ECE 302: Probabilistic Methods in Electrical and Computer Engineering

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PURDUE UNIVERSITY

Past Exam Questions (Fall 2015, Spring 2016, Fall 2016, Fall 2017) Chapter 5 and beyond

Reibman

(November 2018)

partial

SOLUTIONS

These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapter 5, 7, 9, and 10 in our textbook.

I will post the last pages of the final – with the formulas that you'll have available to you – separately.

Problem 1. (YES/NO: 2 POINTS)
Consider a joint CDF $F_{X,Y}(x, y)$. Is it always true that $F_{X,Y}(3, 2) \leq F_{X,Y}(4, 3)$? Yes

Problem 2. (YES/NO: 2 POINTS)
Consider a joint CDF $F_{X,Y}(x, y)$. Is it always true that $F_{X,Y}(83, 84) < F_{X,Y}(84, 85)$? No

Problem 3. (YES/NO: 2 POINTS)
Consider a joint CDF $F_{X,Y}(x, y)$. Is it always true that $\lim_{y\to\infty} F_{X,Y}(5, y) = 1$? No

2: could be equal

3: $\lim_{y \to \infty} F_{XY}(5, y) = F_X(5) \neq 1$

Problem 4.
Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & \text{for } 1 \le x \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

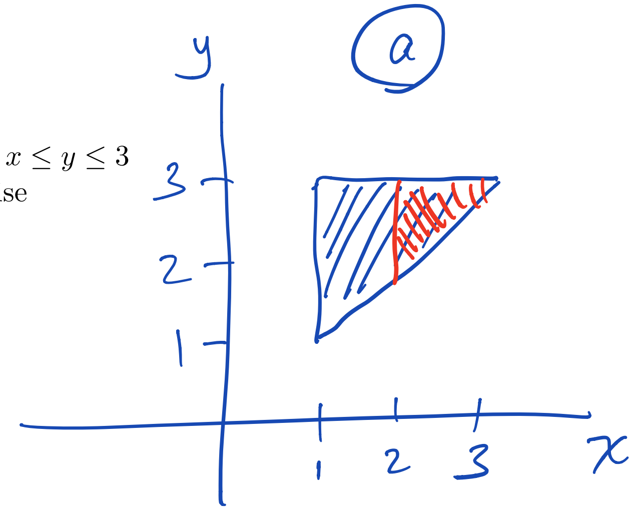
(a) Sketch the region of nonzero probability

(b) What is $P(X > 2)$?

(c) What is $f_X(x)$?

(d) What is $E(X)$? $E(Y)$?

(e) Find $COV(X, Y)$



b) $P(X>2) = \int_{x=2}^{3} \int_{y=x}^{3} \frac{1}{2}\, dy\, dx = \int_{2}^{3} \frac{y}{2}\Big|_x^3 dx = \int_2^3 \frac{3-x}{2} dx$

$= \left(\frac{3x}{2} - \frac{x^2}{4}\right)\Big|_2^3 = \frac{9}{2} - \frac{9}{4} - (3-1) = \frac{9}{4} - 2 = \boxed{\frac{1}{4}}$

or! by an area argument, since the joint pdf is flat, the area of the red triangle above relative to the overall area, $\frac{\frac{1}{2}(1)(1)}{\frac{1}{2}(2)(2)} = \frac{1}{4}$.

c) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$ and when we substitute in this joint pdf

$= \int_x^3 \frac{1}{2} dy = \frac{3-x}{2}$ when $1 \le x \le 3$

and $= 0$ otherwise

Problem 4 (d)

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_1^3 x\,\frac{3-x}{2}\,dx = \frac{3x^2}{4} - \frac{x^3}{6}\Big|_1^3$$

$$= \frac{3\cdot 9}{4} - \frac{27}{6} - \left(\frac{3}{4} - \frac{1}{6}\right) = \frac{9\cdot 9}{12} - \frac{27\cdot 2}{12} - \frac{9}{12} + \frac{2}{12}$$

$$= \frac{1}{12}(81 - 54 - 7) = \frac{20}{12} = \boxed{\frac{5}{3}}$$

4e) $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

Know $E(X) = 5/3$. But we don't yet know $E(Y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_1^y \frac{1}{2}\,dx = \frac{y-1}{2} \text{ when } 1 \le y \le 3$$

and so $$E(Y) = \int_1^3 y\, f_Y(y)\,dy = \int_1^3 \frac{y^2-y}{2}\,dy$$

$$= \left(\frac{y^3}{6} - \frac{y^2}{4}\right|_1^3 = \frac{27}{6} - \frac{9}{4} - \left(\frac{1}{6} - \frac{1}{4}\right)$$

$$= \frac{26}{6} - 2 = \frac{14}{6} = 7/3$$

and finally
$$E(XY) = \int_{x=1}^3 \int_{y=x}^3 \frac{xy}{2}\,dy\,dx = \int_1^3 \left(\frac{9x}{4} - \frac{x^3}{4}\right)dx$$

$$= \frac{9x^2}{8} - \frac{x^4}{16}\Big|_1^3 = \left(\frac{9\cdot 9}{8} - \frac{9\cdot 9}{16}\right) - \left(\frac{9}{8} - \frac{1}{16}\right)$$

$$= \frac{72}{8} - \frac{80}{16} = 9 - 5 = 4$$ so.. $\text{cov}(X,Y) =$

$$4 - \left(\frac{5}{3}\right)\left(\frac{7}{3}\right) = \frac{36-35}{9} = \boxed{\frac{1}{9}}$$

Problem 5. (20 POINTS)
Given the Joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(2-y) & \text{for } 0 < x < 4 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

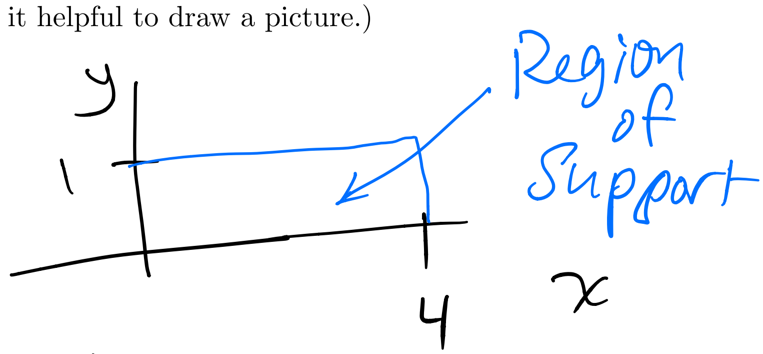
(a) What is the value of c?

(b) Find the marginal PDF of X, $f_X(x)$, and the marginal PDF of Y, $f_Y(y)$.

(c) Are X and Y independent? Explain why or why not.

(d) What is $P(X+Y<2)$? (Hint: you may find it helpful to draw a picture.)

Region of Support:



a) $1 = \int_{x=0}^{4}\int_{y=0}^{1} c(2-y)\,dy\,dx$

$= \int_0^4 c\left(2y - \frac{y^2}{2}\right)\Big|_0^1 dx = \int_0^4 c\left(2 - 1/2\right) dx$

$= c\,\frac{3}{2}x\Big|_0^4$

$= c \cdot 6 = 1$

$\Rightarrow \boxed{c = 1/6}$

b) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$= \int_0^1 c(2-y)\,dy = c\left(2y - \frac{y^2}{2}\right)\Big|_0^1 = c\left(2 - \frac{1}{2}\right) = \frac{1}{6}\left(\frac{3}{2}\right)$

$= \frac{3}{12} = \boxed{\frac{1}{4} \quad \text{when } 0 \le x \le 4}$

c) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^4 c(2-y)\,dx$

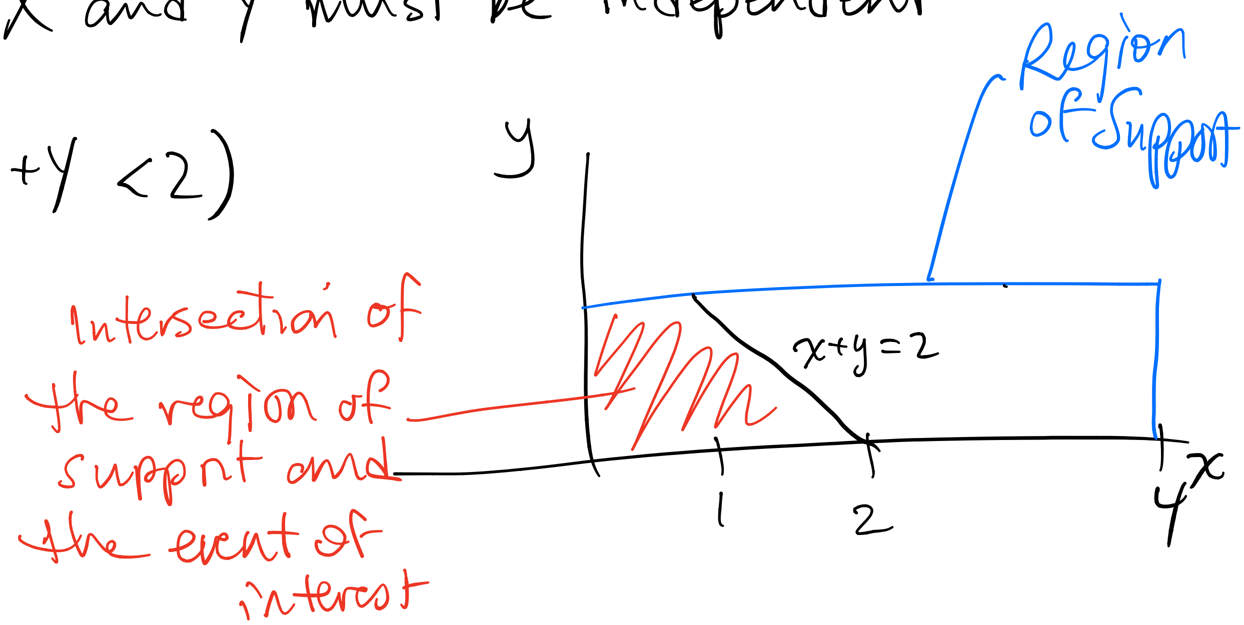
$= c(2-y)\,x \Big|_0^4 = c(2-y)4 = \boxed{\tfrac{2}{3}(2-y) \text{ when } 0 \le y \le 1}$

and $f_{XY}(x,y) = f_X(x) f_Y(y)$ for all x, y

because $\frac{1}{4}\left(\frac{2}{3}(2-y)\right) = \frac{1}{6}(2-y)$

So X and Y must be independent

d) $P(X+Y<2)$



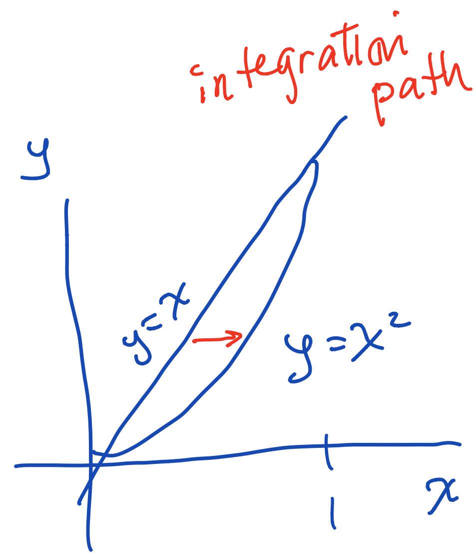
$P(X+Y<2) = \int_{y=0}^{1} \int_{x=0}^{2-y} c(2-y)\,dx\,dy$

$= \int_{y=0}^{1} c(2-y)\,x \Big|_{x=0}^{2-y} dy = \int_0^1 c(2-y)^2\,dy$

$= c\left(4y - 4y^2/2 + y^3/3\right)\Big|_0^1 = \frac{1}{6}\left(4 - 2 + 1/3\right)$

$= \frac{1}{6}\left(7/3\right) = \boxed{7/18}$

Problem 6. (MULTIPLE CHOICE: 5 POINTS)
Let X and Y be continuous random variables with joint density function



$$f_{X,Y}(x,y) = \begin{cases} 15y & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$$

Which represents the marginal density of Y, $f_Y(y)$?
(Hint, draw a clear picture and you may get partial credit.)

(a)
$$f_Y(y) = \begin{cases} 15y & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b)
$$f_Y(y) = \begin{cases} 15y^2/2 & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$$

(c)
$$f_Y(y) = \begin{cases} 15y^2/2 & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(d)
$$f_Y(y) = \begin{cases} 15y^{3/2}(1 - y^{1/2}) & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(e)
$$f_Y(y) = \begin{cases} 15y^{3/2}(1 - y^{1/2}) & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_y^{\sqrt{y}} 15y \, dx = 15y\, x \Big|_{x=y}^{x=\sqrt{y}}$$

$$= 15y\left[\sqrt{y} - y\right]$$

$$= 15y^{3/2}\left(1 - y^{1/2}\right) \qquad 0 \le y \le 1$$

Problem 7. (25 POINTS ((A) IS 7 POINTS; (B) IS 8 POINTS; (C) IS 10 POINTS))
Let X be a discrete RV with sample space $\boxed{S_X = \{1, 4\}}$, each equally likely. Given that we know $X = x$, a second RV Y is exponentially distributed with mean $1/x$.

two and only two possible values

(a) What is the conditional pdf of Y given X?

(b) What is the marginal pdf of Y?

(c) Find $E(Y)$.
(Hint: it will be faster to use the theorem of total expectations, but you may solve it any way you wish.)

a) from the back page, exponential w/ mean $1/x$

$$\Rightarrow f_Y(y \mid X = x) = x \exp(-xy) \qquad x > 0,\ y \geq 0$$

b) $$f_Y(y) = f_Y(y \mid X = 1)\, P(X = 1) + f_Y(y \mid X = 4)\, P(X = 4)$$

by the theorem of total probability,

and $P(X=1) = P(X=4) = 1/2$

$$f_Y(y) = \left[\exp(-y) + 4\exp(-4y)\right]/2 \qquad y \geq 0$$

c) Iterated expectations (or the theorem of total expectations)

$$E(Y) = E(E(Y|X)) = \sum_{x \in S_X} E(Y \mid X = x)\, P_X(x)$$

$E(Y \mid X = x) = 1/x$ from part a.

$$E(Y) = \frac{1}{2}\left[\frac{1}{1} + \frac{1}{4}\right] = \boxed{5/8}$$

Problem 8. (15 POINTS)
Let

$$F_{X.Y}(x,y) = \begin{cases} x(1-e^{-2y})/2 & \text{for } 0 \le y \text{ and } 0 \le x \le 2 \\ (1-e^{-2y}) & \text{for } 0 \le y \text{ and } 2 \le x \\ 0 & \text{otherwise} \end{cases}$$

(a) Use $F_{X,Y}(x,y)$ to compute the probability that $P(1 < X \le 2, Y \le 3)$.

(b) Find $f_X(x)$.

(c) What is the probability that $P(X \le 3)$

a) $P(1 < X \le 2, Y \le 3) = P(1 < X \le 2; -\infty < Y \le 3)$

$= F_{xy}(2,3) - F_{xy}(2,-\infty) - F_{xy}(1,3) + F_{xy}(1,-\infty)$

$= (1-e^{-6}) - 0 - (1-e^{-6})/2 + 0$

$= \boxed{(1-e^{-6})/2}$

b) there are 2 ways to get there.
1. Find $f_{xy}(x,y)$ and integrate to get $f_x(x)$
2. $F_x(x) = \lim_{y \to \infty} F_{xy}(x,y)$ and find $f_x(x)$

The second way is easier.

$$F_x(x) = \begin{cases} 0 & x \le 0 \\ x/2 & 0 < x < 2 \\ 1 & x \ge 2 \end{cases}$$

so $\boxed{f_x(x) = \begin{cases} 1/2 & 0 < x < 2 \\ 0 & \text{else} \end{cases}}$

c) $P(X \le 3) = \int_0^2 f_x(x)\,dx = \int_0^2 \frac{dx}{2} = \boxed{1}$

Problem 9. (25 POINTS)
Let

$$f_{X,Y}(x,y) = \begin{cases} c & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 2 \\ 2c & \text{for } -1 \le x \le 0 \text{ and } -1 \le y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

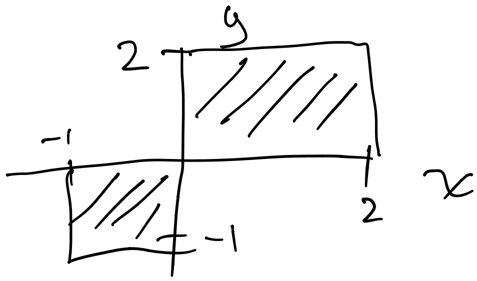
(a) Sketch the region of support for X and Y.

(b) Find c.

(c) Find $f_Y(y|X = 1)$, the conditional PDF of Y given $X = 1$.

(d) Find $f_X(x)$.

a)



both regions are in the region of support

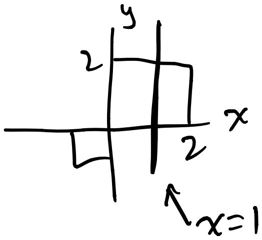
b) $1 = \iint f_{XY}(x,y)\,dx\,dy$ but when you plug in the actual function, you wind up with two

$$1 = \int_{-1}^{0}\int_{-1}^{0} 2c\,dx\,dy + \int_{0}^{2}\int_{0}^{2} c\,dx\,dy = 2c(1) + c(4)$$

(using an area argument)

$\Rightarrow 1 = 6c \Rightarrow \boxed{c = 1/6}$

c) If $X=1$, then Y cannot be negative, so $0 \le y \le 2$.



if $X=1$, Y is a uniform RV in this region

so $f_Y(y|X=1) = \begin{cases} 1/2 & 0 \le y \le 2 \\ 0 & \text{else} \end{cases}$

d) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$

we have to pay attention to the integration limits of y for each of the 2 regions

$$f_X(x) = \begin{cases} \int_{-1}^{0} 2c\, dy & \text{when } -1 \le x \le 0 \\ \int_{0}^{2} c\, dy & \text{when } 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 2c & -1 \le x \le 0 \\ 2c & 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

$$\boxed{f_X(x) = \begin{cases} 1/3 & -1 \le x \le 2 \\ 0 & \text{else} \end{cases}}$$

Problem 13.
X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} k & \text{for } 0 < y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

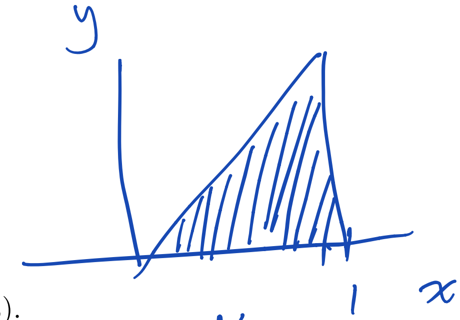
(a) Find the marginal pdf's of X and Y (you don't have to find k yet).

(b) Find k.

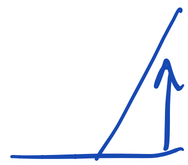
(c) Find $P(0 < X < 1/2; 0 < Y < 1/2)$.

(d) Find the conditional pdf's $f_Y(y|x)$ and $f_X(x|y)$.

(e) Compute the conditional means $E(Y|x)$ and $E(X|y)$.



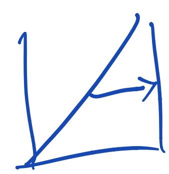
area 1/2 $\Rightarrow K = 2.$



integrate from 0 to x

a) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$= \int_0^x k\,dy = \begin{cases} kx & \text{when } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$

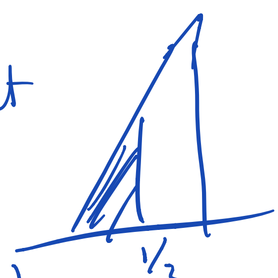


integrate from y to 1

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_y^1 k\,dx = \begin{cases} k(1-y) & \text{when } 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$

b) $K = 2$ by inspection above

c) Solve using areas, since $f_{XY}(x,y)$ is flat



area of shaded triangle is $\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

area of full region of support is $\frac{1}{2}(1)(1)$

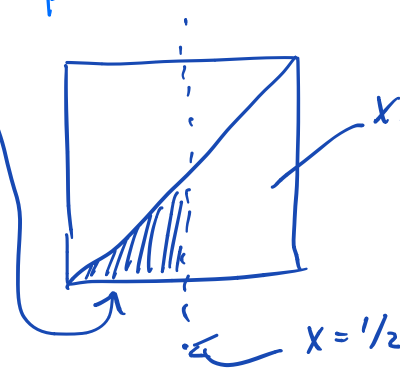
$\Rightarrow P(X < 1/2, Y < 1/2) = \frac{1/8}{1/2} = 1/4$

d) $f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{K}{Kx} = \begin{cases} \frac{1}{x} & \text{when } 0 < y \leq x \leq 1 \\ 0 & \text{else} \end{cases}$

$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-y} & \text{when } 0 < y \leq x < 1 \\ 0 & \text{else} \end{cases}$

$\{X < 1/2 \cap X > Y\}$

picture for part c



Problem 20. (20 POINTS)

Let X and Y be random variables that have the joint pdf

$$f_{X,Y}(x,y) = c(x+y), \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1$$

(square region of support)

(a) Find c.

(b) Find the marginal PDFs of both X and of Y.

(c) What is $P(X > Y | X < 1/2)$?

(d) Find $f_Y(y|x)$.

a) Know $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = \int_0^1\int_0^1 c(x+y)\,dy\,dx$

$= \int_0^1 c\left(xy + \frac{y^2}{2}\right)\Big|_0^1 dx = \int_0^1 c(x + 1/2)\,dx$

$= c\left(\frac{x^2}{2} + \frac{x}{2}\right)\Big|_0^1 = c\left(\frac{1}{2} + \frac{1}{2}\right) = \boxed{c = 1}$

b) $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big|_0^1$

$\boxed{f_X(x) = x + 1/2 \text{ when } 0 \leq x \leq 1}$

x and y are interchangeable, so

$\boxed{f_Y(y) = y + 1/2 \text{ when } 0 \leq y \leq 1}$

c) use Bayes Rule: $P(X > Y | X < 1/2) = \dfrac{P(X < 1/2 \cap X > Y)}{P(X < 1/2)}$

numerator: $= \int_0^{1/2}\int_0^y (x+y)\,dx\,dy = \int_0^{1/2}\left(\frac{x^2}{2} + xy\Big|_0^y\right) dx$

$= \int_0^{1/2} \frac{3y^2}{2}\,dy = \frac{3}{2\cdot 3} y^3\Big|_0^{1/2} = 1/16$

denominator $\int_0^{1/2} f_X(x)\,dx = 3/8$ so answer $= \dfrac{1/16}{3/8} = \boxed{1/6}$

d) $f_Y(y|x) = \dfrac{f_{xy}(x,y)}{f_X(x)} = \dfrac{x+y}{x+1/2} = \boxed{\dfrac{2(x+y)}{2x+1} \text{ when } 0 \leq x \leq 1}$

Problem 21. (15 POINTS)

Suppose X and Y are two random variables with $Y = X^2$. X is uniformly distributed with $S_X = \{-1, 0, 1\}$.

(a) Is X a discrete or continuous random variable? How do you know?

(b) Express the joint PMF of X and Y using a table.

(c) Are X and Y independent? Why or why not?

(d) Are X and Y uncorrelated? Why or why not?

Note: Full credit will only be given for a **complete** answer that includes the correct reason and supporting evidence.

$S_y = \{0, 1\}$

$p_X(x) = 1/3$ when $x \in S_X$

a) X is discrete, because curly braces

b)

y \ x	-1	0	1
0	0	1/3	0
1	1/3	0	1/3

$$p_Y(y) = \begin{cases} 1/3 & y = 0 \\ 2/3 & y = 1 \\ 0 & \text{else} \end{cases}$$

c) Need to have $p_{XY}(x,y) = p_X(x)\,p_Y(y)$ for all x and y

but, for example, $p_{XY}(0,1) = 0$ but

$p_X(0)\,p_Y(1) = (1/3)(2/3) = \frac{2}{9}$

so NO they are not independent

d) $E(X) = 0$, so

$\text{cov}(X,Y) = E(XY) = \frac{1}{3}\left((-1)(1) + (0)(0) + (1)(1)\right)$

$= 0$

xvalue Yvalue

so Yes, X and Y are uncorrelated

Problem 22. (15 POINTS)
Suppose X and Y are two random variables, with means $m_X = 1$ and $m_Y = -3$, and variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 4$, respectively. Suppose we also know that the covariance between X and Y is $\mathrm{COV}(X, Y) = 1/2$.

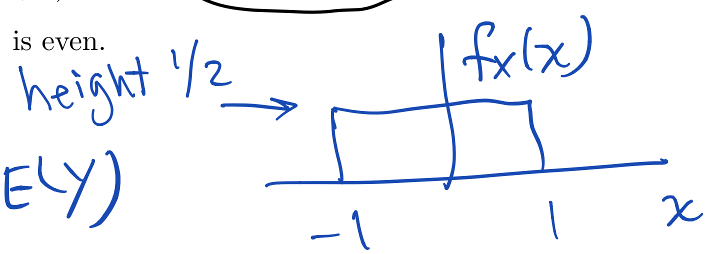
Find the mean and variance of Z when $Z = 3X + Y$.

$$E(Z) = E(3X+Y) = 3E(X) + E(Y)$$
$$= 3\cdot 1 + -3 = 0$$

$$Var(Z) = Var(3X+Y)$$
$$= 9\,var(X) + Var(Y) + 2Cov(3X, Y)$$
$$= 9\cdot 2 + 4 + 2\cdot\tfrac{1}{2}\cdot 3 = \boxed{25}$$

Problem 23. (25 POINTS)
Let X be uniform on $[-1, 1]$. Let $Y = X^n$. Find $COV(X, Y)$.

Hint to save some computation: use the fact that $f_X(x)$ is even.



$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$f_X(x)$ is even, so $E(X) = 0$.

$$Cov(X,Y) = E(XY) = E(X\,X^n) = E(X^{n+1})$$
$$= \int_{-1}^{1} x^{n+1}\,\tfrac{1}{2}\,dx = \frac{x^{n+2}}{2(n+2)}\Bigg|_{-1}^{1} = \begin{cases} \frac{1}{2(n+2)} - \frac{1}{2(n+2)} & \text{if } n \text{ even} \\ \frac{1}{2(n+2)} - \frac{(-1)}{2(n+2)} & \text{if } n \text{ odd} \end{cases}$$

$$\boxed{cov(X,Y) = \begin{cases} 0 & n \text{ even} \\ \frac{1}{n+2} & n \text{ odd} \end{cases}}$$

Problem 24. (10 POINTS)

Suppose X is a RV with zero mean and variance 1, and Y is a RV with mean 1 and variance 4, and suppose the correlation coefficient between X and Y is $1/2$. Find $Var(X+Y)$.

Know: $E(X)=0 \quad Var(X)=1 \quad E(Y)=1 \quad Var(Y)=4$

$$\rho_{XY} = \frac{1}{2} = \frac{cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \quad \text{so} \quad \frac{1}{2} = \frac{cov(X,Y)}{1 \cdot 2}$$

$cov(X,Y) = 1 = E(XY) - E(X)E(Y) = E(XY)$ since $E(X)=0$

want: $Var(X+Y) = VAR(X) + VAR(Y) + 2\,cov(X,Y)$

$$= 1 + 4 + 2(1) = \boxed{7}$$

Problem 25. (10 POINTS)

Suppose X and Y are two random variables with $E(X) = E(Y) = 0$, and $VAR(X) = 1$, and suppose we know X is independent from $X + Y$. What is the covariance between X and Y?

want: $cov(X,Y) = E(XY) - E(X)E(Y) = E(XY)$ since $E(X)=0$

Know: $E(X(X+Y)) = E(X)E(X+Y)$

because X and $X+Y$ are independent

So $E(X^2 + XY) = \underbrace{E(X)}_{\text{zero}} E(X+Y)$

$= E(X^2) + E(XY) = 0$

And since $Var(X) = E(X^2) - E(X)^2 = 1$ from the problem statement

$E(X^2) = 1$ and $E(XY) = -E(X^2) = \boxed{-1}$

covariance between X and Y

Problem 26. (MULTIPLE CHOICE)
Two RVs are uncorrelated.

(a) The correlation is always greater than 0.

(b) The correlation is always less than 0.

(c) The correlation is always equal to 0.

(d) None of the above.

the correlation is $E(XY)$

Uncorrelated means $\rho_{XY} = 0$

or $COV(X,Y) = 0$

$\Rightarrow E(XY) = E(X)E(Y)$

Problem 27. (MULTIPLE CHOICE)
If two RVs are positively correlated, then

(a) $E(XY) > 0$.

(b) $E(XY) < 0$.

(c) $E(XY) = 0$.

(d) $E(XY) = E(X)E(Y)$.

(e) None of the above.

See above

positively corelated means

$\rho_{XY} > 0$ or $COV(X,Y > 0)$

Problem 28. (YES/NO: 4 POINTS)
If the correlation between two random variables is zero, then their correlation coefficient must also be zero.

if $E(X) \neq 0$ and $E(Y) \neq 0$

then $COV(X,Y) = E(XY) - E(X)E(Y) \neq 0$

Problem 29. (YES/NO: 4 POINTS)
If X and Y are independent random variables, then $VAR(3X + 2Y + 1) = VAR(3X - 2Y + 3)$.

$= 9\,Var(X) + 4\,Var(Y)$

Problem 30. (MULTIPLE CHOICE: 5 POINTS)
Let $Z = 3X - Y - 5$, where X and Y are independent random variables with $Var(X) = 1$ and $Var(Y) = 2$. What is $Var(Z)$?

(a) 4

(b) 7

(c) 11

(d) 16

(e) None of the above.

$$Var(Z) = Var(3X - Y - 5)$$
$$= Var(3X) + Var(-Y)$$
$$= 9\,Var(X) + Var(Y)$$
$$= 9 \cdot 1 + 2 = 11$$

Problem 31.
Let $Y = X + 30$ and $Z = 3X - 4$. X is a uniform RV on $[-1, 2]$. Find E(Y), E(Z), VAR(Y), VAR(Z), COV(X,Z), and ρ_{YZ}.

$f_X(x) = \begin{cases} \frac{1}{3} & \text{when } -1 \le x \le 2 \\ 0 & \text{else} \end{cases}$ so $E(X) = \frac{1}{2}$

$Var(X) = \frac{3^2}{12} = \frac{3}{4}$

$E(X^2) = Var(X) + E(X)^2$
$= \frac{3}{4} + \frac{1}{4} = 1$

$E(Y) = E(X + 30) = 30\frac{1}{2}$

$E(Z) = E(3X-4) = 3\left(\frac{1}{2}\right) - 4 = -\frac{5}{2}$

$Var(Y) = E(Y^2) - E(Y)^2$
$= E(X^2 + 60X + 900) - (30.5)^2 = E(X^2) + 60\left(\frac{1}{2}\right) + 900 - (30.5)^2$
$= 1 + 30 + 900 - 930.25 = 3/4$

(or, much easier, $Var(Y) = Var(X+30) = Var(X) = 3/4$)

$Var(Z) = Var(3X-4) = 9\,Var(X) = 27/4$

$COV(X,Z) = E(XZ) - E(X)E(Z) = E(3X^2) - 4E(X) - \frac{1}{2}\left(-\frac{5}{2}\right)$
$\ldots = 9/4$

$\rho_{XZ} = \frac{9/4}{\sqrt{3/4}\sqrt{27/4}} = 1$ (This is not surprising, because Z is a linear function of X)

$COV(Y,Z) = Cov(X+30, 3X-4) = cov(X, 3X) = 3\,var(X) = 9/4$.

$\rho_{YZ} = 1$ also

Problem 32. (5 POINTS)

Let X, Y, and Z be random variables, where X and Y are uncorrelated. The means of the RVs are $E(X) = 1$, $E(Y) = 2$, and $E(Z) = -1$, and $E(XZ) = 5$.

What is $COV(X, Y + 2Z)$?

know $E(X) = 1$

$E(Y) = 2$

$E(Z) = -1$

$E(XY) - E(X)E(Y) = 0$

so $E(XY) = (1)(2) = 2$

and $E(XZ) = 5$.

$$\mathrm{cov}(X, Y+2Z) = E\big(X(Y+2Z)\big) - E(X)E(Y+2Z)$$

$$= E(XY) + E(2XZ) - E(X)\big(E(Y) + 2E(Z)\big)$$

$$= 2 + 2(5) - 1\big(2 + 2(-1)\big)$$

$$= 2 + 10 - 2 + 2 = \boxed{12}$$