SOLUTIONS

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2021
Instructor: Prof. A. R. Reibman

PURDUE UNIVERSITY

Exam 2

Fall 2021, MWF 11:30am-12:20pm
(November 4, 2021)

PURDUE UNIVERSITY

This is a closed book exam with 10 problems. Neither calculators nor help sheets are allowed.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name or initials on every page of the exam and turn in everything when time is up.

Write your answers in the boxes provided. We will be scanning the exams, so **DO NOT WRITE ON THE BACK of the pages!**.

Name: ____________________ PUID: ____________________

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: ____

Problem 1. (TRUE/FALSE: 5 POINTS EACH, TOTAL 20 POINTS)

For each of the following statements, determine which is valid.
If you show your reasoning you might get partial credit.
Finding a counter-example might be helpful if the answer is FALSE.
(Note: if a statement is not always true, then it is FALSE.)

Clearly label each statement T or F in the box to the left of the problem.

T

(a) If X and Y are independent RVs, then X^3 and e^Y are independent RVs.

This fact was mentioned in class

T

$|2-\sqrt{3}| < |1-0| \Rightarrow$ true

(b) If X is a uniform random variable on the interval [0,10], and $Y = g(X) = X^2$, then $P(0 \le Y \le 1) > P(3 \le Y \le 4)$.

$P(0 \le Y \le 1) = P(0 \le X^2 \le 1) = P(0 \le X \le 1)$ for $X \ge 0$, which it is

$P(3 \le Y \le 4) = P(3 \le X^2 \le 4) = P(\sqrt{3} \le X \le 2)$

F

(c) If X is an exponential random variable, then $P(X > 4|X > 3) = P(X > 4)$.

$P(X>4 \mid X>3) = \frac{P(X>4 \cap X>3)}{P(X>3)} = \frac{P(X>4)}{P(X>3)}$

$\neq P(X>4)$

*

(d) If X is a random variable with a PDF that is symmetric about zero, and $Y = |X|$, then $E(Y) = E(X|X > 0)$.

this one is tricky but was not meant to be a "gotcha".

If $f_X(x)$ is symmetric about 0, then

$f_X(-x) = f_X(x)$ for all x, so

$E(|X|) = \int_{-\infty}^{0} (-x) f_X(x)\,dx + \int_0^{\infty} x f_X(x)\,dx$

$E(X|X>0) = \int_0^{\infty} \frac{x}{P(X>0)} f_X(x)\,dx$ so $E(|X|) = \frac{2E(X|X>0)}{P(X>0)}$

continued on next page

Therefore $\underline{\text{if}}$ $P(X>0)=1/2$, $E(|X|) = E(X \mid X>0)$

Note: if $P(X=0)>0$ this is NOT true, but $\underline{\text{if}}$ X is continuous, this $\underline{\text{IS}}$ true

Problem 2. (Multiple choice: 5 points each; 10 points total)

The two questions share the common list of multiple-choice options. I will ignore all circles on the options; please put your answers in the boxes provided.
If you show your reasoning you might get partial credit.

Problem set up: 6 cars compete in a race that takes 200 laps to finish. Suppose the probability that each car breaks down in one lap is $p = 0.1$. After a car breaks down, it is fixed and continues the race. Assume the event of break-down for any car in any lap are independent.

List of possible answers:

(a) Geometric with parameter $p = 0.1$ and sample space starting with 0.

(b) Geometric with parameter $p = 0.9$ and sample space starting with 0.

(c) Geometric with parameter $p = 0.1$ and sample space starting with 1.

(d) Geometric with parameter $p = 0.9$ and sample space starting with 1.

(e) Binomial with parameters $n = 6$ and $p = 0.1$.

(f) Binomial with parameters $n = 200$ and $p = 0.1$.

(g) Binomial with parameters $n = 6$ and $p = 0.9$.

(h) Binomial with parameters $n = 200$ and $p = 0.9$.

(i) Bernoulli with parameter $p = 0.9$

(j) Bernoulli with parameter $p = 0.1$

(k) Insufficient information to determine.

(l) None of the above

Part 1: Let X be a random variable indicating the number of laps a given car completes without breaking down during that lap. (Remember that all cars eventually finish the race.) The random variable X is well-modeled by which of the following distributions?

$\boxed{\text{H}}$ Answer

all cars complete 200 laps, whether they break down or not

$X = \{\#\text{ laps w/out breaking down}\} \rightarrow$ binomial $n=200$, $p(\text{no breakdown}) = 1-0.1 = 0.9$

Part 2: Let X be a random variable indicating the lap number in which a specific car breaks down for the first time. The random variable X is well-modeled by which of the following distributions?

$\boxed{\text{C}}$ Answer

"for the first time" $\Rightarrow$ geometric

stopping criterion = "breakdown"

$\Rightarrow p(\text{success}) = p = 0.1$

lap # $\Rightarrow$ sample space starts @ 1

Problem 3. (Multiple choice: 4 points)

Express the probability $P(\{-1 < X \leq 4\} \cap \{2 < Y \leq 5\})$ in terms of $F_{X,Y}(x, y)$, the joint CDF of the random variables X and Y.

If you show your reasoning you might get partial credit.

D

Answer

(a) $F_{X,Y}(4,5) - F_{X,Y}(-1,5) - F_{X,Y}(4,2) - F_{X,Y}(-1,2)$

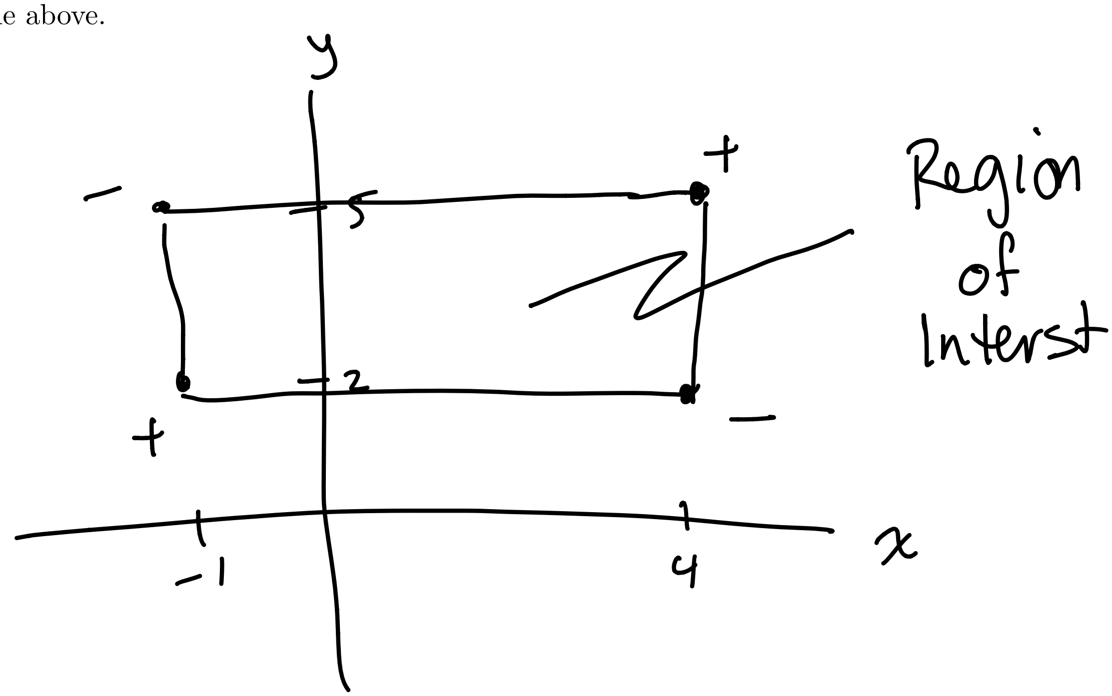
(b) $F_{X,Y}(4,5) + F_{X,Y}(-1,5) + F_{X,Y}(4,2) + F_{X,Y}(-1,2)$

(c) $F_{X,Y}(4,5) - F_{X,Y}(-1,5) + F_{X,Y}(4,2) - F_{X,Y}(-1,2)$

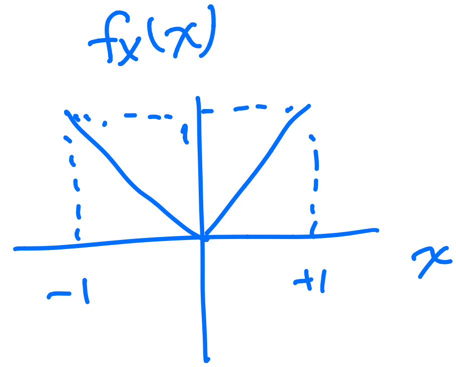
(d) $F_{X,Y}(4,5) - F_{X,Y}(-1,5) - F_{X,Y}(4,2) + F_{X,Y}(-1,2)$

(e) It is not possible to compute the probability of this event using a joint CDF.

(f) None of the above.



Problem 4. (12 POINTS)
Given the Probability Density Function



$$f_X(x) = \begin{cases} |x| & \text{when } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $f_X(x|B)$ when $B = \{X > 0\}$.

(b) Find $E(X|B)$ for the same event B.

a) $$f_X(x \mid X>0) = \begin{cases} \dfrac{f_X(x)}{P(X>0)} & \text{for } x>0 \\ 0 & \text{else} \end{cases}$$

$P(X>0) = \frac{1}{2}$ by symmetry

substituting $P(X>0)$ and $f_X(x)$, answer is as below

b) $E(X \mid X>0)$ = mean of conditional pdf

$$= \int_{-\infty}^{\infty} x \, f_X(x \mid X>0)\, dx = \int_0^1 2x^2\, dx = \frac{2x^3}{3}\Big|_0^1 = 2/3$$

(a): $$f_X(x \mid X>0) = \begin{cases} 2x & \text{for } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

(b): $2/3$

Problem 5. (6 POINTS)
A so-called "camera trap" is designed to observe animals in the wild without human intervention. An automated algorithm inside the camera decides when to start recording; ideally, it starts recording whenever an animal enters the scene, and never starts recording an empty video when there are no animals present. Unfortunately, the algorithm is not perfect, and empty recordings are made. The ability of the automated algorithm to correctly trigger depends on the lighting conditions. Suppose

- between 10am and 2pm, the probability of recording an empty video is 0.2, $E(X|C_1) = \frac{2}{10}$
- between 6pm and 6am the probability of recording an empty video is 0.1, and $E(X|C_2) = 1/10$
- between either 6am-10am or 2pm-6pm, the probability of recording an empty video is 0.4. $E(X|C_3) = \frac{4}{10}$

What is the expected fraction of empty videos recorded over the course of an entire month?

"a fraction of" is always between 0 and 1 inclusive

Use theorem of total expectation: (weighted average of conditional averages)

X = fraction of empties

C_1 = {video recorded during 10-2}
C_2 = {video recorded during 6-6}
C_3 = {video recorded during 6-10 or 2-6}

$$P(C_1) = \frac{4 \text{ hours}}{24 \text{ hours}}, \quad P(C_2) = \frac{12}{24}, \quad P(C_3) = \frac{8}{24}$$

$$E(X) = \sum_{i=1}^{3} E(X|C_i)\,P(C_i) = \frac{2}{10}\cdot\frac{4}{24} + \frac{1}{10}\cdot\frac{12}{24} + \frac{4}{10}\cdot\frac{8}{24}$$

Answer: $\frac{13}{60}$

$$= \frac{8+12+32}{240} = \frac{52}{240} = \frac{13}{60}$$

Problem 6. (6 POINTS)
In a high-quality industrial printer, to achieve 1200 dpi (dots per inch), there are (over) 40,000 ink nozzles. (ASSUME EXACTLY 40,000 NOZZLES FOR THIS PROBLEM!) Each ink nozzle may clog independently of the others. If each nozzle has a probability $5 * 10^{-5}$ of being clogged, what is the probability that more than 2 nozzles are clogged?

You may leave your answers in terms of e **(Euler's number) and/or** $\ln()$ **(the natural logarithm)**, and you do **NOT** need to evaluate a summation.

Find $P(N > 2)$ where $N = \#$ clogged nozzles.

Many many Bernoulli trials (40,000 of them), each w/ a very small prob. of success $(5*10^{-5})$.

$\Rightarrow$ Poisson RV with parameter $\alpha = np$

$$\alpha = (40{,}000)(5*10^{-5}) = 2$$

$$P(N > 2) = \sum_{k=3}^{\infty} P_N(k) = \sum_{k=3}^{\infty} \frac{\alpha^k e^{-\alpha}}{k!}$$

or: $$= 1 - P(N=0) - P(N=1) - P(N=2)$$

Answer: $1 - 5e^{-2}$

$$= 1 - \frac{\alpha^0 e^{-\alpha}}{0!} - \frac{\alpha^1 e^{-\alpha}}{1!} - \frac{\alpha^2 e^{-\alpha}}{2!}$$

$$= 1 - 1 \cdot e^{-\alpha} - 2e^{-\alpha} - \frac{4e^{-\alpha}}{2} = 1 - 5e^{-2}$$

Problem 7. (6 POINTS)
Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find **and sketch** the PDF of the random variable $Y = X^2$.

Use the 2-step process:
find $F_Y(y)$ and differentiate to get $f_Y(y)$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y})$$

(only the one term because X is never negative)

$$= F_X(\sqrt{y})$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(\sqrt{y}) \frac{d\sqrt{y}}{dy} = f_X(\sqrt{y}) \frac{1}{2} \frac{1}{\sqrt{y}}$$

$$= \frac{d}{dy}\left(y^{1/2}\right) = \frac{1}{2} y^{-1/2}$$

Substitute into $f_X(x)$ using $x = \sqrt{y}$

$$f_Y(y) = \begin{cases} \frac{1}{2} \frac{1}{\sqrt{y}} \cdot 3(\sqrt{y})^2 & \text{when } 0 \le \sqrt{y} \le 1 \\ 0 & \text{else} \end{cases}$$

simplifying

$$\boxed{f_Y(y) = \begin{cases} \frac{3\sqrt{y}}{2} & \text{when } 0 \le y \le 1 \\ 0 & \text{else} \end{cases}}$$

Problem 8. (12 POINTS)
Let X be a binary signal, such that $P(X = +1) = P(X = -1) = 0.5$. Suppose X is sent across a noisy channel, with noise N modeled by a zero-mean Gaussian distribution with variance σ^2, where the noise is independent of the signal that was sent. The received signal is $Y = X + N$.

(a) Express the conditional PDFs of Y given both $\{X = +1\}$ and $\{X = -1\}$.
(You may express these using distributional notation if you like; for example, we know that $N \sim \mathcal{N}(0, \sigma^2)$.)
(Hint: what are $E(Y|X = +1)$ and $E(Y|X = -1)$?)

(b) Suppose a detector uses compares the received signal Y to a fixed threshold γ to decide which signal was sent. Specifically, the detector decides that $X = +1$ was sent if $Y \geq \gamma$ and that $X = -1$ was sent otherwise.

For a given threshold γ, express the probability of error in terms of the Φ function, γ, and σ.

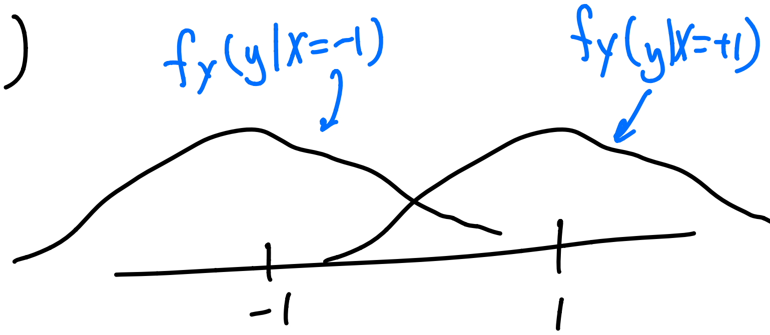
Bonus (2 points): What is the optimal value of γ to minimize the probability of error? You may use intuition or math to determine this. A sketch may help.

$$P_X(x) = \begin{cases} 1/2 & \text{if } x = \pm 1 \\ 0 & \text{else} \end{cases} \qquad N \sim N(0, \sigma^2)$$

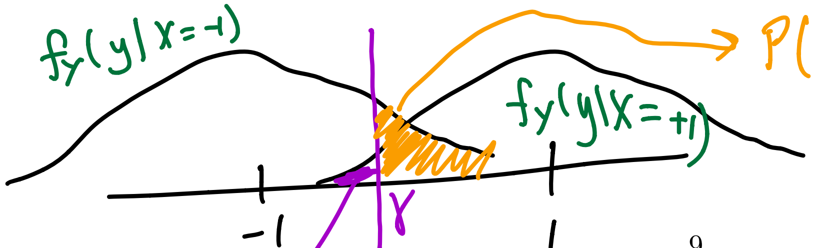
$Y = X + N$, so if $X = +1$, $Y = 1 + N$
and if $X = -1$, $Y = -1 + N$

shifted pdfs of N

a) $(Y \mid X = +1) \sim N(+1, 1)$
$(Y \mid X = -1) \sim N(-1, 1)$



b) Probability of error is defined in terms of the full sample space
→ it is NOT the sum of 2 conditional prob.'s.



$P(Y > \gamma \mid X = -1) = 1 - \Phi\left(\frac{\gamma+1}{\sigma}\right)$

$P(\text{error}) = P(Y > \gamma \mid X = -1)\, P(X = -1) + P(Y < \gamma \mid X = +1)\, P(X = +1)$

recall $P(X = +1) = P(X = -1) = 1/2$

$P(Y < \gamma \mid X = +1) = \Phi\left(\frac{\gamma-1}{\sigma}\right)$

Problem 9. (18 POINTS (6 EACH))
Given the joint Probability Density Function

$$f_{XY}(x,y) = \begin{cases} c(2x+y) & \text{when } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c.

(b) Find $f_X(x)$, the marginal PDF of X.

(c) Are X and Y independent? Why or why not?

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = 1$

$c\int_0^1\int_0^1 (2x+y)\,dx\,dy = c\int_{y=0}^{1} \left(\frac{2x^2}{2} + yx\right)\Big|_0^1 dy$

$= c\int_0^1 (1+y)\,dy = \left[cy + \frac{cy^2}{2}\right]_0^1 = c\left(1+\frac{1}{2}\right) = 1$

$\Rightarrow \boxed{c = 2/3}$

b) $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = c\int_0^1 (2x+y)\,dy$

$= c\left(2xy + \frac{y^2}{2}\right)\Big|_0^1 = c\left(2x + \frac{1}{2}\right) = \frac{4x}{3} + \frac{1}{3}$

but only when $0 \le x \le 1$. $f_x(x) = 0$ otherwise

c) To show independence, you must show $f_{xy}(x,y) = f_x(x)f_y(y)$ for all x and y.
To show lack of independence, you can find one (x,y) pair for which the above does not hold.
"Product form" defines the ROS, and product form is NOT sufficient to prove independence!

$f_y(y) = c\int_0^1 (2x+y)\,dx = c\left(\frac{2x^2}{2} + xy\right) = c(1+y)$ when $0 \le y \le 1$.

Here, $f_{xy}(x,y) \ne f_x(x)f_y(y)$
$\Rightarrow$ NOT independent

Problem 10. (6 POINTS))

Suppose at a call center, the inter-arrival time of calls asking about purchasing a product is an exponential random variable, X, that has parameter $\lambda_1 = 2$, and the inter-arrival times of calls asking about returning a product is another exponential random variable Y with parameter $\lambda_2 = 1$. X and Y are independent random variables.

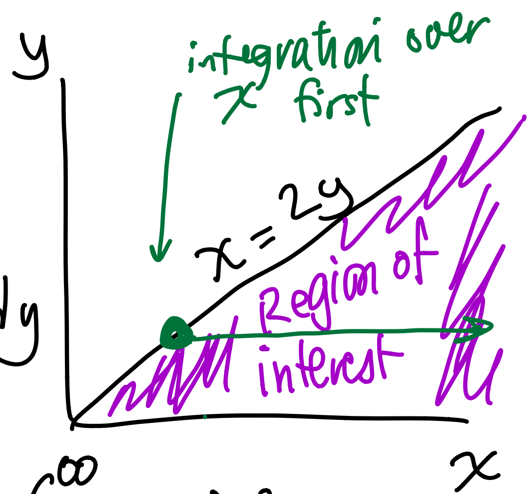
Find the probability that the $X > 2Y$.

2 RVs, X, Y. Probabilities are defined by 2D integration. $\iint_{ROI} f_{XY}(x,y)\,dx\,dy$

Independence $\Rightarrow f_{XY}(x,y) = f_X(x)\, f_Y(y)$

$f_X(x) = \lambda_1 e^{-\lambda_1 x}\, u(x)$

$f_Y(y) = \lambda_2 e^{-\lambda_2 y}\, u(y)$



$$P(X > 2Y) = \int_0^\infty \int_{2y}^\infty \lambda_1 \lambda_2 e^{-\lambda_1 x} e^{-\lambda_2 y}\, dx\, dy$$

$$= \int_0^\infty \lambda_1 \lambda_2 \frac{e^{-\lambda_1 x}}{-\lambda_1} e^{-\lambda_2 y} \Big|_{x=2y}^{\infty} dy = \int_0^\infty \lambda_2 e^{-\lambda_1 2y} e^{-\lambda_2 y}\, dy$$

Answer: $1/5$

$$= \int_0^\infty \lambda_2 e^{-(2\lambda_1 + \lambda_2) y}\, dy = \frac{\lambda_2 e^{-(2\lambda_1+\lambda_2)y}}{-(2\lambda_1+\lambda_2)} \Big|_0^\infty$$

$$= \frac{-\lambda_2}{2\lambda_1 + \lambda_2}(0 - 1) = \frac{\lambda_2}{2\lambda_1 + \lambda_2} = \frac{1}{2\cdot 2+1} = \boxed{\frac{1}{5}}$$

Extra space to solve – label problem clearly so I can give you credit!

ALSO: Write at the bottom of original problem that you're solving here, so I know to look here!

Discrete Random Variables

- Bernoulli Random Variable, parameter p
  $S = \{0, 1\}$
  $p_0 = 1 - p, p_1 = p; 0 \leq p \leq 1$
  $E(X) = p; \mathrm{VAR}(X) = p(1 - p)$

- Binomial Random Variable, parameters (n, p)
  $S = \{0, 1, \ldots, n\}$
  $p_k = \binom{n}{k} p^k (1 - p)^{n-k}; k = 0, 1, \ldots, n; 0 \leq p \leq 1$
  $E(X) = np; \mathrm{VAR}(X) = np(1 - p)$

- Geometric Random Variable, parameter p
  $S = \{0, 1, \ldots\}$
  $p_k = p(1 - p)^k; k = 0, 1, \ldots, ; 0 \leq p \leq 1$
  $E(X) = (1 - p)/p; \mathrm{VAR}(X) = (1 - p)/p^2$

- Poisson Random Variable, parameter α
  $S = \{0, 1, \ldots\}$
  $p_k = \alpha^k e^{-\alpha}/k! \quad k = 0, 1, \ldots,$
  $E(X) = \alpha; \mathrm{VAR}(X) = \alpha$

- Uniform Random Variable
  $S = \{0, 1, \ldots, L\}$
  $p_k = 1/L \quad k = 0, 1, \ldots, L$
  $E(X) = (L + 1)/2; \mathrm{VAR}(X) = (L^2 - 1)/12$

Continuous Random Variables

- Uniform Random Variable
  Equally likely outcomes
  $S = [a, b]$
  $f_X(x) = 1/(b - a), \quad a \leq x \leq b$
  $E(X) = (a + b)/2; \quad \mathrm{VAR}(X) = (b - a)^2/12$

- Exponential Random Variable, parameter λ
  $S = [0, \infty)$
  $f_X(x) = \lambda \exp(-\lambda x), \quad x \geq 0, \ \lambda > 0$
  $E(X) = 1/\lambda; \quad \mathrm{VAR}(X) = 1/\lambda^2$

- One Gaussian Random Variable, parameters μ, σ^2
  $S = (-\infty, \infty)$
  $f_X(x) = \exp(-(x - \mu)^2/(2\sigma^2))/\sqrt{2\pi\sigma^2}$
  $E(X) = \mu; \quad \mathrm{VAR}(X) = \sigma^2$

Other useful formulas

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{if } |r| < 1$$

$$\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2} \qquad \text{if } |r| < 1$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

$$\int x e^{ax}\, dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^{ax}\, dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$

Table 1: Table of the Standard Normal Cumulative Distribution Function $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990