

Past Exam Questions
(Fall 2015, Spring 2016, Fall 2016, Fall 2017)
Chapters 3 and 4

Reibman
(January 2019)

Answers only

These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapters 3 and 4 of our textbook.

The last 3 pages of this document will be provided to you as the last pages of the exam. This will be all the formulas that will be available to you. The rest you must memorize.

The next 5 problems all refer to a discrete random variable X with the following pmf:

$$p_X(x) = \begin{cases} |x|/c & \text{for } x = -2, -1, 0, 1, 2. \\ 0 & \text{otherwise} \end{cases}$$

For these problems use the same set of answers, and clearly mark your answer next to the problem. The answers to each problem may or may not be different answers!

- (a) 0
- (b) 1
- (c) 3
- (d) 4
- (e) 6
- (f) None of the above

e **Problem 1.** (5 POINTS)
Find the value of c .

a **Problem 2.** (5 POINTS)
Find $E(X)$.

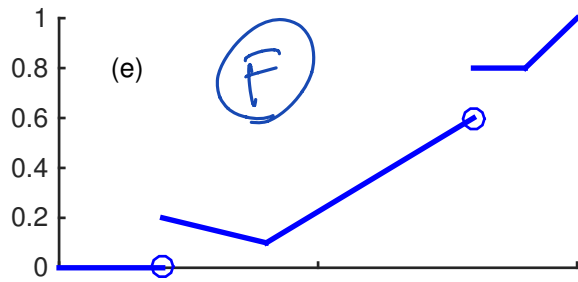
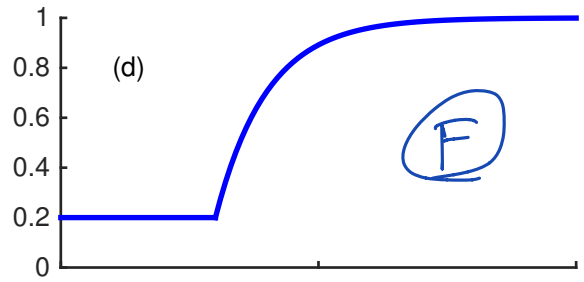
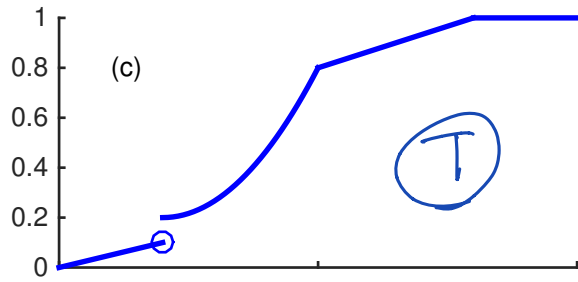
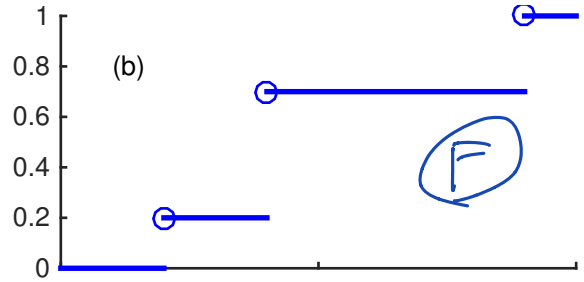
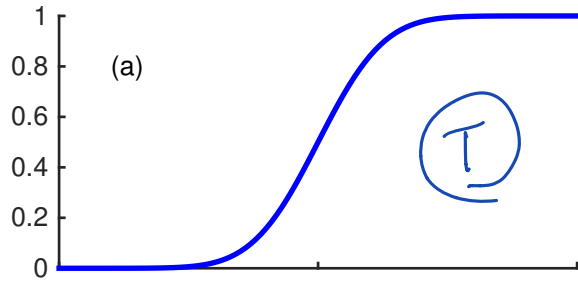
c **Problem 3.** (5 POINTS)
Find $\text{VAR}(X)$.

a **Problem 4.** (5 POINTS)
Consider the random variable $Z = (X - E(X))^2$. Find $P(Z = 9)$.

b **Problem 5.** (5 POINTS)
Consider the event $A = \{X > -1.5\}$. Find $E(X|A)$.

Problem 6. (10 POINTS (2 POINTS EACH))

For each of the following 5 graphs, clearly indicate T (true) if the graph indicates a **valid** cumulative distribution function or F (false) otherwise.



Problem 7. (5 POINTS)

Which of the following is NOT a property that the cumulative distribution function $F_X(x)$ of a random variable X must satisfy?

- (a) $0 \leq F_X(x) \leq 1$ for every x .
- (b) $\lim_{x \rightarrow \infty} F_X(x) = 1$
- (c) $\lim_{x \rightarrow -\infty} F_X(x) = 0$
- (d) $F_X(x)$ is continuous at every x
- (e) $P(X > x) = 1 - F_X(x)$
- (f) $F_X(x)$ is an increasing function of x
- (g) (f) and (d)
- (h) None of the above.
- (i) All of the above.

Problem 8. (5 POINTS)

Which of the following statements are NOT NECESSARILY true about the probability density function $f_X(x)$ of a random variable X ?

- (a) $f_X(x) \leq 1$ for every x .
- (b) $\int_{-\infty}^{\infty} f_X(x) dx = 1$
- (c) $\int_a^b f_X(x) dx = P(a < X \leq b)$ if $a < b$
- (d) $f_X(x) \geq 0$ for every x .
- (e) answers (a) and (c)
- (f) None of the above.
- (g) All of the above.

Problem 9. (5 POINTS)

Which of the following statements is a correct way to compute $P(X > 2)$ for a continuous random variable X ?

(a) $f_X(2)$.

(b) $\int_2^\infty f_X(x)dx$

(c) $\int_2^\infty f_X(X)dX$

(d) $F_X(2)$

(e) $1 - F_X(2)$.

(f) answers (b) and (e)

(g) answers (c) and (d)

(h) None of the above.

Problem 10. (6 POINTS)

Which is the correct mathematical expression for the following probability:

The probability that X varies from its mean μ by no more than 2 is more than 95%.

(You may find it useful to draw a sketch.)

(a) $P(|X - \mu| \leq 2) > 0.95$

(b) $P(|X - \mu| < 2) > 0.95$

(c) $P(|X - \mu| > 2) > 0.95$

(d) $P(|X - \mu| > 2) \leq 0.05$

(e) $P(|X - \mu| \geq 2) \leq 0.05$

(f) answers (a) and (d)

(g) None of the above.

Problem 11. (20 POINTS)

A discrete random variable X has the following probability mass function (PMF):

$$p_X(x) = \begin{cases} cx & \text{for } x = 1, 2, 3, 4. \\ 1/2 & \text{for } x = 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c .

$$1/20$$

(b) Find $E(X)$.

$$4$$

(c) Find $Var(X)$.

$$3/2$$

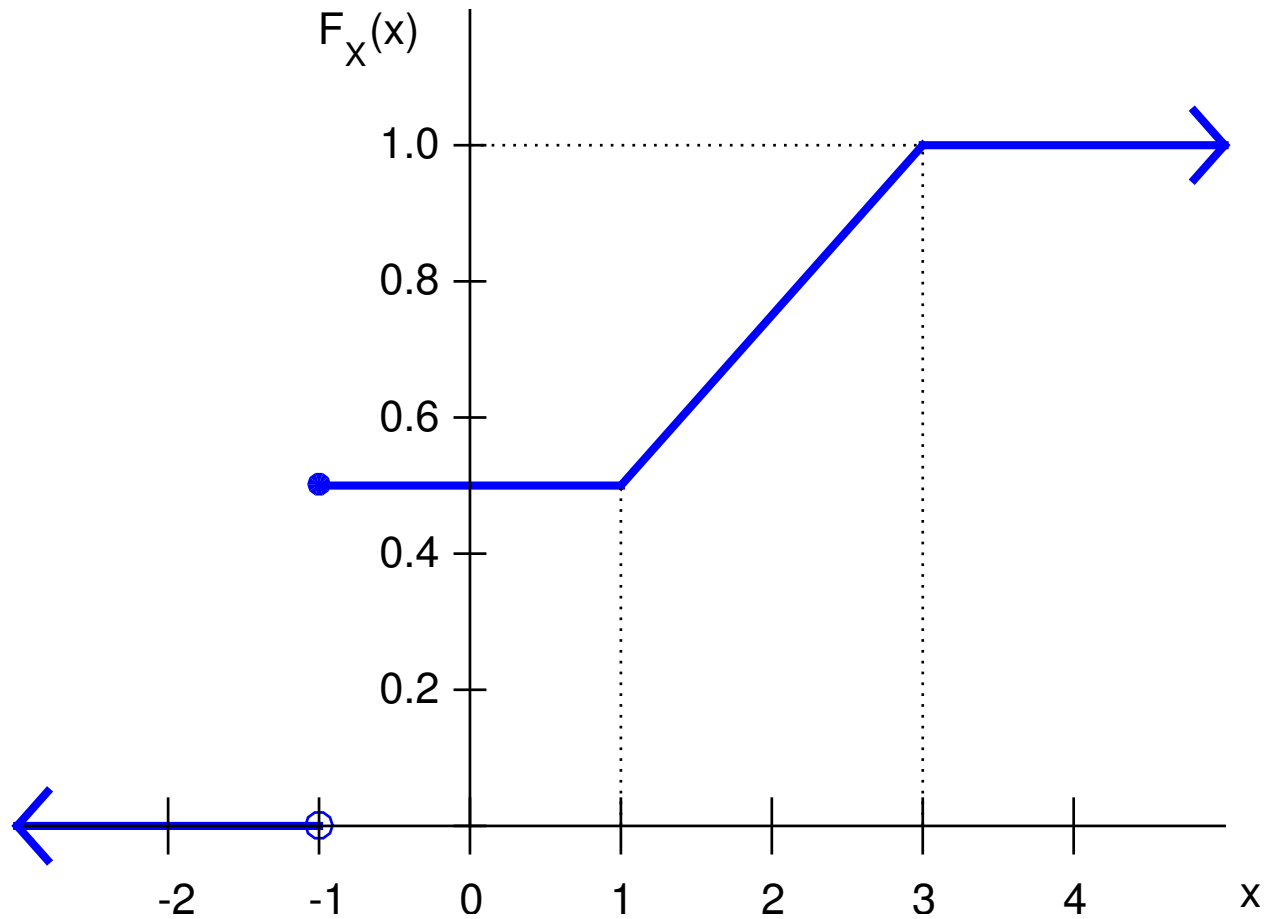
(d) What is the largest value of x for which $P(X > x) > 0.75$? (Hint, a sketch of the PMF can be helpful.)

$$x = 2$$

Problem 12. (15 POINTS)

The figure below shows the cumulative distribution function of a random variable X .

- (a) What is $P(X < -1)$? $\textcircled{0}$
- (b) Find and sketch the PDF of X . Label relevant values.
- (c) Compute $E(X)$.



a) 0

b) $f_X(x) = \frac{1}{2} \delta(x+1) + \frac{1}{4} [u(x-1) - u(x-3)]$

c) $1/2$

Problem 13. (10 POINTS)

The bandwidth of a video being delivered on the internet has a continuous distribution on the interval $(0, 4)$, with a PDF $f_X(x) = a(1+x)^{-2}$ on the interval $(0, 4)$.

(a) Find the constant a .

$$5/4$$

(b) What is the probability that the bandwidth of a video is less than 0.6.

$$15/32$$

Problem 14. (20 POINTS (PART (A) IS 8 POINTS; PART (B) IS 5; PART (C) IS 7))

The probability density function of a random variable X is

$$f_X(x) = (1/4)e^{-x/2}u(x) + \delta(x-1)/2$$

see below

(a) What is $F_X(x)$, the cumulative distribution function of X ?

(b) What is $P(2 \leq X \leq 4)$?

(c) What is $E(X)$, the expected value of X ?

Problem 15. (15 POINTS)

Let X be a continuous RV with PDF

$$f_X(x) = \begin{cases} |x|/10 & \text{for } -2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

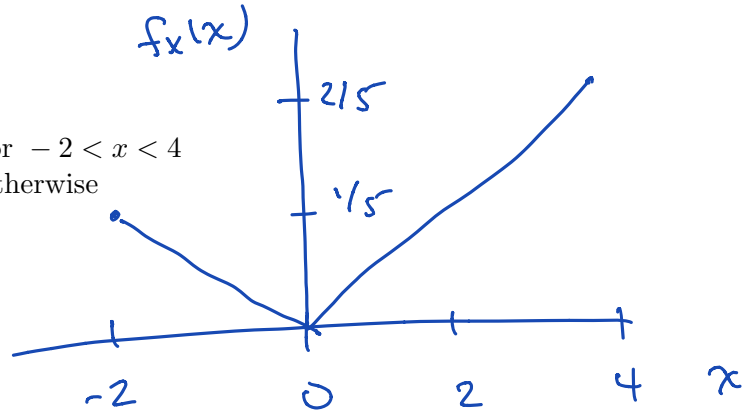
(a) Sketch $f_X(x)$

(b) What is $P(X < 0)$?

$$1/5$$

(c) Find $E(X)$.

$$28/15$$



$$14 \text{ a) } F_X(x) = \frac{1}{2} (1 - e^{-x/2}) u(x) + \frac{1}{2} u(x-1)$$

$$14 \text{ b) } \frac{1}{2} (e^{-1} - e^{-2})$$

$$14 \text{ c) } 3/2$$

The next 4 problems all refer to a random variable X with the following cdf:

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x^2 & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

For these problems use the same set of answers, and clearly mark your answer next to the problem. The answers to each problem may or may not be different answers!

- (a) 0
- (b) 0.1
- (c) 0.2
- (d) $1/3$
- (e) 0.6
- (f) $2/3$
- (g) 1
- (h) None of the above

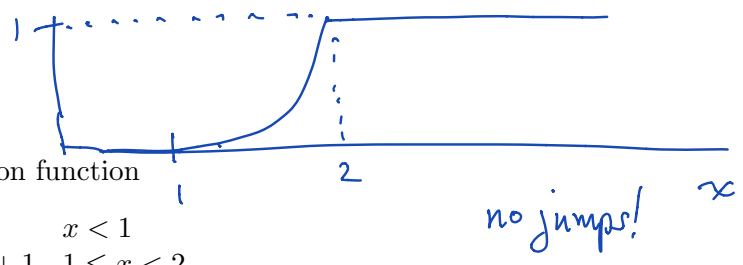
a **Problem 16.** (5 POINTS)
Find $P(X = 0.5)$.

e **Problem 17.** (5 POINTS)
Find $P(0.2 \leq X \leq 0.8)$.

c **Problem 18.** (5 POINTS)
Find $f_X(0.1)$.

f **Problem 19.** (5 POINTS)
Find $E(X)$

$F_X(x)$



Problem 20. (20 POINTS)

A random variable X has the cumulative distribution function

$$F_X(x) = \begin{cases} 0 & x < 1 \\ x^2 - 2x + 1 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

(a) Sketch the CDF, and label the axes and relevant values.

(b) Find the PDF, $f_X(x)$.

(c) Find $E(X)$ $E(X) = 5/3$ $f_X(x) = \begin{cases} 2x - 2 & 1 \leq x \leq 2 \\ 0 & \text{else} \end{cases}$

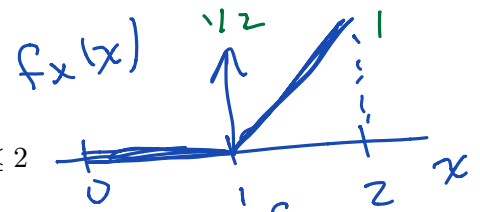
(d) Find the variance of X .

$$\text{var}(X) = 1/18$$

Problem 21. (10 POINTS)

Given the Cumulative Distribution Function

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ (x^2 - 2x + 2)/2 & \text{for } 1 \leq x \leq 2 \\ 1 & \text{for } x > 2 \end{cases}$$



(a) Find and sketch the PDF $f_X(x)$. $f_X(x) = \frac{1}{2} \delta(x-1) + (x-1)[u(x-1) - u(x-2)]$

(b) Find $E(X)$ $E(X) = 4/3$

Problem 22. (28 POINTS (4 POINTS FOR (B), 8 POINTS EACH FOR (A,C,D)))

The probability density function of a random variable X is

$$f_X(x) = \begin{cases} 0.1 + Cx & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of C that makes $f_X(x)$ a valid PDF. $C = 1/25$

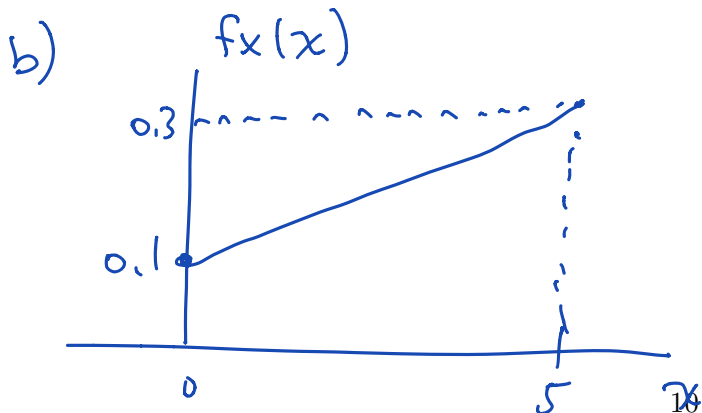
(b) Sketch the PDF. Label axes and relevant values.

(c) What is $P(2 \leq X \leq 4)$?

$$P(2 \leq X \leq 4) = 11/25$$

(d) What is $E(X)$, the expected value of X ?

$$E(X) = 35/12$$



Problem 23. (MULTIPLE CHOICE: 5 POINTS)

Which expression is correct for computing the expected value of a continuous random variable X ?

(a)

$$E(X) = \int_{-\infty}^x t f_X(t) dt$$

(b)

$$E(X) = \int_{-\infty}^{\infty} t f_X(t) dt$$

(c)

$$E(X) = \int_{-\infty}^{\infty} f_X(t) dt$$

(d)

$$E(X) = \int_{-\infty}^{\infty} F_X(t) dt$$

(e) None of the above.

Problem 24. (MULTIPLE CHOICE: 5 POINTS)

The variance of a random variable X with mean μ can be computed using

(a) $E((X - \mu)^2)$

(b) $E(X)^2 - E(X^2)$

(c) $E(X^2) + E(X)^2$

(d) $E(X^2) - E(X)^2$

(e) both (a) and (c)

(f) both (a) and (d)

(g) None of the above

Problem 25. (10 POINTS)

Let X be a random variable with mean μ and variance σ^2 . Define two new random variables, $Y = 2 - X$ and $Z = 3X + 1$. Express $\text{Var}(X + Y + Z)$ in terms of μ and σ^2 .

$$9\sigma^2$$

Problem 26. (10 POINTS)

Let X be a random variable with mean μ and variance σ^2 , and let $Y = 2X + 3$ and $Z = -4X + 5$. Express $\text{Var}(Y + Z)$ in terms of μ and σ .

$$4\sigma^2$$

Problem 27. (10 POINTS)

Let X be a random variable with mean μ and variance σ^2 , and let $Y = 3X + 4X^2$. Express $E(Y)$ in terms of μ and σ .

$$3\mu + 4(\sigma^2 + \mu^2)$$

Problem 48. (TRUE/FALSE: 5 POINTS EACH, TOTAL 20 POINTS)

Label each statement T or F to the left of the problem number.

F (a) $E(g(X)) = g(E(X))$

F (b) Let X be a random variable and let a be a constant. Then $P(X \geq a) = 1 - F_X(a)$.

T (c) A deck of 52 cards is fairly dealt to 2 hands, each with 26 cards. The probability that both hands get 2 aces is $\binom{4}{2} \binom{48}{24} / \binom{52}{26}$

T (d) Let X be a random variable and let $Y = aX + b$, where a, b are constants. Then $\text{VAR}(Y) = a^2 \text{VAR}(X)$.

Discrete Random Variables

- Bernoulli Random Variable, parameter p
 $S = \{0, 1\}$
 $p_0 = 1 - p, p_1 = p; 0 \leq p \leq 1$
 $E(X) = p; \text{VAR}(X) = p(1 - p)$
- Binomial Random Variable, parameters (n, p)
 $S = \{0, 1, \dots, n\}$
 $p_k = \binom{n}{k} p^k (1 - p)^{n-k}; k = 0, 1, \dots, n; 0 \leq p \leq 1$
 $E(X) = np; \text{VAR}(X) = np(1 - p)$
- Geometric Random Variable, parameter p
 $S = \{0, 1, \dots\}$
 $p_k = p(1 - p)^k; k = 0, 1, \dots; 0 \leq p \leq 1$
 $E(X) = (1 - p)/p; \text{VAR}(X) = (1 - p)/p^2$
- Poisson Random Variable, parameter α
 $S = \{0, 1, \dots\}$
 $p_k = \alpha^k e^{-\alpha} / k! \quad k = 0, 1, \dots,$
 $E(X) = \alpha; \text{VAR}(X) = \alpha$
- Uniform Random Variable
 $S = \{0, 1, \dots, L\}$
 $p_k = 1/L \quad k = 0, 1, \dots, L$
 $E(X) = (L + 1)/2; \text{VAR}(X) = (L^2 - 1)/12$

Continuous Random Variables

- Uniform Random Variable
Equally likely outcomes
 $S = [a, b]$
 $f_X(x) = 1/(b - a), \quad a \leq x \leq b$
 $E(X) = (a + b)/2; \quad \text{VAR}(X) = (b - a)^2/12$
- Exponential Random Variable, parameter λ
 $S = [0, \infty)$
 $f_X(x) = \lambda \exp(-\lambda x), \quad x \geq 0, \lambda > 0$
 $E(X) = 1/\lambda; \quad \text{VAR}(X) = 1/\lambda^2$
- One Gaussian Random Variable, parameters μ, σ^2
 $S = (-\infty, \infty)$
 $f_X(x) = \exp(-(x - \mu)^2 / (2\sigma^2)) / \sqrt{2\pi\sigma^2}$
 $E(X) = \mu; \quad \text{VAR}(X) = \sigma^2$

Other useful formulas

$$\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} \quad \text{if } |r| < 1$$

$$\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1 - r)^2} \quad \text{if } |r| < 1$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$\sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = (a + b)^n$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2} \right) e^{ax}$$

$$\int x^2 e^{ax} dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right)$$