ECE 302: Probabilistic Methods in Electrical and Computer Engineering

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Past Exam Questions (Fall 2015, Spring 2016, Fall 2016, Fall 2017) Chapters 3 and 4

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SOLUTIONS

These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapters 3 and 4 of our textbook.

The last 3 pages of this document will be provided to you as the last pages of the exam. This will be all the formulas that will be available to you. The rest you must memorize.

Problems 28-41! solutions only

Problem 28. (5 POINTS (MULTIPLE CHOICE))
A 5-bit codeword is sent over a noisy channel, where bits are flipped with probability p, each independently. An error correcting code is designed such that if two or fewer bits of a codeword are flipped, the codeword can be correctly decode.

What's the probability a codeword cannot be decoded?
(If you show your work, you may receive partial credit.)

(a) $\binom{5}{2}p^2(1-p)^3$

(b) $p^5(1-p)$

(c) $\binom{5}{3}p^3(1-p)^2 + 5p^4(1-p) + p^5$

(d) $p^2(1-p)^3$

(e) $\binom{5}{3}p^3(1-p)^2$

(f) None of the above

X = # flipped bits out of 5 bits

Binomial $n=5$, $p=p$

so $p_X(k) = \binom{5}{k} p^k (1-p)^{5-k}$

BUT we want $P(X \geq 3) = P(X=3) + P(X=4) + P(X=5)$

$$P(X \geq 3) = p_X(3) + p_X(4) + p_X(5)$$
$$= \binom{5}{3} p^3(1-p^2) + \binom{5}{4} p^4(1-p) + \binom{5}{5} p^5 \Rightarrow \boxed{\text{answer } c}$$

Problem 29. (10 POINTS)
A parking lot owner in a major city has a parking lot with 20 spaces. He has already sold monthly parking permits to 21 people, knowing that it is likely that not all cars will ~~want to~~ park there at the same time. In fact, the probability that an individual person will want to park their car there on any given evening is 0.98, independent of all other people.
Parking on any given night of the month is independent of every other night.
Each parking space costs \$5 a night, but if someone with a permit arrives to park and there are no spaces, the parking lot owner will refund \$10 for that night (that is, he will return the \$5 plus a pay a \$5 penalty).
Calculate the expected revenue for the parking lot owner per night.

(NOTE: You do need a complete answer, but you do not have to simplify or use a calculator.
If you choose to simplify it, you may find at least one of the following expressions useful:
$(0.98)^{20} = 0.67$ $(0.98)^{21} = 0.65$ $(0.02)^{20} = 10^{-34}$ $(0.02)^{21} = 2 * 10^{-36}$)

The probability he will refund \$10 is the probability all 21 show up. $= (0.98)^{21}$

But, he has already sold 21 permits, so he has already started with $(21)(5) = \$105$.

So his expected revenue

$$= 105 - 10(0.98)^{21} = 105 - 6.5$$

$$= 98.5$$

So he's not very smart. He could have just sold 20 permits and made \$100.

(This is an example of a resource allocation problem.)

Problem 30. (15 POINTS)
Let X be a random variable that takes values from 0 to 9 with equal probability 1/10. Find the PMF of the random variable $Z = (5 \mod (X+1))$. (Remember, the modulo function $(a \mod b)$ finds the remainder after dividing a by b.)

x	$p_X(x)$	z
0	1/10	5 mod 1 = 0
1	⋮	5 mod 2 = 1
2	⋮	5 mod 3 = 2
3		5 mod 4 = 1
4	⋮	5 mod 5 = 0
5	⋮	5 mod 6 = 5
6		5 mod 7 = 5
7	⋮	5 mod 8 = 5
8	⋮	5 mod 9 = 5
9	1/10	5 mod 10 = 5

z	0	1	2	5	else
$p_Z(z)$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{5}{10}$	0

Problem 31. (Mix and Match: 10 points)
Each of the following world problems (items (1)-(5)) matches one and only one of the list of random variables (items (a)-(e)). For each world problem, identify the appropriate random variable.

e 1. Distance of a cell phone to the nearest base station

b 2. Number of active speakers in a collection of independent conversations

a 3. Fraction of defective items in a production line

d 4. Number of photons received in an optical communication system

c 5. Number of correctly transmitted bits between two erroneous bits

3 (a) Bernoulli random variable?

2 (b) Binomial random variable?

5 (c) Geometric random variable?

4 (d) Poisson random variable?

1 (e) Uniform random variable

Because $X = \#$ failures (not # trials) $P(X=k) = (1-p)^k p$ NOT $(1-p)^{k-1} p$

Problem 32. (5 POINTS)
Suppose X is the number of failures until the first success in a series of independent Bernoulli trials with probability of success 0.2. Find $P(X \geq 10)$.

$X = \#$ failures until 1st success Geometric

$p(\text{success}) = 0.2 = p$

$$P(X \geq 10) = \sum_{k=10}^{\infty} (1-p)^k p$$

change variables:
$j = k - 10$
$k = j + 10$

$$= \sum_{j=0}^{\infty} (1-p)^{j+10} p = p(1-p)^{10} \sum_{j=0}^{\infty} (1-p)^j$$

$$= p(1-p)^{10} \frac{1}{1-(1-p)} = (1-p)^{10} = (0.8)^{10}.$$

Problem 33. (10 POINTS)
Alice and Bob are good friends, who both work at a security company. The company has a design group and an installation group; Alice works in design, Bob in installation. There are 20 people in the design group and 80 people in the installation group. For each project, the company sends 2 from the design group and 5 from the installation group. If each person has the same probability to be selected to a specific project, what is the probability that Alice and Bob will be both be sent to the same project?

ways to form design group $= \binom{20}{2}$

ways to form design group that includes Alice $= \binom{19}{1}$

ways to form installation group $= \binom{80}{5}$

ways to form installation group that includes Bob $= \binom{79}{4}$

$$P(\text{Alice and Bob}) = P(\text{Alice})P(\text{Bob}) = \frac{\binom{19}{1}}{\binom{20}{2}} \cdot \frac{\binom{79}{4}}{\binom{80}{5}}$$

$$= \frac{1}{10} \cdot \frac{1}{16} = \boxed{\frac{1}{160}}$$

Problem 34. (10 POINTS)
Five cars start out on a cross-country race. The probability that a car breaks down and drops out of the race is 0.2. Cars break down independently of each other.

(a) What is the probability that exactly two cars finish the race?

(b) What is the probability that at most two cars finish the race?

(c) What is the probability that at least three cars finish the race?

Binomial $N = \#$ cars that finish.
$p = 0.8$ probability one finishes.

$$P(N=k) = \binom{5}{k} p^k (1-p)^{5-k} \quad \text{for } k=0,1,\dots,5$$

a) $P(N=2) = \binom{5}{2} p^2 (1-p)^3 = \frac{5\cdot 4}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3 = \frac{2\cdot 4^2}{5^4}$
$= 0.0512$

b) $P(N \le 2) = \sum_{k=0}^{2} \binom{5}{k} p^k (1-p)^{5-k} = \frac{1}{5^5}\left[1 + 5\cdot 4 + \frac{5\cdot 4}{2} 4^2\right]$
≈ 0.058

c) $P(N \geq 3) = 1 - P(N \le 2) \approx 1 - 0.058$
$= 0.942$

Problem 35. (TRUE OR FALSE: 5 POINTS)
A fair coin is flipped n times, with independent results each time. The probability there are k heads is $\binom{n}{n-k}(1/2)^n$

$$\binom{n}{n-k} = \binom{n}{k}$$

so $\binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k}$

$= \binom{n}{k}\left(\frac{1}{2}\right)^n = \binom{n}{n-k}\left(\frac{1}{2}\right)^n$ for $k=0,1,\dots,n$

Problem 36. (10 POINTS)
Consider transmitting a signal with a fixed size of 3 bits. The probability of each bit being a "1" is 3/4, independent of the other bits' values. Let N be the number of "1"'s in the signal.

(a) Find and sketch the PMF of N.

(b) What is the probability that either $N = 0$ or $N = 3$.

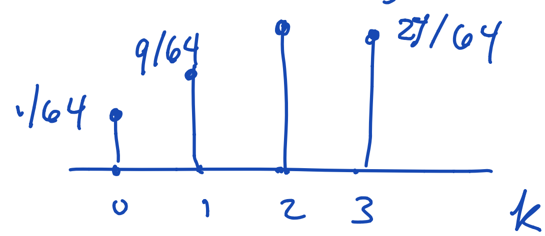
Binomial

$$p_N(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$p = 3/4, \quad n = 3.$$

a)

k	$p_N(k)$	
0	$(1/4)^3$	$= 1/64$
1	$\binom{3}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2$	$= 9/64$
2	$\binom{3}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$	$= 27/64$
3	$\left(\frac{3}{4}\right)^3$	$= 27/64$



b) $P(N=0 \text{ or } N=3) = P(N=0) + P(N=3) = \frac{1}{64} + \frac{27}{64} = \frac{28}{64}$

$= \boxed{7/16}$

Note: "either $N=0$ or $N=3$" does NOT mean you get to choose which of the 2 to tell me. it means $\{N=0 \text{ or } N=3\}$

Problem 37. (5 POINTS)

You flip a fair coin repeatedly until you get a heads. Let X be the number of flips you made up to and including that first heads. What is the probability that $X = 1$ given that $X < 3$?

flip until H $\Rightarrow$ geometric. $k = 1, 2, \ldots$

$p_X(k) = p^{k-1}(1-p)$ where $p = P(\text{Heads}) = 1/2$

pmf NOT conditioned on the event $\{X < 3\}$

$$P(X<3) = P(X=1) + P(X=2) = p^0(1-p) + p(1-p)$$

$$= 1 \cdot \frac{1}{2} + \frac{1}{2}\frac{1}{2} = 3/4$$

$$P(X=1 \mid X<3) = \frac{P(X=1 \cap X<3)}{P(X<3)} = \frac{P(X=1)}{P(X<3)} = \frac{1/2}{3/4}$$

$$= \boxed{\frac{2}{3}}$$

Problem 38. (5 POINTS)
You are designing a video surveillance system on campus, using 6 cameras set up to view 6 independent locations. However, you only buy enough network bandwidth to deliver up to 4 videos at a time to your central monitoring location. So you design a method whereby a video is only sent to the central monitoring location if something "interesting" is happening. If the probability that something "interesting" happens is p, for each camera independently, what is the probability you will not have enough network bandwidth to deliver all "interesting" videos to the central location?

Let X be the number of cameras where something "interesting" is happening.

X is binomial with $n=6$, $p=p$.

$$p_X(k) = \binom{6}{k} p^k (1-p)^{6-k} \qquad k=0,1,\ldots,6$$

P(too many streams to send)

$$= P(X=5) + P(X=6)$$

$$= \binom{6}{5} p^5 (1-p) + p^6 = 6p^5(1-p) + p^6$$

$$= p^5(6-5p)$$

Problem 39. (5 POINTS)
A super-computer has three cooling components that operate independently. Each fails with probability 1/10. The super-computer will overheat if any two (or three) cooling components fail. What is the probability the super-computer overheats?

X = # components that fail.

$P(\text{overheats}) = P(X \geq 2) = P(X=2) + P(X=3)$

pmf of X is binomial with $n=3$ and $p=0.1$

$$P(X=k) = \binom{3}{k} p^k (1-p)^{3-k} \qquad k=0,1,2,3$$

So $P(X \geq 2) = \binom{3}{2}(0.1)^2(0.9) + \binom{3}{3}(0.1)^3$

$$= \frac{3 \cdot 9}{1000} + \frac{1}{1000} = \boxed{\frac{28}{1000}}$$

Problem 40. (5 POINTS (MULTIPLE CHOICE))
Suppose a digital communications channel has a probability of bit error of p, where all errors are independent of each other. If you transmit N bits, whats the probability that exactly k bits are received incorrectly?
(If you show your work, you may receive partial credit.)

(a) $\binom{N}{N-k}p^{N-k}(1-p)^k$

(b) $p^{k-1}(1-p)$

(c) $(1-p)^{k-1}p$

(d) $\binom{N}{k}p^k(1-p)^{N-k}$

(e) None of the above

$$P_X(k) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$k = 0, 1, \dots, N$$

Problem 41. (10 POINTS)
Suppose the probability that a car will have a flat tire while driving on Interstate I-65 is 0.0004. What is the probability that of 10000 cars driving on Interstate I-65, fewer than 3 will have a flat tire. Use the Poisson approximation.

$p = 0.0004$

Poisson

$n = 10\,000$

$\alpha = np = 4$

$$P(X=k) = \frac{4^k e^{-4}}{k!} \qquad k = 0, 1, \dots$$

$$P(X<3) = P(X=0) + P(X=1) + P(X=2)$$

$$= e^{-4}\left[\frac{4^0}{0!} + \frac{4^1}{1} + \frac{4^2}{2}\right]$$

$$= e^{-4}[1 + 4 + 8] \approx 0.238$$

Discrete Random Variables

- Bernoulli Random Variable, parameter p
  $S = \{0, 1\}$
  $p_0 = 1 - p, p_1 = p; 0 \leq p \leq 1$
  $E(X) = p; \mathrm{VAR}(X) = p(1-p)$

- Binomial Random Variable, parameters (n, p)
  $S = \{0, 1, \ldots, n\}$
  $p_k = \binom{n}{k} p^k (1-p)^{n-k}; k = 0, 1, \ldots, n; 0 \leq p \leq 1$
  $E(X) = np; \mathrm{VAR}(X) = np(1-p)$

- Geometric Random Variable, parameter p
  $S = \{0, 1, \ldots\}$
  $p_k = p(1-p)^k; k = 0, 1, \ldots, ; 0 \leq p \leq 1$
  $E(X) = (1-p)/p; \mathrm{VAR}(X) = (1-p)/p^2$

- Poisson Random Variable, parameter α
  $S = \{0, 1, \ldots\}$
  $p_k = \alpha^k e^{-\alpha}/k! \quad k = 0, 1, \ldots,$
  $E(X) = \alpha; \mathrm{VAR}(X) = \alpha$

- Uniform Random Variable
  $S = \{0, 1, \ldots, L\}$
  $p_k = 1/L \quad k = 0, 1, \ldots, L$
  $E(X) = (L+1)/2; \mathrm{VAR}(X) = (L^2 - 1)/12$

Continuous Random Variables

- Uniform Random Variable
  Equally likely outcomes
  $S = [a, b]$
  $f_X(x) = 1/(b-a), \quad a \leq x \leq b$
  $E(X) = (a+b)/2; \quad \mathrm{VAR}(X) = (b-a)^2/12$

- Exponential Random Variable, parameter λ
  $S = [0, \infty)$
  $f_X(x) = \lambda \exp(-\lambda x), \quad x \geq 0,\ \lambda > 0$
  $E(X) = 1/\lambda; \quad \mathrm{VAR}(X) = 1/\lambda^2$

- One Gaussian Random Variable, parameters μ, σ^2
  $S = (-\infty, \infty)$
  $f_X(x) = \exp(-(x-\mu)^2/(2\sigma^2))/\sqrt{2\pi\sigma^2}$
  $E(X) = \mu; \quad \mathrm{VAR}(X) = \sigma^2$

Other useful formulas

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{if } |r|<1$$

$$\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2} \qquad \text{if } |r|<1$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

$$\int xe^{ax}\ dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^{ax}\ dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$