ECE 302: Probabilistic Methods in Electrical and Computer Engineering

Instructor: Prof. A. R. Reibman

Past Exam Questions
(Fall 2015, Spring 2016, Fall 2016, Fall 2017)
Chapters 3 and 4

Reibman

(January 2019)

SOLUTIONS

These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapters 3 and 4 of our textbook.

The last 3 pages of this document will be provided to you as the last pages of the exam. This will be all the formulas that will be available to you. The rest you must memorize.

Solutions to #59-70

only

Problem 59. (Multiple choice: 6 points)
If X is a uniformly distributed RV on the interval $[0,1]$, then the pdf of the RV $Y = 5X$ is

(a)
$$f_Y(y) = \begin{cases} 1/5 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$
(not a pdf)

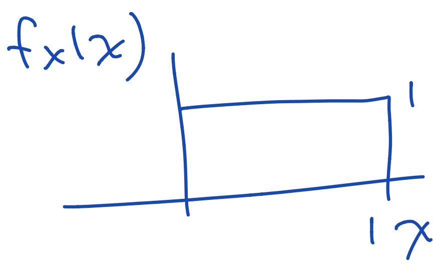
(b)
$$f_Y(y) = \begin{cases} 5 & \text{for } 0 \le y \le 1/5 \\ 0 & \text{otherwise} \end{cases}$$
(wrong height, wrong bounds)

(c)
$$f_Y(y) = \begin{cases} 1/5 & \text{for } 0 \le y \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(d)
$$f_Y(y) = \begin{cases} 5 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$
(not a pdf)

(e)
$$f_Y(y) = \begin{cases} 1 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$
(wrong height + bounds)

Given



$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Recall $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$ if $Y = aX + b$

here, $Y = 5X$, so

$$f_Y(y) = \frac{1}{5} f_X\left(\frac{y}{5}\right) = \begin{cases} 1/5 & 0 \le y \le 5 \\ 0 & \text{else} \end{cases}$$

check! $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$ ✓

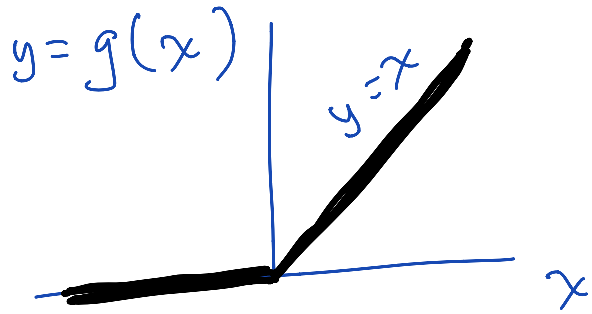
Problem 60. (15 POINTS)
Let X be a voltage input to a rectifier, and let X be a continuous uniform random variable on the interval $[-1, 1]$. The rectifier output is a random variable Y, where

$$Y = g(X) = \begin{cases} 0 & X < 0 \\ X & X \geq 0 \end{cases}$$

Two solution approaches

Find and sketch the PDF of Y, $f_Y(y)$.

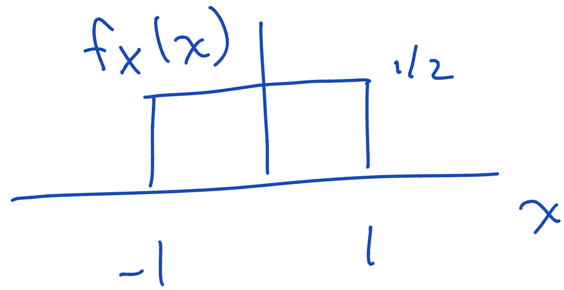
Approach 1:



Two steps:

① Find $F_Y(y)$

② differentiate to get $f_Y(y)$



In step ① there are 3 cases, $y<0$, $y=0$, and $y>0$

case 1: $y<0$

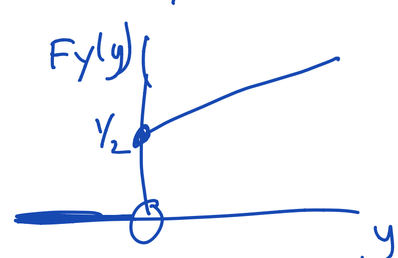
$$F_Y(y) = P(Y \leq y) = 0$$ since Y is never less than 0

case 2: $y=0$

$$F_Y(0) = P(Y \leq 0) = P(X \leq 0) = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-1}^{0} \tfrac{1}{2}\,dx = 1/2$$

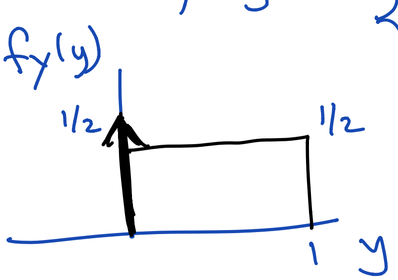
case 3: $y>0$

$$F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y)$$



Differentiate:

$$f_Y(y) = \begin{cases} 0 & y<0 \\ \tfrac{1}{2}\delta(y) & y=0 \\ f_X(y) & y>0 \end{cases} = \begin{cases} 0 & y<0 \text{ or } y>1 \\ \tfrac{1}{2}\delta(y) & y=0 \\ 1/2 & 0<y<1 \end{cases}$$



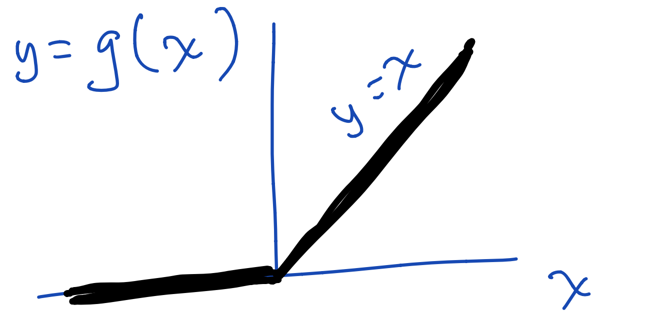
Problem 60. (15 POINTS)
Let X be a voltage input to a rectifier, and let X be a continuous uniform random variable on the interval $[-1, 1]$. The rectifier output is a random variable Y, where

$$Y = g(X) = \begin{cases} 0 & X < 0 \\ X & X \geq 0 \end{cases}$$

Two solution approaches

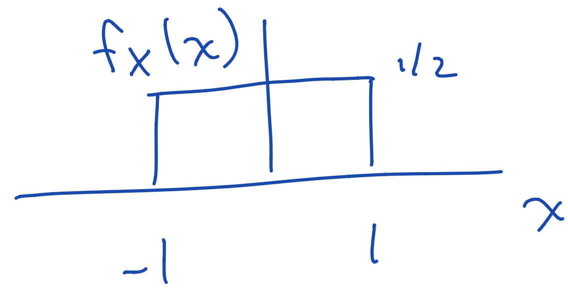
Find and sketch the PDF of Y, $f_Y(y)$.

Approach 2:



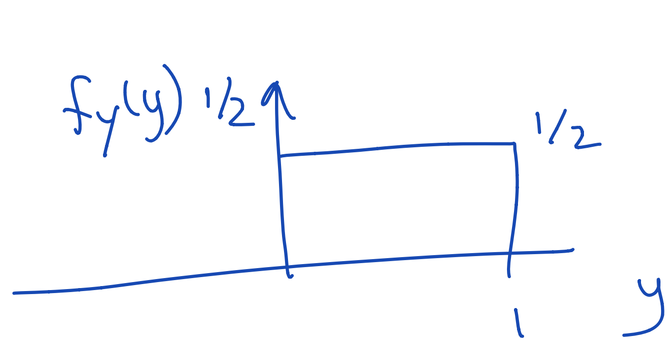
Every value of $X \leq 0$ gets mapped into the same value, $Y = 0$.
So $Y = 0$ happens half the time.

$\Rightarrow \frac{1}{2}\delta(y)$ at zero



The positive values of X are unchanged, so there's no change to the density there.

so



Problem 61. (20 POINTS)
Suppose self-driving cars have become a reality, in which you, personally, have no control over the speed. You program your car with a destination, and the car drives at a constant speed throughout the trip. This speed is chosen by the car before the trip starts, and the speed is held constant throughout the trip.

Suppose you "drive" (in your self-driving car) from West Lafayette, IN to Louisville KY at a constant speed that is uniformly distributed between 30 and 60 miles per hour (mph). The trip is exactly 180 miles. What is the PDF of the **duration** of the trip?

Another description of the problem that is completely equivalent:
Suppose X and Y are random variables, where X is the speed and $Y = 180/X$ is the duration. If X is uniformly distributed between $[30, 60]$, what is the PDF of Y?

(Hint: first decide: is X a continuous or discrete RV?)

Recall
$$f_X(x) = \begin{cases} 1/30 & 30 \le x \le 60 \\ 0 & \text{else} \end{cases}$$

Use 2-step process
1. Find $F_Y(y)$
2. differentiate wrt y.

$$F_Y(y) = P(Y \le y) = P\left(\frac{180}{X} \le y\right) = P\left(\frac{180}{y} \le X\right)$$
$$= 1 - P\left(X < \frac{180}{y}\right) = 1 - F_X\left(\frac{180}{y}\right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = -f_X\left(\frac{180}{y}\right) \frac{d}{dy}\left(\frac{180}{y}\right)$$
$$= -f_X\left(\frac{180}{y}\right)\left(-\frac{180}{y^2}\right) = \frac{180}{y^2} f_X\left(\frac{180}{y}\right) \quad \text{(using the chain rule)}$$

Range for Y: when $X = 30$, $Y = 6$; when $X = 60$, $Y = 3$

Now substitute $f_X(x)$ to get

$$f_Y(y) = \begin{cases} \frac{180}{y^2} \frac{1}{30} & 3 \le y \le 6 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{6}{y^2} & 3 \le y \le 6 \\ 0 & \text{else} \end{cases}$$

Problem 62.
Let $Y = 2X + 3$. Find the PDF of Y if X is a uniform RV on $[-1, 2]$.

when $Y = aX + b$, $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

here, $a = 2$, $b = 3$

$$f_Y(y) = \frac{1}{2} \underbrace{f_X\left(\frac{y-3}{2}\right)}$$

this is 0 when $x = \frac{y-3}{2} < -1$

or $x = \frac{y-3}{2} > 2$

so it's zero for $y < 1$

or for $y > 7$

The problem statement tells us $f_X(x) = \begin{cases} \frac{1}{3} & -1 \le x \le 2 \\ 0 & \text{else} \end{cases}$

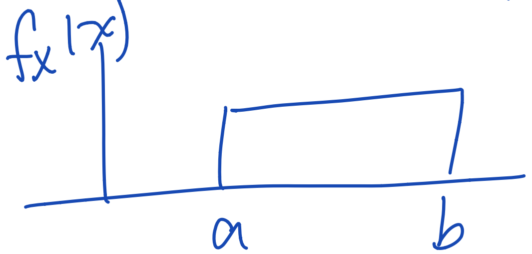
so $f_Y(y) = \begin{cases} \frac{1}{6} & 1 \le y \le 7 \\ 0 & \text{else} \end{cases}$

and yes, $\int_{-\infty}^{\infty} f_Y(y)\, dy = 1$

Problem 63.
If X is a positive random variable with density $f_X(x)$, find the density of $+\sqrt{X}$. Apply this to find the pdf of the length of a side of a square when the area of the square is uniformly distributed in $[a, b]$.

let $Y = g(X) = +\sqrt{X}$

if X is the area of a square,
then Y is the length of its side



$$f_X(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{else} \end{cases}$$

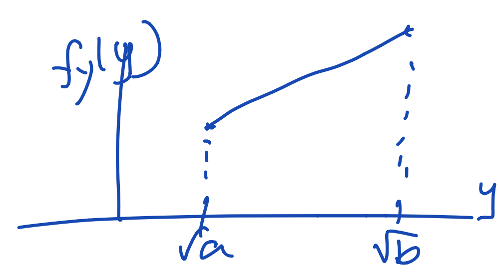
we know $f_Y(y) = 0$ when $y < 0$.

when $y \geq 0$, $F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y)$
$$= P(X \le y^2) = F_X(y^2)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(y^2) \frac{d}{dy}(y^2)$$
$$= f_X(y^2)(2y)$$ (using the chain rule)

Substituting $f_X(x)$ from the problem statement,

$$f_Y(y) = \begin{cases} \frac{2y}{b-a} & \text{when } \sqrt{a} \le y \le \sqrt{b} \\ 0 & \text{else} \end{cases}$$



$Y = X^{1/3}$

Problem 64. (Multiple choice: 5 points)

Let $Y = X^{1/3}$, and let X be a continuous random variable that is uniformly distributed on the interval $[-1, 8]$. What is the PDF of Y?

(Note: if you show your work you may get partial credit.)

(a)
$$f_Y(y) = \begin{cases} 3y^2/513 & \text{for } -1 < y < 8 \\ 0 & \text{otherwise} \end{cases}$$
(wrong range)

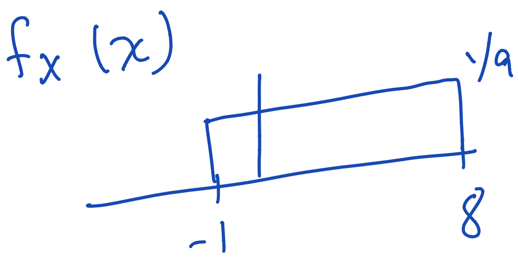
(b)
$$f_Y(y) = \begin{cases} y^2/3 & \text{for } -1 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(c)
$$f_Y(y) = \begin{cases} 1/3 & \text{for } -1 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$
(since $Y = g(X)$ is nonlinear, the uniform pdf is altered)

(d)
$$f_Y(y) = \begin{cases} 1/9 & \text{for } -1 < y < 8 \\ 0 & \text{otherwise} \end{cases}$$
(wrong range)

(e)
$$f_Y(y) = \begin{cases} 3y^2 & \text{for } -1 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$
(not a pdf)

(f) None of the above.



2-step process

$F_Y(y) = P(Y \le y)$
$= P(X^{1/3} \le y)$
$= P(X \le y^3) = F_X(y^3)$

differentiate: $f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(y^3)(3y^2)$
(by chain rule)

$= \frac{1}{9}(3y^2)$ ← in the proper range only
$= y^2/3$

range:
when $X=-1$, $Y=-1$ and when $X=8$, $Y=2$
so $\boxed{f_Y(y) = y^2/3 \text{ when } -1 \le y \le 2}$ ⟹ answer b

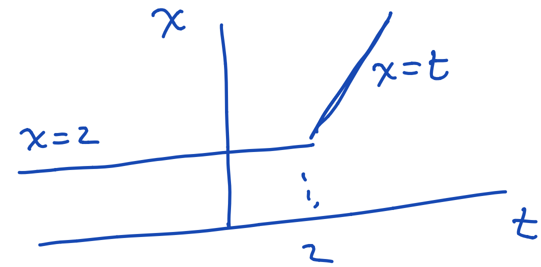
Problem 65. (5 POINTS)
A device is deployed in a remote region. The time, T, to failure, is exponentially distributed with mean 3 years. The device will not be monitored during the first 2 years, so the time before failure can be discovered is $X = max(T, 2)$. What is $E(X)$?

(Hint: Draw a sketch of the PDF of X. If you can break it into 2 parts and apply principles we learned in class, only one integration is necessary. If not, you may find the following integral (without limits) helpful.)

$$\int xe^{ax}\,dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$$

X = time failure detected
T = time failure happens

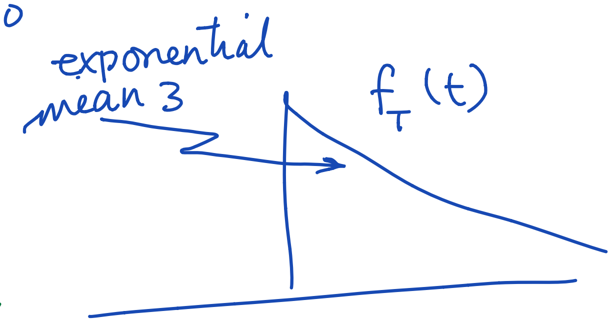
$g(t) = \max(t, 2)$



Because T is exponential mean 3,

$f_T(t) = \frac{1}{3}\exp(-t/3)$ for $t \geq 0$

$F_T(t) = 1 - \exp(-t/3)$ for $t \geq 0$



There are 2 regions of interest: $T < 2$ and $T \geq 2$.

Apply theorem of total expectation

$$E(X) = E(X \mid T < 2)\,P(T<2) + E(X \mid T \geq 2)\,P(T \geq 2)$$

Examine all 4 components:

$E(X \mid T < 2) = 2$, since if $T<2$, $X = 2$ always

$E(X \mid T \geq 2)$ can be found by applying the memoryless property of exponential RVs

$= 2 + E(T) = 2 + 3 = 5$

$P(T < 2) = F_T(2) = 1 - e^{-2/3}$

$P(T \geq 2) = 1 - P(T<2)$

Combining, $E(X) = 2\left(1 - e^{-2/3}\right) + 5e^{-2/3}$

$= \boxed{2 + 3e^{-2/3}}$

mistakes to avoid:

if $Y=X^2$ $F_Y(y)=F_X(x)^2$

if $Y=X^2$ $f_Y(y)=f_X(x)^2$

Both are wrong

Problem 66. (5 POINTS)
Given the CDF of X,

$$F_X(x) = \begin{cases} 1-(2/x)^2 & \text{for } x > 2 \\ 0 & \text{otherwise} \end{cases}$$

what is the PDF of $Y = X^2$, $f_Y(y)$.

There are at least 2 approaches, with a common start.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X < \sqrt{y})$$

(only need the positive part because X always positive)

$$= F_X(\sqrt{y})$$

Approach 1: substitute $F_X(\sqrt{y})$ and then differentiate

$$F_Y(y) = \begin{cases} 1-\left(\frac{2}{\sqrt{y}}\right)^2 & y>2 \\ 0 & \text{else} \end{cases} = \begin{cases} 1-\frac{4}{y} & y>4 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(-4y^{-1}\right) = \boxed{4y^{-2} \quad \text{when } y>4}$$

Approach 2: differentiate $F_Y(y)$ using chain rule and $F_X(x)$ to get $f_X(x)$ and substitute.

$$f_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y})\right) = f_X(\sqrt{y})\frac{d}{dy}(\sqrt{y}) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}}$$

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\left(-\frac{4}{x^2}\right) = 8x^{-3} \quad \text{when } x>2$$

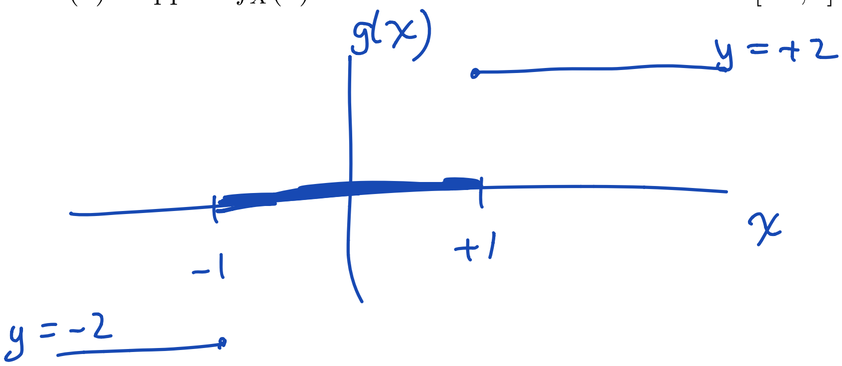
combine $f_Y(y) = f_X(\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right) = \frac{8}{2\sqrt{y}(\sqrt{y})^3} = \boxed{\frac{4}{y^2} \quad \text{when } y>4}$

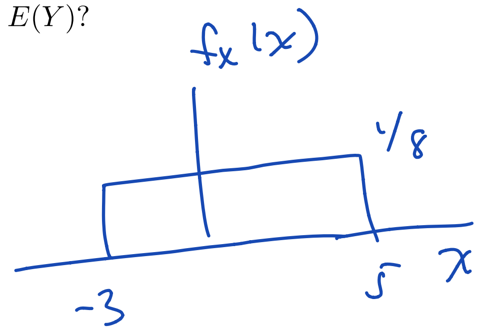
Problem 67. (15 POINTS)
Suppose a continuous random variable X is input to a quantizer to create Y, where

$$Y = g(X) = \begin{cases} -2 & \text{for } x < -1 \\ 0 & \text{for } -1 \le x < 1 \\ +2 & \text{for } x \ge 1 \end{cases}$$

(a) Express the PDF of the random variable Y in terms of the PDF $f_X(x)$ of the input X, using an expression that will be true for any $f_X(x)$. You may find it useful to sketch $g(X)$.

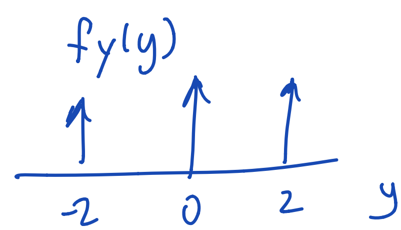
(b) Suppose $f_X(x)$ is a uniform distribution between $[-3, 5]$. What is $E(Y)$?





a) For any pdf, the event $\{Y = -2\}$ corresponds to the event $\{X \le -1\}$; event $\{Y = 0\}$ to $\{-1 < X \le 1\}$; event $\{Y = 2\}$ to $\{X > 1\}$

$$f_Y(y) = F_X(-1)\,\delta(y+2) + [F_X(1) - F_X(-1)]\,\delta(y) + (1 - F_X(1))\,\delta(y-2)$$



b) for the specific pdf,

$F_X(-1) = \frac{1}{8}(-3 - -1) = \frac{1}{4}$

$[F_X(1) - F_X(-1)] = \frac{1}{8}(1 - -1) = 1/4$

$1 - F_X(1) = \frac{1}{8}(4) = 1/2$

so $E(Y) = (-2)\left(\frac{1}{4}\right) + 0\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right) = \boxed{\frac{1}{2}}$

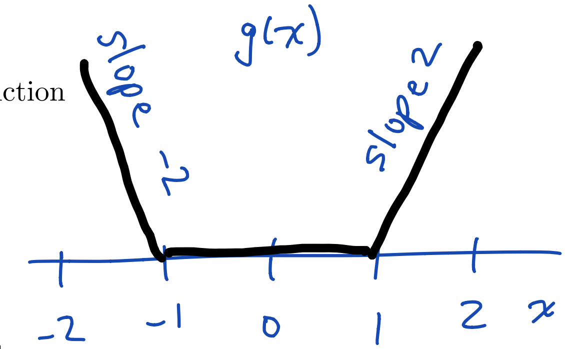
Problem 68. (15 POINTS)
Let X be a continuous random variable with probability density function

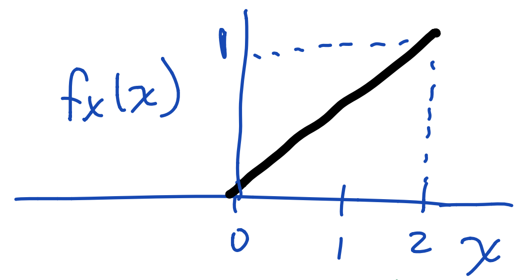
$$f_X(x) = \begin{cases} x/2 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

X is input to a system with output $Y = g(X)$, where

$$g(X) = \begin{cases} -2(X+1) & \text{for } X < -1 \\ 0 & \text{for } -1 \le X < 1 \\ +2(X-1) & \text{for } X \ge 1 \end{cases}$$

Find the PDF of the output random variable Y.





Two step process:
1) Find $F_Y(y)$
2) differentiate to get $f_Y(y)$

3 cases: $y<0$, $y=0$, $y>0$ (by examining $g(x)$)

$y<0$: $F_Y(y)=0$ because Y can never be less than 0

$y=0$: $F_Y(0) = P(Y \le 0) = P(0 \le X \le 1) = \int_0^1 f_X(x)\,dx$

$= \int_0^1 x/2\,dx = 1/4 \leftarrow$ a jump in the CDF @ $y=0$

$y>0$: This particular input X is never less than zero, so this simplifies things a bit

$F_Y(y) = P(Y \le y) = P(2(X-1) \le y)$ (just use the 3rd piece of $g(x)$)

$= P(X \le \frac{y}{2}+1) = F_X(\frac{y}{2}+1)$

differentiate:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} 0 & y<0 \\ 1/4\,\delta(y) & y=0 \\ \frac{d}{dy}F_X\left(\frac{y}{2}+1\right) & y>0 \end{cases} = \begin{cases} 0 & y<0 \\ 1/4\,\delta(y) & y=0 \\ f_X\left(\frac{y+2}{2}\right)\frac{1}{2} & y>0 \end{cases}$$

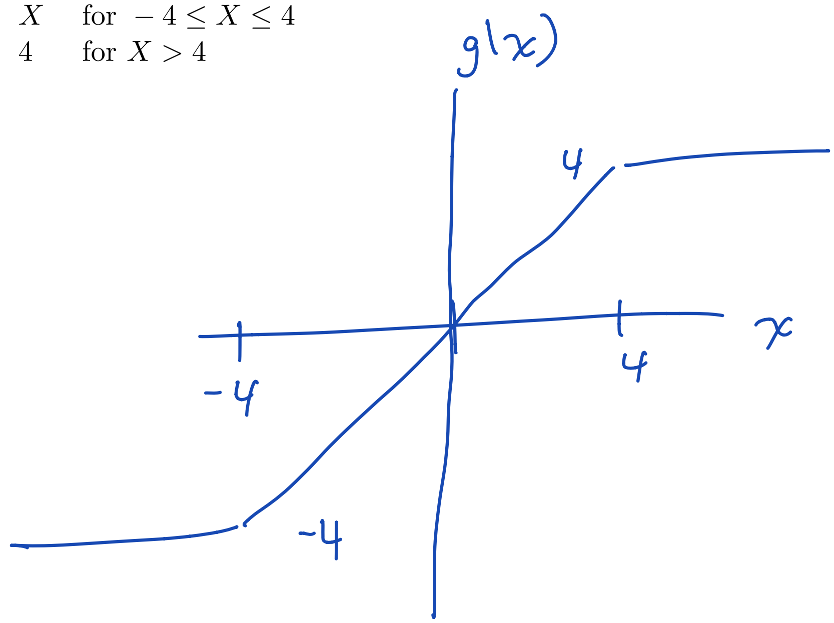
$$= \begin{cases} 0 & y<0 \\ 1/4\,\delta(y) & y=0 \\ (y+2)/8 & 0<y<2 \end{cases}$$

after substituting for f_X ...

Problem 69. (15 POINTS)
Let X be the voltage output from a microphone which is uniform on the interval from $[-5, 5]$. Then X is input to a limiter circuit, with cut-off ± 4. Thus the output of the limiter Y is given by

$$Y = g(X) = \begin{cases} -4 & \text{for } X < -4 \\ X & \text{for } -4 \leq X \leq 4 \\ 4 & \text{for } X > 4 \end{cases}$$

Find and sketch the PDF of Y, $f_Y(y)$.

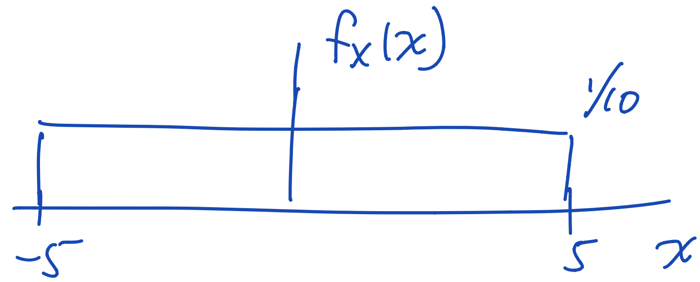


3 regions of interest:

$y \leq -4$

$4 < y < 4$

$y \geq 4$



Y can never be less than -4 or greater than $+4$

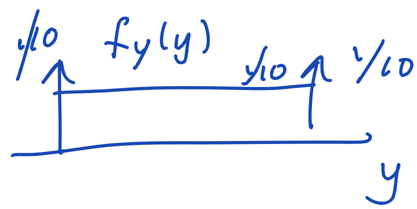
so $f_Y(y) = 0$ for $y < -4$ and $y > 4$.

for $y = -4$ exactly, $P(Y = -4) = P(-5 < X \leq -4) = \frac{1}{10}$

for $y = +4$ exactly, $P(Y = 4) = P(4 \leq X < 5) = \frac{1}{10}$

for $-4 < y < 4$, $f_Y(y) = f_X(y)$ because g(x) has slope 1 in this region

So $f_Y(y) = \begin{cases} \frac{1}{10}\delta(y+4) & y = -4 \\ 1/10 & -4 \leq y < 4 \\ \frac{1}{10}\delta(y-4) & y = 4 \end{cases}$



NOTE: this is the solution when $Y = \max(X, 10)$, not $Y = \min(X, 10)$ as stated

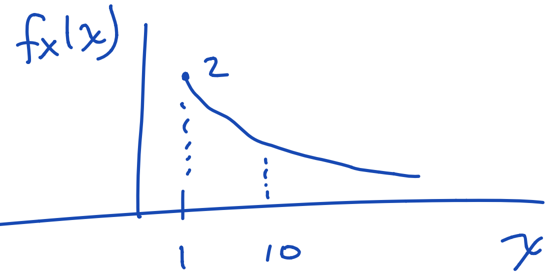
Problem 70. (5 POINTS)
Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 2/x^3 & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = g(X) = \max(X, 10)$. (This means that Y is either X or 10, whichever is less.) What is the expected value of Y?

$g(x) = \max(X, 10)$

break into 2 pieces:
$1 < x \leq 10$ and $x > 10$



$$E(Y) = E(Y \mid X \leq 10) P(X \leq 10) + E(Y \mid X > 10) P(X > 10)$$

4 quantities:

$$P(X \leq 10) = \int_1^{10} f_X(x)\,dx = 1 - P(X > 10)$$

$$P(X > 10) = \int_{10}^{\infty} f_X(x)\,dx = \int_{10}^{\infty} 2x^{-3}\,dx = \frac{2x^{-2}}{-2}\Big|_{10}^{\infty} = \frac{1}{100}$$

$$E(Y \mid X \leq 10) = 10$$

$$E(Y \mid X > 10) = \int_{10}^{\infty} \frac{x f_X(x)}{P(X > 10)}\,dx = 100 \int_{10}^{\infty} 2x^{-2}\,dx$$

$$= 100\left(\frac{2}{-1} x^{-1}\Big|_{10}^{\infty}\right) = \frac{200}{10} = 20$$

So $E(Y) = \frac{99}{100}(10) + \frac{1}{100}(20) = \boxed{\frac{101}{10}}$

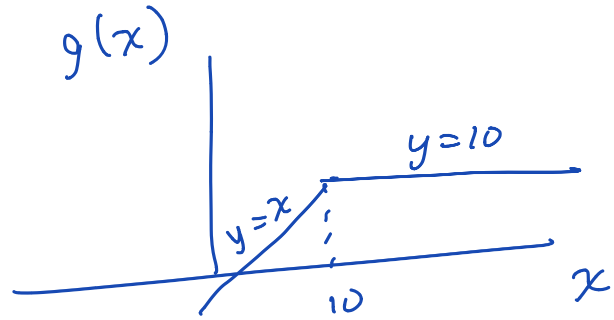
just a little bigger than 10 - makes sense

This is the correct solution

Problem 70. (5 POINTS)
Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 2/x^3 & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

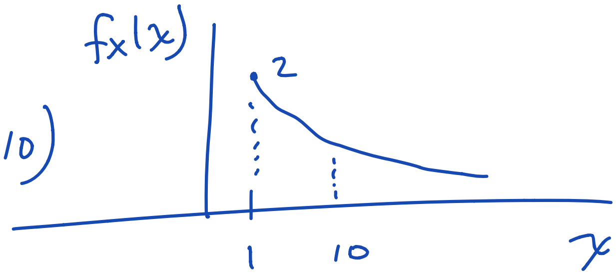
Let $Y = g(X) = \min(X, 10)$. (This means that Y is either X or 10, whichever is less.) What is the expected value of Y?



break into 2 pieces:

$1 < x \leq 10$ and $x > 10$

$$E(Y) = E(Y \mid X \leq 10) P(X \leq 10) + E(Y \mid X > 10) P(X > 10)$$



4 quantities:

$$P(X \leq 10) = \int_1^{10} f_X(x)\,dx = 1 - P(X > 10)$$

$$P(X > 10) = \int_{10}^{\infty} f_X(x)\,dx = \int_{10}^{\infty} 2x^{-3}\,dx = \frac{2x^{-2}}{-2}\Big|_{10}^{\infty} = \frac{1}{100}$$

$$E(Y \mid X \leq 10) = \int_1^{10} \frac{x f_X(x)}{P(X \leq 10)}\,dx = \frac{100}{99}\int_1^{10} \frac{2}{x^2}\,dx$$

$$= \frac{100}{99}\left(\frac{2}{-1} x^{-1}\Big|_1^{10}\right) = \frac{200}{99}\left(1 - \frac{1}{10}\right) = \frac{20}{11}$$

$$E(Y \mid X > 10) = 10$$

$$\text{So } E(Y) = \frac{99}{100}\left(\frac{20}{11}\right) + \frac{1}{100}(10) = \frac{9}{5} + \frac{1}{10} = \boxed{\frac{19}{10}}$$

much less than 10 - makes sense