Solutions

ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2021
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PURDUE UNIVERSITY

Exam 1

Fall 2021, MWF 11:30am-12:20pm
(September 28, 2021)

PURDUE UNIVERSITY

This is a closed book exam with 8 multi-part problems. Neither calculators nor help sheets are allowed.

Cheating will result in a zero on the exam and possibly failure of the class. Do not cheat!

Use of any electronics is considered cheating.

Put your name or initials on every page of the exam and turn in everything when time is up.

Write your answers in the boxes provided. We will be scanning the exams, so **DO NOT WRITE ON THE BACK of the pages!**.

Name: ____________________ PUID: ____________________

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: ____

Problem 1. (True/False: 4 points each, total 24 points)

Clearly label each statement T or F in the box to the left of the problem. (Note: if a statement is not always true, then it is FALSE.) **If you show your reasoning you might get partial credit.** Finding a counter-example might be helpful if the answer is FALSE.

F

(a) The function $F_X(x)$ below is a valid CDF:

when $\sqrt{2} < x < 2$, then $F_X(x) > 1$.

$$F_X(x) = \begin{cases} 0. & \text{if } x < 0 \\ \frac{x^2}{2} & \text{if } 0 \le x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

T

(b) If A and B are disjoint, and $P(B) > 0$ then $P(A|B) = 0$.

$\to A \cap B = \emptyset \Rightarrow P(A \cap B) = 0 = \frac{P(A|B)}{P(B)}$

$\Rightarrow P(A|B) = 0$

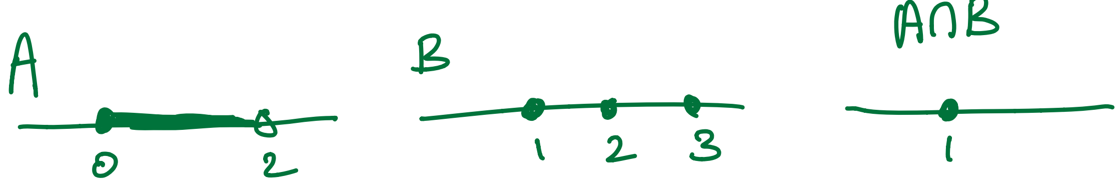
F

(c) $P(A|B) + P(A|B^c) = 1$ for any two sets A and B.

Find a counter-example: Suppose A and B are independent, so $P(A|B) = P(A) = P(A|B^c)$. And suppose $P(A) \neq 1/2$. Then $P(A|B) + P(A|B^c) = 2P(A) \neq 1$

F

(d) If $A = [0, 2)$ and $B = \{1, 2, 3\}$ then $A \cap B = \{1, 2\}$.



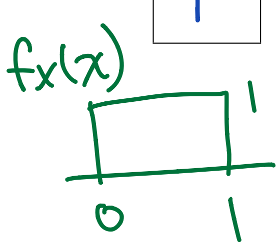
F

(e) If events A and B, are independent, then $P(A^c \cup B) = (1 - P(A))P(B)$.

A True statement : $P(A^c \cap B) = (1 - P(A))P(B)$

F

(f) If X is a uniform random variable on the interval from 0 to 1, then $E(\exp(X)) = 1-e$.



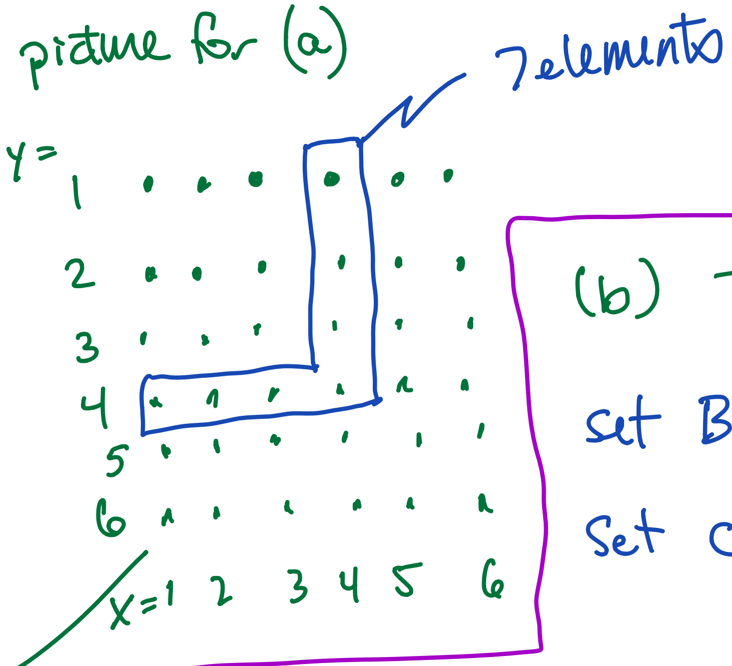
$E(e^X) = \int_0^1 1 \cdot e^x \, dx = e^x \Big|_0^1 = e - 1$

Problem 2. (12 POINTS (6 POINTS FOR EACH PART))
Two dice are rolled; both dice are fair. Let X be the number on the first die, and let Y be the number on the second die.

(a) Find $P(\max(X,Y)=4)$. set $A = \{(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1)\}$
(Note: the function $\max(a,b)$ returns the larger of the two values a or b.)

(b) Find $P(\ \{|X-Y|\le 2\}\ |\ X=4)$.

picture for (a)



$$P(A) = \frac{|A|}{|S|} = \frac{7}{6\cdot 6} = \frac{7}{36}$$

(b) Two approaches

set $B = \{|X-Y| \le 2\} = \{-2 \le X-Y \le 2\}$

Set $C = \{X=4\}$ Want $P(B|C)$

Approach 1: enumerate the full set B, then restrict to find set $B \cap C$. Then compute

$$P(B\cap C) = \frac{|B\cap C|}{|S|}, \quad P(C) = \frac{|C|}{|S|}, \quad P(B|C) = \frac{P(B\cap C)}{P(C)}$$

(a)	7/36
(b)	5/6

Approach 2: Ignore the full set B, because we care only about the condition that set C happened. Within set C, enumerate elements that are also in $B \equiv B\cap C$
Then compute $|B\cap C| / |C|$

Problem 3. (12 POINTS (6 POINTS EACH PART))
On any given day, let B be the event that a particular phone's battery lasts all day without recharging, and
let M be the event the phone's user listens to **music** for more than an hour during that same day.
Suppose: $P(B|M) = 0.3$, $P(M|B) = 0.6$, and $P(B) = 0.4$.

(a) What is the probability the phone has been used to listen to more than an hour of music in a day?

(b) What is the probability the phone's battery does not last the full day and yet the phone has not been used to listen to more than an hour of music that day?

$P(B|m) = 0.3$
$P(m|B) = 0.6$
$P(B) = 0.4$

a) $P(m \cap B) = P(m|B)\,P(B)$
$= (0.6)(0.4) = 0.24$

and $P(m \cap B) = P(B|m)\,P(m)$

$\Rightarrow P(m) = \dfrac{P(m \cap B)}{P(B|m)} = \dfrac{0.24}{0.30} = \boxed{\dfrac{4}{5}}$

b) want $P(B^c \cap m^c)$
$= 1 - P(m \cup B)$ by DeMorgan's Rule
$= 1 - [P(m) + P(B) - P(m \cap B)]$
$= 1 - (0.8 + 0.4 - 0.24) = 0.04$

or fill out the table ...

(a) $4/5 = 0.8$	Table
(b) 0.04	

Table

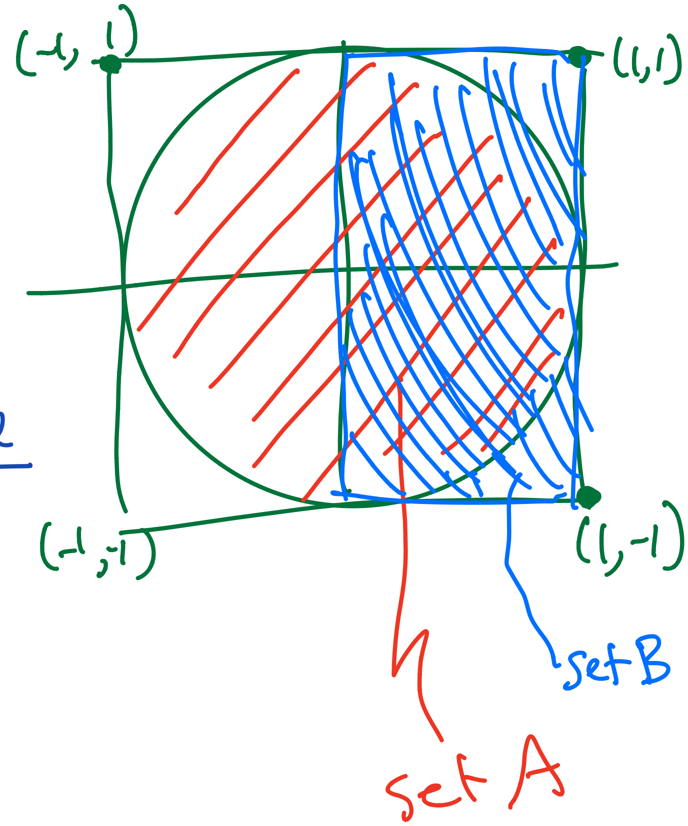
	B	B^c	
m	0.24	0.56	4/5
m^c	0.16	$\boxed{0.04}$	1/5
	0.4	0.6	

Answers

Problem 4. (6 POINTS)
Suppose the outcome of an experiment is equally likely to be anywhere in a square defined by corners $(x, y) = (-1, -1)$, $(x, y) = (1, -1)$, $(x, y) = (1, 1)$, $(x, y) = (-1, 1)$. Define event A to be the event that the outcome is somewhere inside the unit circle, and define the event B to be the event that the first component is greater than zero.
Are events A and B independent? Justify your answer. (No points without justification.)

Need to show whether or not $P(A \cap B) = P(A)P(B)$

$$P(A) = \frac{|A|}{|S|} = \frac{\pi r^2}{4} = \frac{\pi}{4}$$



$$P(B) = \frac{|B|}{|S|} = \frac{2}{4} = \frac{1}{2}$$

$$P(A \cap B) = \frac{\text{area of half circle}}{|S|}$$

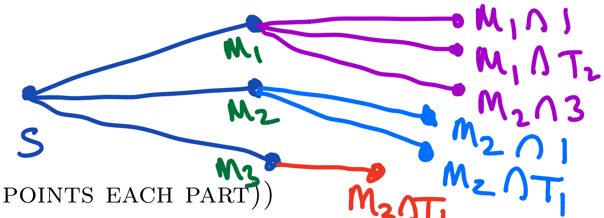
$$= \frac{\frac{1}{2}\pi r^2}{4} = \frac{\pi}{8}$$

$$P(A)\,P(B) = \left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right) = \frac{\pi}{8} \checkmark$$

Answer: Yes, Independent because

$P(A \cap B) = P(A)P(B)$

$T_1 = \{$pull two on 1st pull$\}$



$P(k|M_1) = \frac{1}{3}$ $k=1,2,3$

$P(k|M_2) = \frac{1}{2}$ $k=1,2$

$P(k|M_3) = 1$ $k=2$

Problem 5. (18 points (6 points each part))
A casino has 3 machines where, when you pull the lever, you will be shown a number between 1 and 3. On the outside, the machines look identical, and they are in random locations. However, for one of the machines (M_1), when the lever is pulled it is equally likely to show either a 1, 2, or 3. For the second machine (M_2), it is equally likely you will be shown a 1 or a 2. The third machine (M_3) always shows a 2. $P(M_i) = 1/3$ for $i = 1,2,3$

(a) You approach a machine at random, and pull the lever. What is the probability you are shown a 2?

(b) Given that you were shown a 2, what is the probability you had pulled the lever on the M_3?

(c) To learn more about which machine you chose, you pull the lever once more **on the same machine**. Given that you were shown a 2 the first time, what is the probability you are shown a second 2?

a) $P(T_1) = P(T_1|M_1)P(M_1) + P(T_1|M_2)P(M_2) + P(T_1|M_3)P(M_3)$

$= \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{1}{3}\left(\frac{1}{3}+\frac{1}{2}+1\right) = \frac{1}{3}\left(\frac{2}{6}+\frac{3}{6}+\frac{6}{6}\right) = \boxed{\frac{11}{18}}$

b) $P(M_3|T_1) = P(T_1|M_3)P(M_3)/P(T_1)$

$= (1)(\frac{1}{3})/(11/18) = \frac{18}{3\cdot 11} = \boxed{\frac{6}{11}}$

c) $T_2 = \{$pull two on second pull$\}$

$P(T_2|T_1) = P(T_1 \cap T_2)/P(T_1)$

$P(T_1 \cap T_2) = \sum_{i=1}^{3} P(T_1 \cap T_2 | M_i)P(M_i) = \frac{1}{3}\left[\left(\frac{1}{3}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2\right]$

$= \frac{1}{3}\left[\frac{1}{9}+\frac{1}{4}+1\right] = \frac{1}{3}\left(\frac{4}{36}+\frac{9}{36}+\frac{36}{36}\right)$

$= \frac{1}{3}\left(\frac{49}{36}\right)$

then $P(T_2|T_1) = \frac{1}{3}\left(\frac{49}{36}\right)/\frac{11}{18}$

$= 49/66$

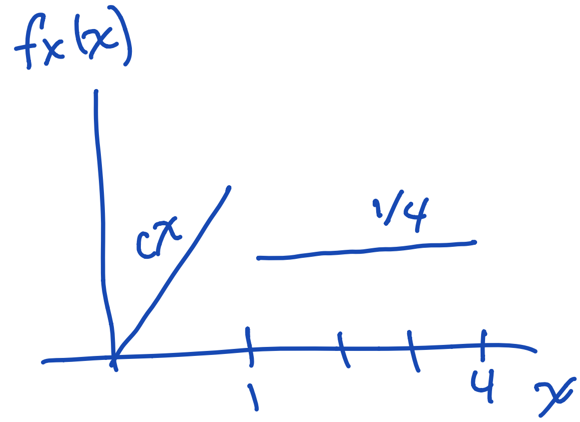
(a)	11/18
(b)	6/11
(c)	49/66

Answers

Problem 6. (12 points (6 points each part))

Suppose the PDF of X is given by

continuous RV $$f_X(x) = \begin{cases} cx & \text{if } 0 \le x < 1 \\ 1/4 & \text{if } 1 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$



(a) Find the value of the constant c.

(b) Find the cumulative distribution function of X: $F_X(x)$.

a) Set c so $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f_X(x) = \int_0^1 cx\,dx + 3\left(\tfrac{1}{4}\right) = \frac{cx^2}{2}\Big|_0^1 + \frac{3}{4}$$

$$= \frac{c}{2} + \frac{3}{4} = 1 \Rightarrow \boxed{c = 1/2}$$

b) $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ Break into regions

Region $x < 0$ $F_X(x) = 0$ Region $x > 4$ $F_X(x) = 1$

Region $0 \le x \le 1$ $F_X(x) = \int_0^x ct\,dt = \frac{ct^2}{2}\Big|_0^x = \frac{cx^2}{2} = \frac{x^2}{4}$

(a)	$c = 1/2$
(b)	$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \le x \le 1 \\ x/4 & 1 < x \le 4 \\ 1 & x > 4 \end{cases}$

Region $1 < x \le 4$ $F_X(x) = \frac{1}{4} + \int_1^x \frac{1}{4}\,dt = \frac{1}{4} + \frac{t}{4}\Big|_1^x = \frac{1}{4} + \frac{x}{4} - \frac{1}{4}$

Problem 7. (12 POINTS (6 POINTS EACH PART))

Suppose the PDF of X is given by

continuous RV

$$f_X(x) = \begin{cases} 1/2 & \text{if } \ 0 \le x < 1 \\ 1/6 & \text{if } \ 1 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean of X: $E(X)$

(b) Find the variance of X: $\text{VAR}(X)$

a) $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 \frac{1}{2} x\,dx + \int_1^4 \frac{1}{6} x\,dx$

$= \left.\frac{x^2}{2\cdot 2}\right|_0^1 + \left.\frac{x^2}{6\cdot 2}\right|_1^4 = \frac{1}{4}(1-0) + \frac{1}{12}(16-1)$

$= \frac{1}{4} + \frac{15}{12} = \frac{1}{4} + \frac{5}{4} = \frac{6}{4} = \boxed{\frac{3}{2}}$

b) $\text{Var}(X) = E(X^2) - E(X)^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 \frac{1}{2} x^2\,dx + \int_1^4 \frac{1}{6} x^2\,dx$

$= \left.\frac{x^3}{2\cdot 3}\right|_0^1 + \left.\frac{x^3}{6\cdot 3}\right|_1^4 = \frac{1}{6}(1-0) + \frac{1}{18}(64-1)$

$= \frac{1}{6} + \frac{63}{18} = \frac{1}{6} + \frac{21}{6} = \frac{22}{6} = \frac{11}{3}$

(a)	3/2
(b)	17/12

$\text{Var}(X) = E(X^2) - E(X)^2$

$= \frac{11}{3} - \left(\frac{3}{2}\right)^2 = \frac{11}{3} - \frac{9}{4}$

$= \frac{44}{12} - \frac{27}{12} = \boxed{\frac{17}{12}}$

Problem 8. (4 POINTS)

Let X be a random variable with mean μ and variance σ^2, and let $Y = 5X - 3X^2 + 2$. Express the mean of Y, $E(Y)$, in terms of μ and σ.

$$Y = 5X - 3X^2 + 2$$

$$E(Y) = 5E(X) - 3E(X^2) + 2$$

by linearity of $E(\cdot)$ function

we know $Var(X) = \sigma^2$

$$= E(X^2) - E(X)^2$$

so $E(X^2) = \sigma^2 + E(X)^2$

and that $E(X) = \mu$, so

$$E(Y) = 5\mu - 3(\sigma^2 + \mu^2) + 2$$

$VAR(Y) =$ $5\mu - 3\sigma^2 - 3\mu^2 + 2$

Extra space to solve – label problem clearly so I can give you credit!

ALSO: Write at the bottom of original problem that you're solving here, so I know to look here!