_{XY}(x,y)$ is <u>NOT</u> flat, we can't use the area approach.

(continued)

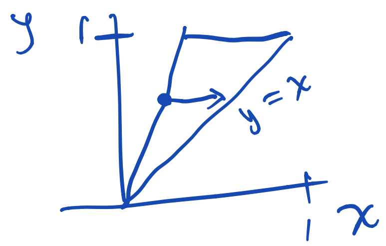
\#15c continued.

There are 3 possible approaches.
All require double integration.

1. inner integral is x outer is y
2. inner integral is y outer is x
3. find $f_Y(y)$ then integrate over y

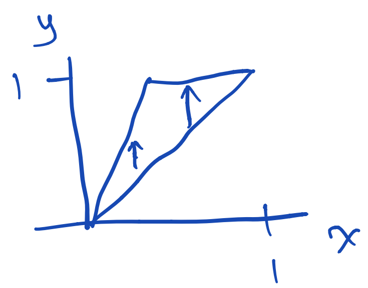
A quick triage of which approach is fastest:

Approach 1:
inner x
outer y



just one set of integration bounds

Approach 2:
inner y
outer x



need 2 sets of integration bounds

Approach 3
find $f_Y(y)$
integrate it

need to integrate the full region over x

from top picture, there are 2 regions, but in fact we only need one of them..

Decision: Approach 1 is easiest.

$$P(Y<1) = \int_{y=0}^{1} \int_{x=y/2}^{y} \frac{1}{2x}\, dx\, dy = \int_0^1 \frac{1}{2} \ln x \Big|_{2y}^{y} dy$$

$$= \frac{1}{2} \int_0^1 \ln\left(\frac{y}{y/2}\right) dy = \frac{1}{2} \int_0^1 \ln 2\, dy = \boxed{\frac{1}{2} \ln 2}$$

15 d   what is $E(Y)$?

Long super tedious approach:

1st find $f_Y(y)$ – do the 2-part double integration mentioned above

2nd compute $\int y\, f_Y(y)\, dy$

You will find this second step, for the part where $2 < y < 4$ to be NO FUN AT ALL.

So use iterated expectations.

$$E(Y) = E(E(Y|X))$$

$$E(Y|X) = \frac{X + 2X}{2} = \frac{3X}{2}$$

since Y is uniformly distributed from X to $2X$.

Then $E(Y) = E\left(\frac{3X}{2}\right) = \frac{3}{2} E(X) = \boxed{\frac{3}{2}}$

because X is uniform between 0 and 2

Problem 16. (5 POINTS)

The number of "likes" to a social-media post on any given day is a Poisson random variable, with mean α. However, the parameter α is a random variable that depends on the amount of sunshine outside and is uniformly distributed on the continuous interval $[0, 3]$.
What is the probability there is exactly one "like" on a specific day?
(Hint: you may find the following integral useful.)

$$\int xe^{ax}\,dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$$

For a given α, the Poisson pmf is

$$p_X(x|\alpha) = \frac{\alpha^x e^{-\alpha}}{x!}$$ where X is the # likes conditioned on α value.

α is uniform, height $1/3$, from $[0,3]$.

$$p_X(x) = \int_\alpha p_X(x|\alpha) f_\alpha(\alpha)\,d\alpha$$

and we want $p_X(1) = P(X=1)$

$$p_X(1|\alpha) = \frac{\alpha^1 e^{-\alpha}}{1!}$$

and $$p_X(1) = \int_0^3 \alpha e^{-\alpha}\left(\frac{1}{3}\right)d\alpha$$

Solve using integration by parts, or applying the given integral, by mapping $x \to \alpha$ and $a \to -1$

$$p_X(1) = \frac{1}{3}\left(\frac{\alpha}{-1} - \frac{1}{1}\right)e^{-\alpha}\Big|_0^3 = \frac{1}{3}\left[(-3-1)e^{-3} - (0-1)e^{0}\right]$$

$$= \frac{1}{3}\left(-4e^{-3}+1\right) = \boxed{1/3\left(1-4e^{-3}\right)}$$

region of support

Problem 17. (15 POINTS)

Background story: You order many boxes of pizza for a gathering of Purdue students. When you check the pizza boxes some time after the gathering starts, there is a fraction X of pizza remaining, where X is a random variable between 0 and 1. You check the boxes of pizza again at a later time during the gathering, and observe a fraction Y of pizza still remaining. Y is a random variable between 0 and X. You ordered enough boxes of pizza that you can safely assume that both X and Y are continuous random variables.

Math problem: Suppose X has the PDF

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and Y is uniformly distributed between $[0, X]$.

$$f_Y(y|x) = \begin{cases} \frac{1}{x}, & 0<y<x \\ 0 & \text{else} \end{cases}$$

(a) Find the joint PDF, $f_{XY}(x, y)$, **including its region of support**.

(b) Find the marginal PDF of Y.

(c) What is $P(Y < 1/2)$?

a) $$f_{XY}(x,y) = f_Y(y|x) f_X(x) = \boxed{\begin{cases} \frac{1}{x} 3x^2 & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le x \\ 0 & \text{else} \end{cases}}$$

b) $$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_y^1 3x\,dx = \frac{3x^2}{2}\Big|_y^1$$

see sketch above

$$= \boxed{\begin{cases} \frac{3}{2}(1-y^2) & \text{for } 0 \le y \le 1 \\ 0 & \text{else} \end{cases}}$$

c) $$P(Y<1/2) = \int_0^{1/2} f_Y(y)\,dy = \int_0^{1/2} \frac{3}{2}(1-y^2)\,dy$$

$$= \frac{3}{2}\left(y - \frac{y^3}{3}\right)\Big|_0^{1/2} = \frac{3}{2}\left[\frac{1}{2} - \left(\frac{1}{2}\right)^3 \frac{1}{3}\right]$$

$$= \frac{3}{4}\left[1 - \frac{1}{12}\right] = \frac{3}{4} \cdot \frac{11}{12} = \boxed{\frac{33}{48}} = \boxed{\frac{11}{16}}$$

Problem 18. (20 POINTS)
Suppose you build a system with a part that comes from either Company 1 (with probability 2/3) or Company 2 (with probability 1/3). Let N indicate the company, so that N is a discrete RV with PMF

$$p_N(n) = \begin{cases} 2/3 & \text{for } n = 1 \\ 1/3 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let X be the lifetime of the part, which has a different distribution whether the part comes from Company 1 or Company 2. In particular, let X be a continuous RV such that when $N = 1$, X is exponentially distributed with mean 5, and when $N = 2$, X is exponentially distributed with mean 8.

(a) What is the conditional PDF of X given N=2?
(In other words, what is the PDF of X when the part comes from Company 2?)

(b) Find the marginal PDF of X.

(c) What is $P(X > 8)$?

(d) If your device has lasted long enough that $X > 8$ already, what is the probability it is from Company 1, i.e., that $N = 1$?

a) from the problem statement, $\boxed{f_X(x|N=2) = \frac{1}{8}\exp\left(-\frac{x}{8}\right) \text{ when } x\geq 0}$

b) $f_X(x) = P(N=1)\,f_X(x|N=1) + P(N=2)\,f_X(x|N=2)$

$= \frac{2}{3}\left(\frac{1}{5}\exp\left(-\frac{x}{5}\right)\right) + \frac{1}{3}\left(\frac{1}{8}\exp\left(-\frac{x}{8}\right)\right) \quad x\geq 0$

$\boxed{= \frac{2}{15}\exp\left(-\frac{x}{5}\right) + \frac{1}{24}\exp\left(-\frac{x}{8}\right) \quad x \geq 0}$

c) $P(X>8) = \int_8^\infty f_X(x) = \frac{2}{15}(-5)\exp\left(-\frac{x}{5}\right)\Big|_8^\infty + \frac{1}{24}(-8)\exp\left(-\frac{x}{8}\right)\Big|_8^\infty$

$= \boxed{2/3\exp(-8/5) + 1/3\exp(-1)}$

d) Bayes Rule
$P(N=1|X>8) = \frac{P(X>8 \cap N=1)}{P(X>8)} = \frac{P(X>8|N=1)P(N=1)}{P(X>8)}$

$P(X>8|N=1) = \exp(-8/5)$, $P(N=1) = 2/3$, $P(X>8)$ above.

$P(N=1|X>8) = \frac{2/3\exp(-8/5)}{2/3\exp(-8/5) + 1/3\exp(-1)} = \boxed{\frac{2}{2+\exp(-3/5)}}$

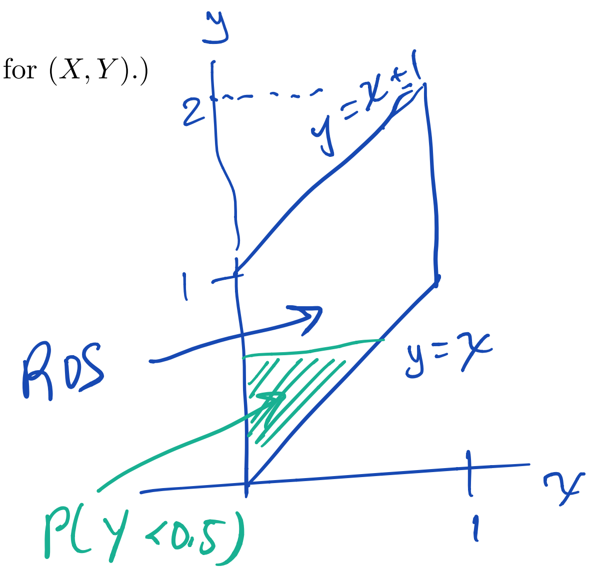
Problem 19. (20 POINTS)
Let X be a continuous uniform RV on the interval $[0,1]$. Conditioned on X, then Y is a continuous RV that is uniformly distributed on the interval $[x, x+1]$.

(a) What is $P(Y > 0.5)$?
(Hint, draw a clear diagram of the region of support for (X,Y).)

(b) Find the marginal PDF of Y.

(c) Find $E(Y)$. (Hint: are you able to use symmetry?)

(d) Find $COV(X,Y)$.



a) $P(Y > 0.5) = 1 - P(Y < 0.5)$

$= 1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 7/8$

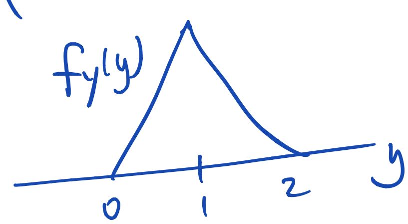
b) $f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\,dx$

There are 2 different regions of y we care about

When $0<y<1$, x varies between 0 and y

when $1<y<2$ x varies between $y-1$ and 1

$$f_Y(y) = \begin{cases} \int_0^y 1\,dx & 0 \le y \le 1 \\ \int_{y-1}^1 1\,dx & 1 \le y \le 2 \end{cases} = \begin{cases} y & 0 \le y \le 1 \\ 2-y & 1 \le y \le 2 \end{cases}$$



c) $E(Y) = 1$ because $f_Y(y)$ is symmetric about 1.

d). $Cov(XY) = E(XY) - E(X)E(Y)$, where $E(X) = 1/2$ and we just found $E(Y) = 1$.

$$E(XY) = \int_0^1 \int_x^{x+1} xy\,dy\,dx = \int_0^1 \frac{xy^2}{2}\Big|_x^{x+1} dx = \int_0^1 \frac{2x^2+x}{2}\,dx$$

$$= \frac{2x^3}{6} + \frac{x^2}{4}\Big|_0^1 = \frac{1}{3} + \frac{1}{4} = 7/12, \text{ so } cov(X,Y) = \frac{7}{12} - \frac{1}{2} = \boxed{1/12}$$

Problem 33. (Yes/No: 2 points) The sum of 2 Gaussians is always Gaussian,
If X and Y are jointly Gaussian RVs then $X + Y$ is a Gaussian RV. whether they are correlated or not

Problem 34. (Yes/No: 2 points)
If X and Y are ~~jointly~~ Gaussian RVs then X and Y are independent only if X and Y are uncorrelated.

Problem 35. (Yes/No: 2 points) ← they must be independent, not just uncorrelated
If X and Y are uncorrelated RVs then the PDF of Z, where $Z = X+Y$, can be found by convolving the PDF's of X and Y.

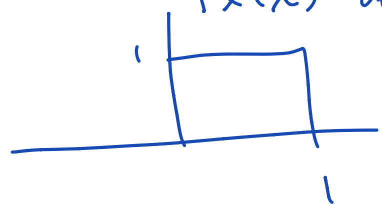
Problem 36. (Yes/No: 4 points)
If X and Y are independent random variables, then the CDF of $Z = \max(X, Y)$ is
$F_Z(z) = F_X(z)F_Y(z)$.

$$F_Z(z) = P(Z \le z)$$
$$= P(X \le z \cap Y \le z)$$
$$= P(X \le z)\,P(Y \le z) = F_X(z)F_Y(z)$$

Problem 37.
Let X and Y be independent RVs each uniformly distributed on $[0, 1]$. Let $Z = X + Y$ and $W = X - Y$. Find the marginal PDF's of Z and W.

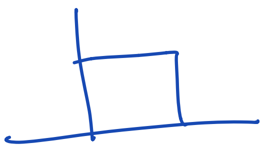
(convolution)

$f_X(x)$ and $f_Y(y)$

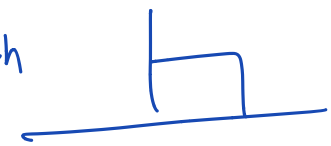


$Z = X + Y$

convolution of



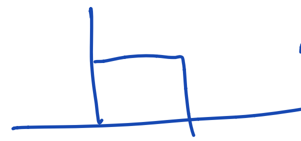
with



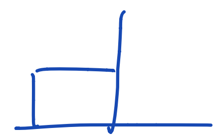
mean $E(Z) = E(X) + E(Y) = 1$

$W = X - Y$

convolution of



with



mean $E(W) = E(X) - E(Y)$
$= 0$