ECE 302: Probabilistic Methods in Electrical and Computer Engineering

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PURDUE UNIVERSITY

Part 3:

42-47

Past Exam Questions
(Fall 2015, Spring 2016, Fall 2016, Fall 2017)
Chapter 5 and beyond

Reibman

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SOLUTIONS

These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapter 5, 7, 9, and 10 in our textbook.

I will post the last pages of the final – with the formulas that you'll have available to you – separately.

Problem 42. (15 POINTS)

You want to watch a 10-minute video. Due to the current network conditions, in each (discrete) minute of video watching, there is a chance p of having one "rebuffering event", and probability $1-p$ of having none. A "rebuffering event" happens when there is not enough bandwidth to play the video continuously, and so the video playback stalls. The "rebuffering events" in each minute are independent of each other.

(a) What is the PMF of the random variable X_{10}, which is the total number of "rebuffering events" experienced during the 10-minute video? (Note: You may assume that each playback stall is so short in time that it is negligible and can be ignored, when counting the length of the video.)

Now, let X_k be the discrete-time random process which counts the cumulative number of "rebuffering events" experienced up until the current time. For example, X_3 is the total number of "rebuffering events" that happened during minutes 1, 2, and 3, and X_{10} is as described above.

(b) Find the PMF for X_k for $k = 1, 2, \ldots, 10$.

(c) What is the mean of the random process as a function of time k.

a) 10 independent Bernoulli RVs each w/ prob p

$\Rightarrow$ binomial $(10, p)$, where $n=10$

$$\boxed{P_{X_{10}}(x) = \binom{10}{x} p^x (1-p)^{10-x} \quad \text{for } x = 0,1,2,\ldots 10}$$

b) for each time k, we have k independent Bernoullis each w/ probability p

$\Rightarrow$ for each time k, a binomial (k, p)

$$\boxed{P_{X_k}(x) = \binom{k}{x} p^x (1-p)^{k-x} \quad \text{for } k = 1,2,\ldots,10 \text{ and } x = 0,1,\ldots,k.}$$

c) The mean of a binomial (n,p) is np.

So the mean of each time step's binomial is kp

$$\boxed{E(X_k) = kp}$$

Problem 43. (YES/NO: 4 POINTS)
If a WSS random process $X(t)$ is input to a stable linear time invariant system, the output $Y(t)$ is a WSS random process with power spectral density $S_Y(f) = |H(f)|^2 S_X(f)$.

Problem 44. (YES/NO: 4 POINTS)
If $X(t)$ is a zero-mean random process, then its auto-correlation function equals its auto-covariance function.

Problem 45. (MULTIPLE CHOICE: 5 POINTS)
Suppose $X(t)$ is a WSS random process with mean $m_X = 0$ and autocorrelation function $R_X(\tau) = 2\exp(-2|\tau|)$. What is the **correlation coefficient** between $X(2)$ and $X(-1)$?

(a) e^{-3}

(b) 0

(c) $e^{-2} - e^{-1}$

(d) e^{-6}

(e) None of the above.

$$\rho = \frac{Cov(X(2), X(-1))}{\sqrt{Var(X(2))\, Var(X(-1))}}$$

Because WSS, $Var(X(t)) = R_X(0)$ for all t
$= 2$

$$Cov(X(2), X(-1)) = E(X(2)X(-1)) - E(X(2))E(X(-1))$$
$$= R_X(2-(-1)) - 0$$
$$= R_X(3) = 2e^{-6}$$

So $\rho = \frac{2e^{-6}}{2} = \boxed{e^{-6}}$

Problem 46. (10 POINTS)

Given a WSS random process $X(t)$ with autocorrelation function $R_X(\tau)$, we want to estimate the value of $X(t_0)$ from a sample taken Δ seconds previously, namely from $X(t_0 - \Delta)$. One common solution is to use a linear predictor, such that the estimate

$$\hat{X}(t_0) = aX(t_0 - \Delta).$$

(a) Write an expression – in terms of $R_X(\tau)$ – of the expected squared prediction error

$$E\left[(X(t_0) - aX(t_0 - \Delta))^2\right].$$

(b) Find the value of a to minimize the expected squared prediction error, if $R_X(\tau) = \exp(-b|\tau|)$.

a) $E((X(t_0) - a X(t_0 - \Delta))^2)$

$= E[X^2(t_0) - 2a X(t_0) X(t_0 - \Delta) + X^2(t_0 - \Delta)]$

$= E(X^2(t_0)) - 2a E(X(t_0) X(t_0 - \Delta)) + a^2 E(X^2(t_0 - \Delta))$

Express in terms of $R_X(\tau)$, where τ is the time difference between the samples

$= R_X(0) - 2a R_X(\Delta) + a^2 R_X(0)$

$= (1 + a^2) R_X(0) - 2a R_X(\Delta)$

b) Take derivative with respect to unknown a, and set to zero and solve.

$\frac{d}{da}(\text{expected squared error}) = 2a R_X(0) - 2 R_X(\Delta) = 0$

$\Rightarrow a = \frac{R_X(\Delta)}{R_X(0)}$

For $R_X(\tau) = e^{-b|\tau|}$

$\boxed{a = e^{-\Delta b}}$

Problem 47. (15 POINTS)
Suppose X_n is a discrete-time random process with $E(X_n) = 0$ and $\text{VAR}(X_n) = 3$ for all n, where $E(X_i X_j) = 0$ for $i \neq j$. This random process is input to a digital filter to create Y_n, where

$$Y_n = X_n - X_{n-1} \quad \text{for all } n.$$

(a) Find $E(Y_n)$.

(b) Find the auto-correlation of Y_n, which is denoted $R_Y(m,k)$.

(c) Is Y_n a wide-sense stationary random process? Explain your answer by providing justification.

a) $E(Y_n) = E(X_n - X_{n-1}) = E(X_n) - E(X_{n-1}) = 0$

b) $R_Y(m,k) = E(Y_m Y_k) = E((X_m - X_{m-1})(X_k - X_{k-1}))$

$= E(X_m X_k) - E(X_m)(X_{k-1}) - E(X_{m-1}) E(X_k) + E(X_{m-1}) E(X_{k-1})$

4 situations to pay attention to.

1) $m=k$. $E(X_m X_k) = E(X_m^2) = Var(X_m) = 3$.

$E(X_m X_{k-1}) = E(X_{m-1} X_k) = 0$

$E(X_{m-1} X_{k-1}) = Var(X_{k-1}) = 3$.

so $R_Y(m,k) = 6$ when $m=k$.

2) $m=k-1$. $E(X_m X_k) = E(X_m X_{k+1}) = 0$

$E(X_m X_{k-1}) = E(X_m^2) = Var(X_m) = 3$

$E(X_{m-1} X_k) = E(X_{k-2} X_k) = 0$

so $R_Y(m,k) = 3$ when $m = k-1$

3) $m = k+1 \longrightarrow$ similar to case 2 $R_Y(m,k) = 3$

4) $|m-k| \geq 2$ all terms are 0 so $R_Y(m,k) = 0$

Combined, $R_Y(m,k) = \begin{cases} 6 & |m-k| = 0 \\ 3 & |m-k| = 1 \\ 0 & |m-k| \geq 2 \end{cases} \rightarrow$ depends only on $m-k$.