Past Exam Questions
(Fall 2015, Spring 2016, Fall 2016, Fall 2017)
Chapter 5 and beyond

Reibman

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These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapter 5, 7, 9, and 10 in our textbook.

I will post the last pages of the final – with the formulas that you'll have available to you – separately.

Problem 1. (Yes/No: 2 points)
Consider a joint CDF $F_{X,Y}(x, y)$. Is it always true that $F_{X,Y}(3, 2) \leq F_{X,Y}(4, 3)$?

Problem 2. (Yes/No: 2 points)
Consider a joint CDF $F_{X,Y}(x, y)$. Is it always true that $F_{X,Y}(83, 84) < F_{X,Y}(84, 85)$?

Problem 3. (Yes/No: 2 points)
Consider a joint CDF $F_{X,Y}(x, y)$. Is it always true that $\lim_{y\to\infty} F_{X,Y}(5, y) = 1$?

Problem 4.
Random variables X and Y have the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 1/2 & \text{for } 1 \leq x \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the region of nonzero probability

(b) What is $P(X > 2)$?

(c) What is $f_X(x)$?

(d) What is $E(X)$? $E(Y)$?

(e) Find $COV(X, Y)$

Problem 5. (20 points)
Given the Joint PDF

$$f_{X,Y}(x, y) = \begin{cases} c(2 - y) & \text{for } 0 < x < 4 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) Find the marginal PDF of X, $f_X(x)$, and the marginal PDF of Y, $f_Y(y)$.

(c) Are X and Y independent? Explain why or why not.

(d) What is $P(X + Y < 2)$? (Hint: you may find it helpful to draw a picture.)

Problem 6. (Multiple choice: 5 points)
Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} 15y & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$$

Which represents the marginal density of Y, $f_Y(y)$?
(Hint, draw a clear picture and you may get partial credit.)

(a)

$$f_Y(y) = \begin{cases} 15y & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_Y(y) = \begin{cases} 15y^2/2 & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_Y(y) = \begin{cases} 15y^2/2 & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$f_Y(y) = \begin{cases} 15y^{3/2}(1 - y^{1/2}) & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(e)

$$f_Y(y) = \begin{cases} 15y^{3/2}(1 - y^{1/2}) & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$$

Problem 7. (25 points ((a) is 7 points; (b) is 8 points; (c) is 10 points))
Let X be a discrete RV with sample space $S_X = \{1, 4\}$, each equally likely. Given that we know $X = x$, a second RV Y is exponentially distributed with mean $1/x$.

(a) What is the conditional pdf of Y given X?

(b) What is the marginal pdf of Y?

(c) Find $E(Y)$.
(Hint: it will be faster to use the theorem of total expectations, but you may solve it any way you wish.)

Problem 8. (15 POINTS)
Let

$$F_{X.Y}(x,y) = \begin{cases} x(1-e^{-2y})/2 & \text{for } 0 \le y \text{ and } 0 \le x \le 2 \\ (1-e^{-2y}) & \text{for } 0 \le y \text{ and } 2 \le x \\ 0 & \text{otherwise} \end{cases}$$

(a) Use $F_{X,Y}(x,y)$ to compute the probability that $P(1 < X \le 2, Y \le 3)$.

(b) Find $f_X(x)$.

(c) What is the probability that $P(X \le 3)$

Problem 9. (25 POINTS)
Let

$$f_{X,Y}(x,y) = \begin{cases} c & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 2 \\ 2c & \text{for } -1 \le x \le 0 \text{ and } -1 \le y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the region of support for X and Y.

(b) Find c.

(c) Find $f_Y(y|X=1)$, the conditional PDF of Y given $X = 1$.

(d) Find $f_X(x)$.

Problem 10. (15 POINTS)
Given the Joint PDF

$$f_{X.Y}(x,y) = \begin{cases} 0.50 & \text{for } 0 < x < 0.5 \text{ and } 0 < y < 0.5 \\ 1.25 & \text{for } 0.5 < x < 1 \text{ and } 0 < y < 0.5 \\ 1.50 & \text{for } 0 < x < 0.5 \text{ and } 0.5 < y < 1 \\ 0.75 & \text{for } 0.5 < x < 1 \text{ and } 0.5 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find and sketch the conditional PDF of X given that $Y = 3/4$.

(b) What is $E(X|Y = 3/4)$?

Problem 11. (5 POINTS)
Let X and Y be two discrete RV's with joint PMF given by

$$p_{X,Y}(x,y) = \begin{cases} 0.800 & \text{for } x = 0 \text{ and } y = 0 \\ 0.050 & \text{for } x = 1 \text{ and } y = 0 \\ 0.025 & \text{for } x = 0 \text{ and } y = 1 \\ 0.125 & \text{for } x = 1 \text{ and } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $Var(X|Y = 1)$?

Problem 12.
X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (4x+2y)/3 & \text{for } 0 \le x \le 1; 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) For which values of y is $f_X(x|y)$ defined?

(b) Find $f_X(x|y)$

(c) For which values of x is $f_Y(y|x)$ defined?

(d) Find $f_Y(y|x)$

Problem 13.
X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} k & \text{for } 0 < y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Find the marginal pdf's of X and Y (you don't have to find k yet).

(b) Find k.

(c) Find $P(0 < X < 1/2; 0 < Y < 1/2)$.

(d) Find the conditional pdf's $f_Y(y|x)$ and $f_X(x|y)$.

(e) Compute the conditional means $E(Y|x)$ and $E(X|y)$.

Problem 14. (15 POINTS)

Let X be the number of bugs in the first draft of a piece of software, and let Y be the number of bugs remaining in the second draft of the same software. Their joint PMF is given by the following table:

		Y=		
		0	1	2
X=	0	1/6	0	0
	1	1/3	1/6	0
	2	1/12	1/12	1/6

(a) What is the conditional PMF of Y given $X = 2$?

(b) What is the marginal PMF of X?

(c) Find the Variance of X.

Problem 15. (20 POINTS)
Given X with PDF

$$f_X(x) = \begin{cases} 1/2 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Then Y is uniformly distributed between x and $2x$.

(a) Sketch the region of support. That is, indicate where $f_{XY}(x, y)$ is nonzero.

(b) Find the joint PDF $f_{XY}(x, y)$.

(c) What is $P(Y < 1)$?

(d) What is $E(Y)$? (Hint: you may use the law of iterated expectations.)

Problem 16. (5 POINTS)

The number of "likes" to a social-media post on any given day is a Poisson random variable, with mean α. However, the parameter α is a random variable that depends on the amount of sunshine outside and is uniformly distributed on the continuous interval $[0, 3]$.
What is the probability there is exactly one "like" on a specific day?
(Hint: you may find the following integral useful.)

$$\int xe^{ax}\, dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

Problem 17. (15 POINTS)

Background story: You order many boxes of pizza for a gathering of Purdue students. When you check the pizza boxes some time after the gathering starts, there is a fraction X of pizza remaining, where X is a random variable between 0 and 1. You check the boxes of pizza again at a later time during the gathering, and observe a fraction Y of pizza still remaining. Y is a random variable between 0 and X. You ordered enough boxes of pizza that you can safely assume that both X and Y are continuous random variables.

Math problem: Suppose X has the PDF

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and Y is uniformly distributed between $[0, X]$.

(a) Find the joint PDF, $f_{XY}(x, y)$, **including its region of support**.

(b) Find the marginal PDF of Y.

(c) What is $P(Y < 1/2)$?

Problem 18. (20 POINTS)
Suppose you build a system with a part that comes from either Company 1 (with probability 2/3) or Company 2 (with probability 1/3). Let N indicate the company, so that N is a discrete RV with PMF

$$p_N(n) = \begin{cases} 2/3 & \text{for } n = 1 \\ 1/3 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let X be the lifetime of the part, which has a different distribution whether the part comes from Company 1 or Company 2. In particular, let X be a continuous RV such that when $N = 1$, X is exponentially distributed with mean 5, and when $N = 2$, X is exponentially distributed with mean 8.

(a) What is the conditional PDF of X given N=2?
(In other words, what is the PDF of X when the part comes from Company 2?)

(b) Find the marginal PDF of X.

(c) What is $P(X > 8)$?

(d) If your device has lasted long enough that $X > 8$ already, what is the probability it is from Company 1, i.e., that $N = 1$?

Problem 19. (20 POINTS)
Let X be a continuous uniform RV on the interval $[0, 1]$. Conditioned on X, then Y is a continuous RV that is uniformly distributed on the interval $[x, x + 1]$.

(a) What is $P(Y > 0.5)$?
(Hint, draw a clear diagram of the region of support for (X, Y).)

(b) Find the marginal PDF of Y.

(c) Find $E(Y)$. (Hint: are you able to use symmetry?)

(d) Find $COV(X, Y)$.

Problem 20. (20 POINTS)

Let X and Y be random variables that have the joint pdf

$$f_{X,Y}(x,y) = c(x+y), \text{for } 0 \le x \le 1, 0 \le y \le 1$$

(a) Find c.

(b) Find the marginal PDFs of both X and of Y.

(c) What is $P(X > Y | X < 1/2)$?

(d) Find $f_Y(y|x)$.

Problem 21. (15 POINTS)

Suppose X and Y are two random variables with $Y = X^2$. X is uniformly distributed with $S_X = \{-1, 0, 1\}$.

(a) Is X a discrete or continuous random variable? How do you know?

(b) Express the joint PMF of X and Y using a table.

(c) Are X and Y independent? Why or why not?

(d) Are X and Y uncorrelated? Why or why not?

Note: Full credit will only be given for a **complete** answer that includes the correct reason and supporting evidence.

Problem 22. (15 POINTS)
Suppose X and Y are two random variables, with means $m_X = 1$ and $m_Y = -3$, and variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 4$, respectively. Suppose we also know that the covariance between X and Y is $\mathrm{COV}(X, Y) = 1/2$.

Find the mean and variance of Z when $Z = 3X + Y$.

Problem 23. (25 POINTS)
Let X be uniform on $[-1, 1]$. Let $Y = X^n$. Find $COV(X, Y)$.

Hint to save some computation: use the fact that $f_X(x)$ is even.

Problem 24. (10 POINTS)

Suppose X is a RV with zero mean and variance 1, and Y is a RV with mean 1 and variance 4, and suppose the correlation coefficient between X and Y is 1/2. Find $Var(X + Y)$.

Problem 25. (10 POINTS)

Suppose X and Y are two random variables with $E(X) = E(Y) = 0$, and $VAR(X) = 1$, and suppose we know X is independent from $X + Y$. What is the covariance between X and Y?

Problem 26. (MULTIPLE CHOICE)
Two RVs are uncorrelated.

(a) The correlation is always greater than 0.

(b) The correlation is always less than 0.

(c) The correlation is always equal to 0.

(d) None of the above.

Problem 27. (MULTIPLE CHOICE)
If two RVs are positively correlated, then

(a) $E(XY) > 0$.

(b) $E(XY) < 0$.

(c) $E(XY) = 0$.

(d) $E(XY) = E(X)E(Y)$.

(e) None of the above.

Problem 28. (YES/NO: 4 POINTS)
If the correlation between two random variables is zero, then their correlation coefficient must also be zero.

Problem 29. (YES/NO: 4 POINTS)
If X and Y are independent random variables, then $VAR(3X + 2Y + 1) = VAR(3X - 2Y + 3)$.

Problem 30. (MULTIPLE CHOICE: 5 POINTS)
Let $Z = 3X - Y - 5$, where X and Y are independent random variables with $Var(X) = 1$ and $Var(Y) = 2$. What is $Var(Z)$?

(a) 4

(b) 7

(c) 11

(d) 16

(e) None of the above.

Problem 31.
Let $Y = X + 30$ and $Z = 3X - 4$. X is a uniform RV on $[-1, 2]$. Find E(Y), E(Z), VAR(Y), VAR(Z), COV(X,Z), and ρ_{YZ}.

Problem 32. (5 POINTS)

Let X, Y, and Z be random variables, where X and Y are uncorrelated. The means of the RVs are $E(X) = 1$, $E(Y) = 2$, and $E(Z) = -1$, and $E(XZ) = 5$.

What is $COV(X, Y + 2Z)$?

Problem 33. (YES/NO: 2 POINTS)
If X and Y are jointly Gaussian RVs then $X + Y$ is a Gaussian RV.

Problem 34. (YES/NO: 2 POINTS)
If X and Y are jointly Gaussian RVs then X and Y are independent only if X and Y are uncorrelated.

Problem 35. (YES/NO: 2 POINTS)
If X and Y are uncorrelated RVs then the PDF of Z, where $Z = X + Y$, can be found by convolving the PDF's of X and Y.

Problem 36. (YES/NO: 4 POINTS)
If X and Y are independent random variables, then the CDF of $Z = \max(X, Y)$ is
$F_Z(z) = F_X(z)F_Y(z)$.

Problem 37.
Let X and Y be independent RVs each uniformly distributed on $[0, 1]$. Let $Z = X + Y$ and $W = X - Y$. Find the marginal PDF's of Z and W.

Problem 38. (Yes/No: 4 points)
Let $M_X''(s)$ be the second derivative with respect to s of the moment generating function $M_X(s)$. Then $M_X''(0) = E(X)^2 + VAR(X)$.

Problem 39. (Multiple choice: 5 points)

Let X and Y be independent random variables, with moment generating functions

$$M_X(s) = 0.8\exp(s) + 0.2\exp(2s)$$

and

$$M_Y(s) = 0.5\exp(s) + 0.5\exp(2s)$$

respectively, for $-\infty < s < \infty$. What is the $E(Z^2)$ where the random variable Z is defined by $Z = X + Y$?

(If you show your work you may receive partial credit.)

(a) 0.5

(b) 2.7

(c) 4.1

(d) 7.7

(e) None of the above

Problem 40. (5 points)

Let X and Y be independent random variables, with moment generating functions

$$M_X(s) = \frac{-3}{s-6} + \frac{-1}{s-2}$$

and

$$M_Y(s) = \frac{-2}{s-2}$$

respectively, for $s < 2$. What is the $E(Z)$ where the random variable Z is defined by $Z = X + Y$?

Problem 41. (5 points)
X and Y are independent random variables, with moment generating functions $M_X(s) = (1-2s)^{-3}$ and $M_Y(s) = (1-2s)^{-2}$. What is the second moment of $Z = X + Y$?

Problem 42. (15 POINTS)

You want to watch a 10-minute video. Due to the current network conditions, in each (discrete) minute of video watching, there is a chance p of having one "rebuffering event", and probability $1-p$ of having none. A "rebuffering event" happens when there is not enough bandwidth to play the video continuously, and so the video playback stalls. The "rebuffering events" in each minute are independent of each other.

(a) What is the PMF of the random variable X_{10}, which is the total number of "rebuffering events" experienced during the 10-minute video? (Note: You may assume that each playback stall is so short in time that it is negligible and can be ignored, when counting the length of the video.)

Now, let X_k be the discrete-time random process which counts the cumulative number of "rebuffering events" experienced up until the current time. For example, X_3 is the total number of "rebuffering events" that happened during minutes 1, 2, and 3, and X_{10} is as described above.

(b) Find the PMF for X_k for $k = 1, 2, \ldots, 10$.

(c) What is the mean of the random process as a function of time k.

Problem 43. (YES/NO: 4 POINTS)

If a WSS random process $X(t)$ is input to a stable linear time invariant system, the output $Y(t)$ is a WSS random process with power spectral density $S_Y(f) = |H(f)|^2 S_X(f)$.

Problem 44. (YES/NO: 4 POINTS)

If $X(t)$ is a zero-mean random process, then its auto-correlation function equals its auto-covariance function.

Problem 45. (MULTIPLE CHOICE: 5 POINTS)

Suppose $X(t)$ is a WSS random process with mean $m_X = 0$ and autocorrelation function $R_X(\tau) = 2\exp(-2|\tau|)$. What is the **correlation coefficient** between $X(2)$ and $X(-1)$?

(a) e^{-3}

(b) 0

(c) $e^{-2} - e^{-1}$

(d) e^{-6}

(e) None of the above.

Problem 46. (10 POINTS)

Given a WSS random process $X(t)$ with autocorrelation function $R_X(\tau)$, we want to estimate the value of $X(t_0)$ from a sample taken Δ seconds previously, namely from $X(t_0 - \Delta)$. One common solution is to use a linear predictor, such that the estimate

$$\hat{X}(t_0) = aX(t_0 - \Delta).$$

(a) Write an expression – in terms of $R_X(\tau)$ – of the expected squared prediction error

$$E\left[(X(t_0) - aX(t_0 - \Delta))^2\right].$$

(b) Find the value of a to minimize the expected squared prediction error, if $R_X(\tau) = \exp(-b|\tau|)$.

Problem 47. (15 POINTS)

Suppose X_n is a discrete-time random process with $E(X_n) = 0$ and $\mathrm{VAR}(X_n) = 3$ for all n, where $E(X_i X_j) = 0$ for $i \neq j$. This random process is input to a digital filter to create Y_n, where

$$Y_n = X_n - X_{n-1} \quad \text{for all } n.$$

(a) Find $E(Y_n)$.

(b) Find the auto-correlation of Y_n, which is denoted $R_Y(m, k)$.

(c) Is Y_n a wide-sense stationary random process? Explain your answer by providing justification.