Past Exam Questions
(Fall 2015, Spring 2016, Fall 2016, Fall 2017)
Chapters 3 and 4

Reibman

(January 2019)

These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapters 3 and 4 of our textbook.

The last 3 pages of this document will be provided to you as the last pages of the exam. This will be all the formulas that will be available to you. The rest you must memorize.

The next 5 problems all refer to a discrete random variable X with the following pmf:

$$p_X(x) = \begin{cases} |x|/c & \text{for } x = -2, -1, 0, 1, 2. \\ 0 & \text{otherwise} \end{cases}$$

For these problems use the same set of answers, and clearly mark your answer next to the problem. The answers to each problem may or may not be different answers!

(a) 0

(b) 1

(c) 3

(d) 4

(e) 6

(f) None of the above

Problem 1. (5 POINTS)
Find the value of c.

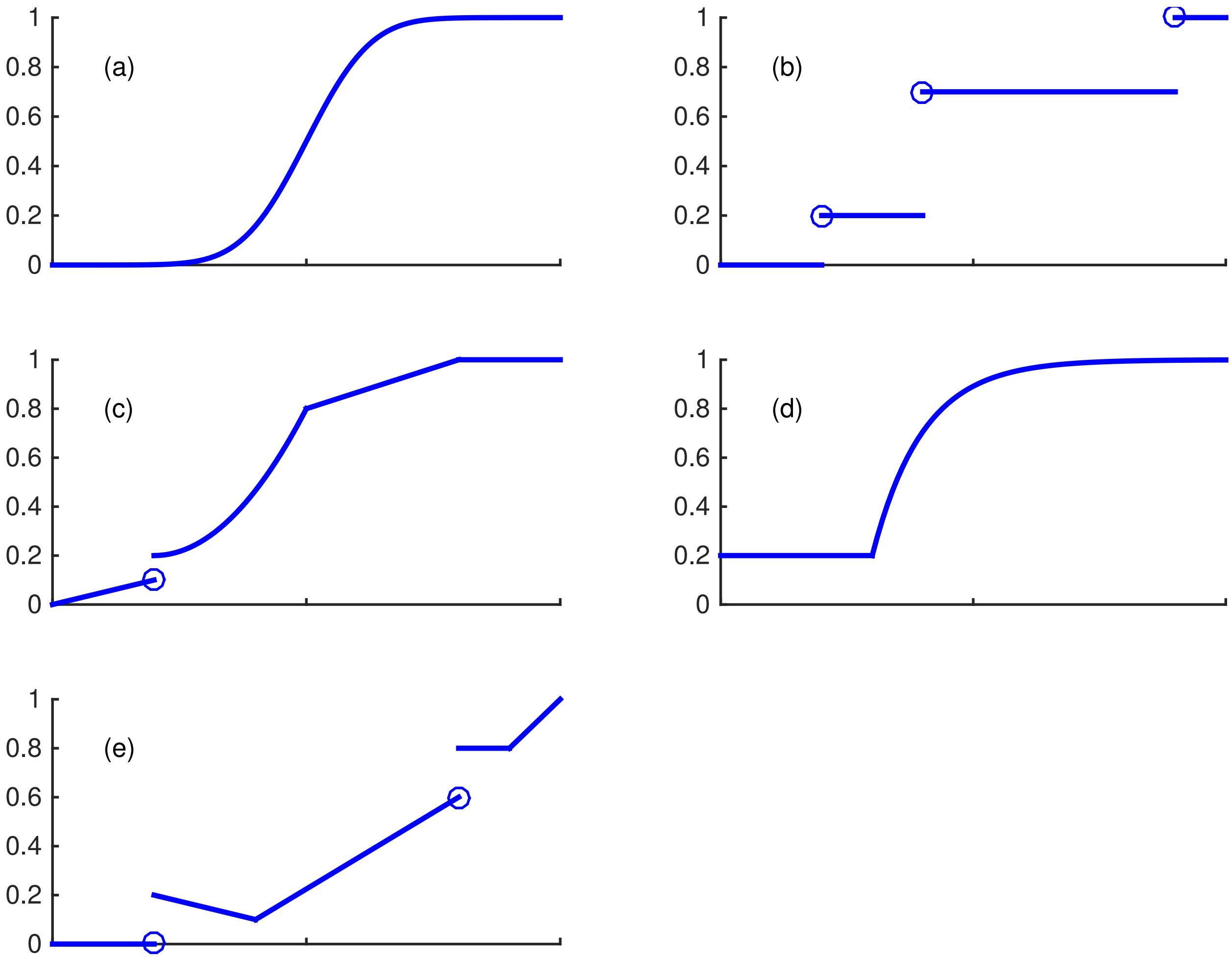
Problem 2. (5 POINTS)
Find $E(X)$.

Problem 3. (5 POINTS)
Find VAR(X).

Problem 4. (5 POINTS)
Consider the random variable $Z = (X - E(X))^2$. Find $P(Z = 9)$.

Problem 5. (5 POINTS)
Consider the event $A = \{X > -1.5\}$. Find $E(X|A)$.

Problem 6. (10 POINTS (2 POINTS EACH))
For each of the following 5 graphs, clearly indicate T (true) if the graph indicates a **valid** cumulative distribution function or F (false) otherwise.



Problem 7. (5 POINTS)
Which of the following is NOT a property that the cumulative distribution function $F_X(x)$ of a random variable X must satisfy?

(a) $0 \leq F_X(x) \leq 1$ for every x.

(b) $\lim_{x\to\infty} F_X(x) = 1$

(c) $\lim_{x\to-\infty} F_X(x) = 0$

(d) $F_X(x)$ is continuous at every x

(e) $P(X > x) = 1 - F_X(x)$

(f) $F_X(x)$ is an increasing function of x

(g) (f) and (d)

(h) None of the above.

(i) All of the above.

Problem 8. (5 POINTS)
Which of the following statements are NOT NECESSARILY true about the probability density function $f_X(x)$ of a random variable X?

(a) $f_X(x) \leq 1$ for every x.

(b) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

(c) $\int_a^b f_X(x)dx = P(a < X \leq b)$ if $a < b$

(d) $f_X(x) \geq 0$ for every x.

(e) answers (a) and (c)

(f) None of the above.

(g) All of the above.

Problem 9. (5 POINTS)
Which of the following statements is a correct way to compute $P(X > 2)$ for a continuous random variable X?

(a) $f_X(2)$.

(b) $\int_2^\infty f_X(x)dx$

(c) $\int_2^\infty f_X(X)dX$

(d) $F_X(2)$

(e) $1 - F_X(2)$.

(f) answers (b) and (e)

(g) answers (c) and (d)

(h) None of the above.

Problem 10. (6 POINTS)
Which is the correct mathematical expression for the following probability:
The probability that X varies from its mean μ by no more than 2 is more than 95%.
(You may find it useful to draw a sketch.)

(a) $P(|X - \mu| \leq 2) > 0.95$

(b) $P(|X - \mu| < 2) > 0.95$

(c) $P(|X - \mu| > 2) > 0.95$

(d) $P(|X - \mu| > 2) \leq 0.05$

(e) $P(|X - \mu| \geq 2) \leq 0.05$

(f) answers (a) and (d)

(g) None of the above.

Problem 11. (20 POINTS)
A discrete random variable X has the following probability mass function (PMF):

$$p_X(x) = \begin{cases} cx & \text{for } x = 1, 2, 3, 4. \\ 1/2 & \text{for } x = 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $E(X)$.

(c) Find $Var(X)$.

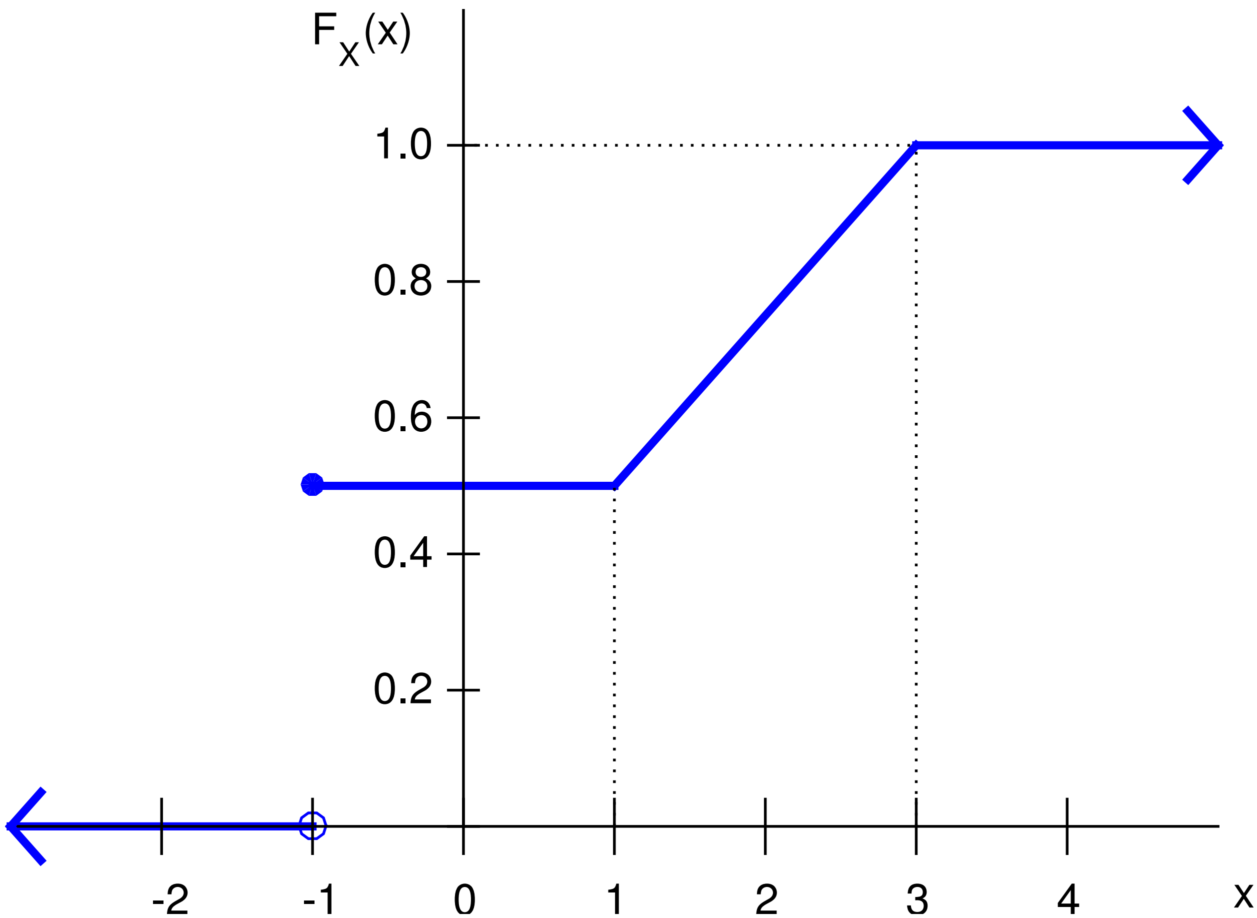
(d) What is the largest value of x for which $P(X > x) > 0.75$? (Hint, a sketch of the PMF can be helpful.)

Problem 12. (15 POINTS)
The figure below shows the cumulative distribution function of a random variable X.

(a) What is $P(X < -1)$?

(b) Find and sketch the PDF of X. Label relevant values.

(c) Compute $E(X)$.



Problem 13. (10 POINTS)
The bandwidth of a video being delivered on the internet has a continuous distribution on the interval $(0, 4)$, with a PDF $f_X(x) = a(1 + x)^{-2}$ on the interval $(0, 4)$.

(a) Find the constant a.

(b) What is the probability that the bandwidth of a video is less than 0.6.

Problem 14. (20 POINTS (PART (A) IS 8 POINTS; PART (B) IS 5; PART (C) IS 7))
The probability density function of a random variable X is

$$f_X(x) = (1/4)e^{-x/2}u(x) + \delta(x - 1)/2$$

(a) What is $F_X(x)$, the cumulative distribution function of X?

(b) What is $P(2 \leq X \leq 4)$?

(c) What is $E(X)$, the expected value of X?

Problem 15. (15 POINTS)
Let X be a continuous RV with PDF

$$f_X(x) = \begin{cases} |x|/10 & \text{for } -2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch $f_X(x)$

(b) What is $P(X < 0)$?

(c) Find $E(X)$.

The next 4 problems all refer to a random variable X with the following cdf:

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x^2 & \text{for } 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

For these problems use the same set of answers, and clearly mark your answer next to the problem. The answers to each problem may or may not be different answers!

(a) 0

(b) 0.1

(c) 0.2

(d) 1/3

(e) 0.6

(f) 2/3

(g) 1

(h) None of the above

Problem 16. (5 POINTS)
Find $P(X = 0.5)$.

Problem 17. (5 POINTS)
Find $P(0.2 \leq X \leq 0.8)$.

Problem 18. (5 POINTS)
Find $f_X(0.1)$.

Problem 19. (5 POINTS)
Find $E(X)$

Problem 20. (20 POINTS)
A random variable X has the cumulative distribution function

$$F_X(x) = \begin{cases} 0 & x < 1 \\ x^2 - 2x + 1 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

(a) Sketch the CDF, and label the axes and relevant values.

(b) Find the PDF, $f_X(x)$.

(c) Find $E(X)$

(d) Find the variance of X.

Problem 21. (10 POINTS)
Given the Cumulative Distribution Function

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ (x^2 - 2x + 2)/2 & \text{for } 1 \leq x \leq 2 \\ 1 & \text{for } x > 2 \end{cases}$$

(a) Find and sketch the PDF $f_X(x)$.

(b) Find $E(X)$

Problem 22. (28 POINTS (4 POINTS FOR (B), 8 POINTS EACH FOR (A,C,D))
The probability density function of a random variable X is

$$f_X(x) = \begin{cases} 0.1 + Cx & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find the value of C that makes $f_X(x)$ a valid PDF.

(b) Sketch the PDF. Label axes and relevant values.

(c) What is $P(2 \leq X \leq 4)$?

(e) What is $E(X)$, the expected value of X?

Problem 23. (Multiple choice: 5 points)

Which expression is correct for computing the expected value of a continuous random variable X?

(a)
$$E(X) = \int_{-\infty}^{x} t f_X(t) dt$$

(b)
$$E(X) = \int_{-\infty}^{\infty} t f_X(t) dt$$

(c)
$$E(X) = \int_{-\infty}^{\infty} f_X(t) dt$$

(d)
$$E(X) = \int_{-\infty}^{\infty} F_X(t) dt$$

(e) None of the above.

Problem 24. (Multiple choice: 5 points)
The variance of a random variable X with mean μ can be computed using

(a) $E((X-\mu)^2)$

(b) $E(X)^2 - E(X^2)$

(c) $E(X^2) + E(X)^2$

(d) $E(X^2) - E(X)^2$

(e) both (a) and (c)

(f) both (a) and (d)

(g) None of the above

Problem 25. (10 POINTS)
Let X be a random variable with mean μ and variance σ^2. Define two new random variables, $Y = 2 - X$ and $Z = 3X + 1$. Express $Var(X + Y + Z)$ in terms of μ and σ^2.

Problem 26. (10 POINTS)
Let X be a random variable with mean μ and variance σ^2, and let $Y = 2X + 3$ and $Z = -4X + 5$. Express $Var(Y + Z)$ in terms of μ and σ.

Problem 27. (10 POINTS)
Let X be a random variable with mean μ and variance σ^2, and let $Y = 3X + 4X^2$. Express $E(Y)$ in terms of μ and σ.

Problem 28. (5 POINTS (MULTIPLE CHOICE))
A 5-bit codeword is sent over a noisy channel, where bits are flipped with probability p, each independently. An error correcting code is designed such that if two or fewer bits of a codeword are flipped, the codeword can be correctly decode.

What's the probability a codeword cannot be decoded?
(If you show your work, you may receive partial credit.)

(a) $\binom{5}{2}p^2(1-p)^3$

(b) $p^5(1-p)$

(c) $\binom{5}{3}p^3(1-p)^2 + 5p^4(1-p) + p^5$

(d) $p^2(1-p)^3$

(e) $\binom{5}{3}p^3(1-p)^2$

(f) None of the above

Problem 29. (10 POINTS)
A parking lot owner in a major city has a parking lot with 20 spaces. He has already sold monthly parking permits to 21 people, knowing that it is likely that not all cars will want to park there at the same time. In fact, the probability that an individual person will want to park their car there on any given evening is 0.98, independent of all other people.
Parking on any given night of the month is independent of every other night.
Each parking space costs $5 a night, but if someone with a permit arrives to park and there are no spaces, the parking lot owner will refund $10 for that night (that is, he will return the $5 plus a pay a $5 penalty).
Calculate the expected revenue for the parking lot owner per night.

(NOTE: You do need a complete answer, but you do not have to simplify or use a calculator.
If you choose to simplify it, you may find at least one of the following expressions useful:
$(0.98)^{20} = 0.67$ $\quad$ $(0.98)^{21} = 0.65$ $\quad$ $(0.02)^{20} = 10^{-34}$ $\quad$ $(0.02)^{21} = 2 * 10^{-36}$)

Problem 30. (15 POINTS)
Let X be a random variable that takes values from 0 to 9 with equal probability 1/10. Find the PMF of the random variable $Z = (5 \mod (X+1))$. (Remember, the modulo function ($a \mod b$) finds the remainder after dividing a by b.)

Problem 31. (Mix and Match: 10 points)
Each of the following world problems (items (1)-(5)) matches one and only one of the list of random variables (items (a)-(e)). For each world problem, identify the appropriate random variable.

1. Distance of a cell phone to the nearest base station
2. Number of active speakers in a collection of independent conversations
3. Fraction of defective items in a production line
4. Number of photons received in an optical communication system
5. Number of correctly transmitted bits between two erroneous bits

(a) Bernoulli random variable?

(b) Binomial random variable?

(c) Geometric random variable?

(d) Poisson random variable?

(e) Uniform random variable

Problem 32. (5 POINTS)
Suppose X is the number of failures until the first success in a series of independent Bernoulli trials with probability of success 0.2. Find $P(X \geq 10)$.

Problem 33. (10 POINTS)
Alice and Bob are good friends, who both work at a security company. The company has a design group and an installation group; Alice works in design, Bob in installation. There are 20 people in the design group and 80 people in the installation group. For each project, the company sends 2 from the design group and 5 from the installation group. If each person has the same probability to be selected to a specific project, what is the probability that Alice and Bob will be both be sent to the same project?

Problem 34. (10 POINTS)
Five cars start out on a cross-country race. The probability that a car breaks down and drops out of the race is 0.2. Cars break down independently of each other.

(a) What is the probability that exactly two cars finish the race?

(b) What is the probability that at most two cars finish the race?

(c) What is the probability that at least three cars finish the race?

Problem 35. (TRUE OR FALSE: 5 POINTS)
A fair coin is flipped n times, with independent results each time. The probability there are k heads is $\binom{n}{n-k}(1/2)^n$

Problem 36. (10 POINTS)
Consider transmitting a signal with a fixed size of 3 bits. The probability of each bit being a "1" is 3/4, independent of the other bits' values. Let N be the number of "1"'s in the signal.

(a) Find and sketch the PMF of N.

(b) What is the probability that either $N = 0$ or $N = 3$.

Problem 37. (5 POINTS)

You flip a fair coin repeatedly until you get a heads. Let X be the number of flips you made up to and including that first heads. What is the probability that $X = 1$ given that $X < 3$?

Problem 38. (5 POINTS)
You are designing a video surveillance system on campus, using 6 cameras set up to view 6 independent locations. However, you only buy enough network bandwidth to deliver up to 4 videos at a time to your central monitoring location. So you design a method whereby a video is only sent to the central monitoring location if something "interesting" is happening. If the probability that something "interesting" happens is p, for each camera independently, what is the probability you will not have enough network bandwidth to deliver all "interesting" videos to the central location?

Problem 39. (5 POINTS)
A super-computer has three cooling components that operate independently. Each fails with probability 1/10. The super-computer will overheat if any two (or three) cooling components fail. What is the probability the super-computer overheats?

Problem 40. (5 POINTS (MULTIPLE CHOICE))
Suppose a digital communications channel has a probability of bit error of p, where all errors are independent of each other. If you transmit N bits, whats the probability that exactly k bits are received incorrectly?
(If you show your work, you may receive partial credit.)

(a) $\binom{N}{N-k} p^{N-k}(1-p)^k$

(b) $p^{k-1}(1-p)$

(c) $(1-p)^{k-1}p$

(d) $\binom{N}{k} p^k (1-p)^{N-k}$

(e) None of the above

Problem 41. (10 POINTS)
Suppose the probability that a car will have a flat tire while driving on Interstate I-65 is 0.0004. What is the probability that of 10000 cars driving on Interstate I-65, fewer than 3 will have a flat tire. Use the Poisson approximation.

Problem 42. (15 POINTS)

Let X be a random variable with PDF

$$f_X(x) = c(2x - x^2), \text{for } 0 \leq x \leq 2$$

(a) Find c.

(b) What is $P(X < 1)$?

(c) Find $E(X|A)$ for the event $A = \{X < 1\}$.

Problem 43. (15 POINTS)
Suppose X is a continuous random variable with PDF

$$f_X(x) = \begin{cases} 3(4 - x^2)/32 & \text{for } -2 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the probability of event A, where $A = \{0 \leq X \leq 1\}$?

(b) Find the conditional PDF of X, conditioned on the event A.

(c) What is $E(X|A)$?

Problem 44. (20 POINTS ((A) IS 4 POINTS; (B,C) ARE 8 POINTS EACH))
Consider the random variable X with PDF given by

$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch $f_X(x)$. Label axes and relevant values.

(b) Find and sketch the conditional density $f_X(x|A)$ for the event $A = \{X < 1/4\}$.

(c) What is the conditional mean $E(X|B)$ for the event $B = \{2/3 < X < 4/3\}$?
(Hint: you do not need to find $P(B)$ to solve part (c)!)

Problem 45. (25 POINTS ((A) IS 7 POINTS; (B) IS 8 POINTS; (C) IS 10 POINTS))
Let X be a discrete RV with sample space $S_X = \{1, 4\}$, each equally likely. Given that we know $X = x$, a second RV Y is exponentially distributed with mean $1/x$.

(a) What is the conditional pdf of Y given X?

(b) What is the marginal pdf of Y?

(c) Find $E(Y)$.
(Hint: it will be faster to use the theorem of total expectations, but you may solve it any way you wish.)

Problem 46. (MULTIPLE CHOICE: 5 POINTS)
Three engineers, Jan, Pat, and Rory, are processing work orders. The time it takes each to finish one work order is an exponential random variable. Jan takes an average of 3 hours; Pat takes an average of 1 hours, and Rory takes an average of 4 hours. Because of their speed, Pat processes 50% of the work orders, while Jan and Rory each process 25% of them.
What is the mean time (in hours) it takes any given work order to be completed?

(a) 2

(b) 9/4

(c) 8/3

(d) 8

(e) None of the above

(f) Too little information to solve.

Problem 47. (16 POINTS)
A customer walks into a store and is equally likely to be served by one of three clerks. The time taken by the first clerk is an exponetial RV with mean 2; the time taken by the second clerk is an constant RV with mean 1; and the time taken by the third clerk is a uniform RV between zero and two.

(a) Express the PDF of T the time to serve the customer.

(b) Find $E(T)$.

Problem 48. (True/False: 5 points each, total 20 points)
Label each statement T or F to the left of the problem number.

(a) $E(g(X)) = g(E(X))$

(b) Let X be a random variable and let a be a constant. Then $P(X \geq a) = 1 - F_X(a)$.

(c) A deck of 52 cards is fairly dealt to 2 hands, each with 26 cards. The probability that both hands get 2 aces is $\binom{4}{2}\binom{48}{24}/\binom{52}{26}$

(d) Let X be a random variable and let $Y = aX + b$, where a, b are constants.
Then VAR(Y)=a^2VAR(X).

Problem 49. (15 points)
The time it takes a computer program to execute is exponentially distributed with a mean of 5 minutes. Calculate the mean execution time, given that it is at least 4 minutes.

Problem 50. (15 points)

Suppose Y is Gaussian with mean 1 and variance 9. Find the value c such that

$$P(Y > c) = 2P(Y \leq c)$$

You may want to use the facts that $\Phi(-0.96) = 1/6$, $\phi(-0.435) = 1/3$, and $\Phi(-0.675) = 0.25$, and $\Phi(x) = 1 - \Phi(-x)$. A sketch may also be helpful.

Problem 51. (15 points)

Suppose Y is Gaussian with mean 0 and variance 4. What value of y corresponds to $P(|Y| \geq y) = 0.1$.

Hint: it may be helpful to sketch the pdf and the area under the curve. Also, you may want to use the facts that $\phi(-0.84) = 0.2$, $\Phi(-1.28) = 0.1$, $\Phi(-1.64) = 0.05$, and $\Phi(-1.96) = 0.025$.

Problem 52. (Multiple choice: 5 points)
The lifetime of a machine, X, is a Gaussian random variable with mean 10 and variance 4. What is the value of x for which the machine has an 12% chance of surviving x or more years?
(You may use (and detach) the Φ-function table on the last page of the exam. If you draw a clear picture you may get partial credit.)

(a) 5.30

(b) 7.65

(c) 8.41

(d) 12.35

(e) 14.70

Problem 53. (15 points)

You have two devices, each of whose lifetimes are modeled by a Gaussian distribution. Let D_1 be the lifetime of Device 1, which is Gaussian with mean 60 and variance 9. Let D_2 be the lifetime of Device 2, which is Gaussian with mean 56 and variance 36.

(a) If you are to choose one device for your system that needs to operate for 62 hours, which of these two devices should you choose?

(b) If your system needs to operate for 65 hours, which of these two devices should you choose?

Note: Full credit will only be given for a **complete** answer that includes the correct reason and supporting evidence. You may use (and detach) the Φ-function table on the last page of the exam.

Problem 54.
Let X be a Gaussian variable with $E(X) = 0$ and $P(|X| \leq 10) = 0.1$. What is the standard deviation of X? You may want to use the facts that $\Phi(-1.28) = 0.1$, $\Phi(-1.64) = 0.05$, and $\Phi(-1.96) = 0.025$.

Problem 55. (5 POINTS)

Let X be a Gaussian random variable with mean 5 and variance 4. Use the $\Phi(x)$ function to write an expression for the probability that X takes on a value between 3 and 6?

(It may be helpful to draw a **clear** picture of the PDF and how you compute the probability from it.)
You can leave your answer in terms of the Φ function (because no table is provided!).

Problem 56.
The peak temperature T in degrees Farenheit, on a July day in Antarctica is a Gaussian RV with variance 225. With probabilty 1/2, the peak temperature T exceeds 10 degrees. What is $P(T > 32)$, the probability the peak temperature is above freezing?

Problem 57. (5 POINTS)

Let X be a Gaussian random variable with mean 52 and variance 4. What is the probability that $X > 57$ given that $X > 54$?

(You may leave the (approximate) answer in terms of a ratio of integers.)

Problem 58. (15 POINTS)

Two manufacturers label resistors as 100-ohm resistors, but in reality the actual resistence is a random variable, X. For one company, X is Gaussian with mean 100 and variance 9. For the second company, X is Gaussian with mean 100 and variance 4. Equal numbers of resistors from each company are purchased and then mixed together.

(a) If you pick a resistor at random, what is the probability its resistance is between 97 and 100?

(b) If you pick a resistor with resistance $X = 97$, what is the probability it was made by Company 2?

Note: You may use (and detach) the Φ-function table on the last page of the exam.

Problem 59. (Multiple choice: 6 points)
If X is a uniformly distributed RV on the interval $[0, 1]$, then the pdf of the RV $Y = 5X$ is

(a)

$$f_Y(y) = \begin{cases} 1/5 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_Y(y) = \begin{cases} 5 & \text{for } 0 \le y \le 1/5 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_Y(y) = \begin{cases} 1/5 & \text{for } 0 \le y \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$f_Y(y) = \begin{cases} 5 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(e)

$$f_Y(y) = \begin{cases} 1 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 60. (15 points)
Let X be a voltage input to a rectifier, and let X be a continuous uniform random variable on the interval $[-1, 1]$. The rectifier output is a random variable Y, where

$$Y = g(X) = \begin{cases} 0 & X < 0 \\ X & X \ge 0 \end{cases}$$

Find and sketch the PDF of Y, $f_Y(y)$.

Problem 61. (20 points)
Suppose self-driving cars have become a reality, in which you, personally, have no control over the speed. You program your car with a destination, and the car drives at a constant speed throughout the trip. This speed is chosen by the car before the trip starts, and the speed is held constant throughout the trip.

Suppose you "drive" (in your self-driving car) from West Lafayette, IN to Louisville KY at a constant speed that is uniformly distributed between 30 and 60 miles per hour (mph). The trip is exactly 180 miles. What is the PDF of the **duration** of the trip?

Another description of the problem that is completely equivalent:
Suppose X and Y are random variables, where X is the speed and $Y = 180/X$ is the duration. If X is uniformly distributed between $[30, 60]$, what is the PDF of Y?

(Hint: first decide: is X a continuous or discrete RV?)

Problem 62.
Let $Y = 2X + 3$. Find the PDF of Y if X is a uniform RV on $[-1, 2]$.

Problem 63.
If X is a positive random variable with density $f_X(x)$, find the density of $+\sqrt{X}$. Apply this to find the pdf of the length of a side of a square when the area of the square is uniformly distributed in $[a, b]$.

Problem 64. (Multiple choice: 5 points)

Let $Y = X^{1/3}$, and let X be a continuous random variable that is uniformly distributed on the interval $[-1, 8]$. What is the PDF of Y?

(Note: if you show your work you may get partial credit.)

(a)

$$f_Y(y) = \begin{cases} 3y^2/513 & \text{for } -1 < y < 8 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_Y(y) = \begin{cases} y^2/3 & \text{for } -1 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_Y(y) = \begin{cases} 1/3 & \text{for } -1 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$f_Y(y) = \begin{cases} 1/9 & \text{for } -1 < y < 8 \\ 0 & \text{otherwise} \end{cases}$$

(e)

$$f_Y(y) = \begin{cases} 3y^2 & \text{for } -1 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(f) None of the above.

Problem 65. (5 POINTS)
A device is deployed in a remote region. The time, T, to failure, is exponentially distributed with mean 3 years. The device will not be monitored during the first 2 years, so the time before failure can be discovered is $X = max(T, 2)$. What is $E(X)$?

(Hint: Draw a sketch of the PDF of X. If you can break it into 2 parts and apply principles we learned in class, only one integration is necessary. If not, you may find the following integral (without limits) helpful.)

$$\int xe^{ax}\, dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

Problem 66. (5 POINTS)
Given the CDF of X,

$$F_X(x) = \begin{cases} 1 - (2/x)^2 & \text{for } x > 2 \\ 0 & \text{otherwise} \end{cases}$$

what is the PDF of $Y = X^2$, $f_Y(y)$.

Problem 67. (15 POINTS)
Suppose a continuous random variable X is input to a quantizer to create Y, where

$$Y = g(X) = \begin{cases} -2 & \text{for } x < -1 \\ 0 & \text{for } -1 \le x < 1 \\ +2 & \text{for } x \ge 1 \end{cases}$$

(a) Express the PDF of the random variable Y in terms of the PDF $f_X(x)$ of the input X, using an expression that will be true for any $f_X(x)$. You may find it useful to sketch $g(X)$.

(b) Suppose $f_X(x)$ is a uniform distribution between $[-3, 5]$. What is $E(Y)$?

Problem 68. (15 POINTS)
Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} x/2 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

X is input to a system with output $Y = g(X)$, where

$$g(X) = \begin{cases} -2(X+1) & \text{for } X < -1 \\ 0 & \text{for } -1 \le X < 1 \\ +2(X-1) & \text{for } X \ge 1 \end{cases}$$

Find the PDF of the output random variable Y.

Problem 69. (15 POINTS)
Let X be the voltage output from a microphone which is uniform on the interval from $[-5, 5]$. Then X is input to a limiter circuit, with cut-off ± 4. Thus the output of the limiter Y is given by

$$Y = g(X) = \begin{cases} -4 & \text{for } X < -4 \\ X & \text{for } -4 \leq X \leq 4 \\ 4 & \text{for } X > 4 \end{cases}$$

Find and sketch the PDF of Y, $f_Y(y)$.

Problem 70. (5 POINTS)
Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 2/x^3 & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = g(X) = \min(X, 10)$. (This means that Y is either X or 10, whichever is less.) What is the expected value of Y?

Discrete Random Variables

- Bernoulli Random Variable, parameter p
  $S = \{0, 1\}$
  $p_0 = 1 - p, p_1 = p; 0 \leq p \leq 1$
  $E(X) = p; \text{VAR}(X) = p(1-p)$

- Binomial Random Variable, parameters (n, p)
  $S = \{0, 1, \ldots, n\}$
  $p_k = \binom{n}{k} p^k (1-p)^{n-k}; k = 0, 1, \ldots, n; 0 \leq p \leq 1$
  $E(X) = np; \text{VAR}(X) = np(1-p)$

- Geometric Random Variable, parameter p
  $S = \{0, 1, \ldots\}$
  $p_k = p(1-p)^k; k = 0, 1, \ldots, ; 0 \leq p \leq 1$
  $E(X) = (1-p)/p; \text{VAR}(X) = (1-p)/p^2$

- Poisson Random Variable, parameter α
  $S = \{0, 1, \ldots\}$
  $p_k = \alpha^k e^{-\alpha}/k! \quad k = 0, 1, \ldots,$
  $E(X) = \alpha; \text{VAR}(X) = \alpha$

- Uniform Random Variable
  $S = \{0, 1, \ldots, L\}$
  $p_k = 1/L \quad k = 0, 1, \ldots, L$
  $E(X) = (L+1)/2; \text{VAR}(X) = (L^2 - 1)/12$

Continuous Random Variables

- Uniform Random Variable
  Equally likely outcomes
  $S = [a, b]$
  $f_X(x) = 1/(b-a), \quad a \leq x \leq b$
  $E(X) = (a+b)/2; \quad \text{VAR}(X) = (b-a)^2/12$

- Exponential Random Variable, parameter λ
  $S = [0, \infty)$
  $f_X(x) = \lambda \exp(-\lambda x), \quad x \geq 0, \ \lambda > 0$
  $E(X) = 1/\lambda; \quad \text{VAR}(X) = 1/\lambda^2$

- One Gaussian Random Variable, parameters μ, σ^2
  $S = (-\infty, \infty)$
  $f_X(x) = \exp(-(x-\mu)^2/(2\sigma^2))/\sqrt{2\pi\sigma^2}$
  $E(X) = \mu; \quad \text{VAR}(X) = \sigma^2$

Other useful formulas

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{if } |r| < 1$$

$$\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2} \qquad \text{if } |r| < 1$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

$$\int xe^{ax}\ dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^{ax}\ dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$

Table 1: Table of the Standard Normal Cumulative Distribution Function $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990