Past Exam Questions
(Fall 2015, Spring 2016, Fall 2016, Fall 2017)
Chapters 3 and 4

Reibman

(January 2019)

SOLUTIONS

These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapters 3 and 4 of our textbook.

The last 3 pages of this document will be provided to you as the last pages of the exam. This will be all the formulas that will be available to you. The rest you must memorize.

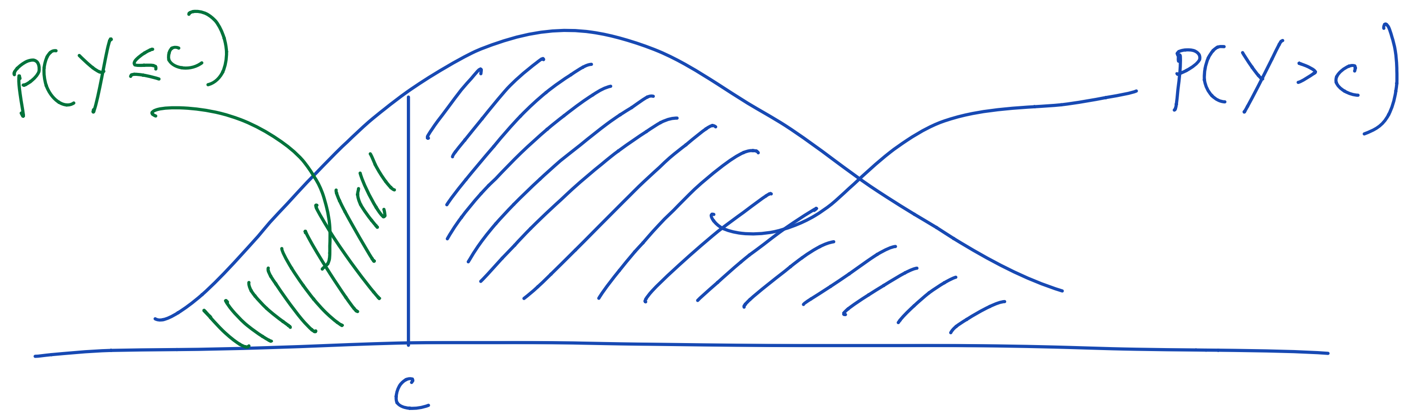
Solutions to #50-58 only

Problem 50. (15 POINTS)

Suppose Y is Gaussian with mean 1 and variance 9. Find the value c such that

$$P(Y > c) = 2P(Y \leq c)$$

You may want to use the facts that $\Phi(-0.96) = 1/6$, $\phi(-0.435) = 1/3$, and $\Phi(-0.675) = 0.25$, and $\Phi(x) = 1 - \Phi(-x)$. A sketch may also be helpful.



$P(Y > c) = 2P(Y \leq c)$ (given)

$P(Y > c) = 1 - P(Y \leq c)$ (by symmetry)

$\Rightarrow P(Y \leq c) = 1/3$

$= P\left(\frac{Y-\mu}{\sigma} \leq \frac{c-\mu}{\sigma}\right) = P\left(Z \leq \frac{c-1}{3}\right)$

where Z is a Gaussian RV w/ mean 0 and variance 1

So $P(Y \leq c) = \Phi\left(\frac{c-1}{3}\right) = \frac{1}{3}$

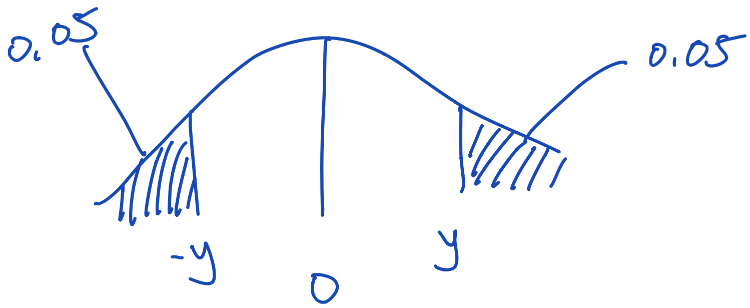
Using the given facts, $\frac{c-1}{3} = -0.435$

$\Rightarrow$ $\boxed{c = -0.305}$

Problem 51. (15 POINTS)

Suppose Y is Gaussian with mean 0 and variance 4. What value of y corresponds to $P(|Y| \geq y) = 0.1$.

Hint: it may be helpful to sketch the pdf and the area under the curve. Also, you may want to use the facts that $\phi(-0.84) = 0.2$, $\Phi(-1.28) = 0.1$, $\Phi(-1.64) = 0.05$, and $\Phi(-1.96) = 0.025$.



$Y \sim N(0,4)$

$$P(|Y| \geq y) = P(Y \leq -y) + P(Y \geq y)$$

$$= P(Y \leq -y) + 1 - P(Y \leq y)$$

or $= 2P(Y \leq -y)$ using symmetry

$P(|Y| \leq y) = 0.1$ from the problem statement

so $P(Y \leq -y) = 0.05$

$Z = \frac{Y-0}{2}$ is a Gaussian RV with mean 0, variance 1.

$$P(Y \leq -y) = P\left(\frac{Y}{2} \leq \frac{-y}{2}\right) = P\left(Z \leq \frac{-y}{2}\right)$$

$$= \Phi\left(-\frac{y}{2}\right) = 0.05$$

$\Rightarrow -\frac{y}{2} = -1.64$ so $y = 2(1.64) = \boxed{3.28}$

Problem 52. (Multiple choice: 5 points)
The lifetime of a machine, X, is a Gaussian random variable with mean 10 and variance 4. What is the value of x for which the machine has an 12% chance of surviving x or more years?
(You may use (and detach) the Φ-function table on the last page of the exam. If you draw a clear picture you may get partial credit.)

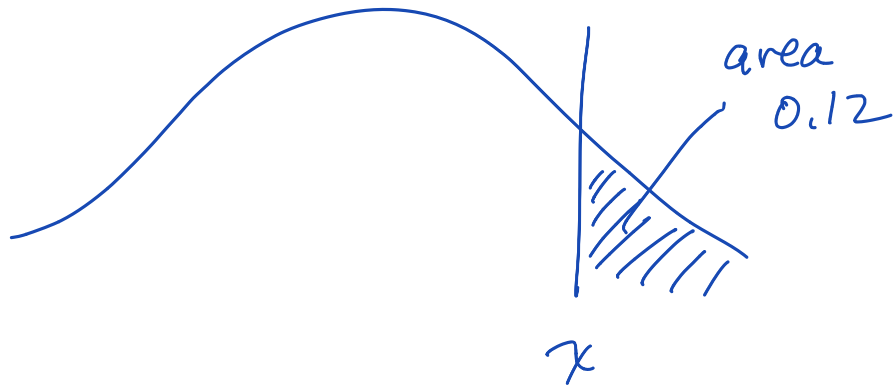
(a) 5.30

(b) 7.65

(c) 8.41

(d) 12.35

(e) 14.70



$Z = \frac{X-10}{\sqrt{4}}$ is Gaussian RV with mean 0 variance 1

$$1 - \Phi\left(\frac{x-10}{2}\right) = 0.12 \Rightarrow \Phi\left(\frac{x-10}{2}\right) = 0.88$$

$$\frac{x-10}{2} = 1.175$$

$$x - 10 = 2.35$$

$$\boxed{x = 12.35}$$

Problem 53. (15 POINTS)

You have two devices, each of whose lifetimes are modeled by a Gaussian distribution. Let D_1 be the lifetime of Device 1, which is Gaussian with mean 60 and variance 9. Let D_2 be the lifetime of Device 2, which is Gaussian with mean 56 and variance 36.

(a) If you are to choose one device for your system that needs to operate for 62 hours, which of these two devices should you choose?

(b) If your system needs to operate for 65 hours, which of these two devices should you choose?

Note: Full credit will only be given for a **complete** answer that includes the correct reason and supporting evidence. You may use (and detach) the Φ-function table on the last page of the exam.

a) P(Device 1 still working after 62 hours)

$$= P(D_1 > 62) = P\left(\frac{D_1 - 60}{\sqrt{9}} > \frac{62-60}{\sqrt{9}}\right)$$

$$= P(Z > 2/3) = 1 - \Phi(2/3)$$

P(Device 2 still working after 62 hours)

$$= P(D_2 > 62) = P\left(\frac{D_2 - 56}{\sqrt{36}} > \frac{62-56}{6}\right)$$

$$= P(Z > 1) = 1 - \Phi(1)$$

In each case, Z is a Gaussian RV w/ mean 0 variance 1.

Clearly, $P(Z > 1) < P(Z > 2/3)$, so **choose Device 1.**

b) Similarly, $P(D_1 > 65) = P\left(Z > \frac{65-60}{3}\right) = P\left(Z > \frac{5}{3}\right)$

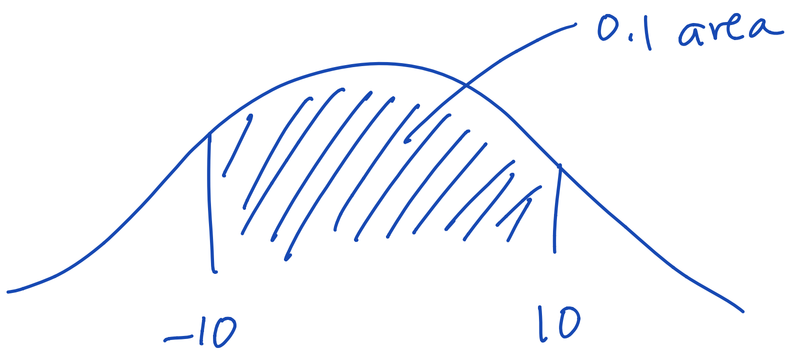
and $P(D_2 > 65) = P\left(Z > \frac{65-56}{6}\right) = P(Z > 9/6)$

$\frac{9}{6} < \frac{5}{3}$ so $P(Z > 9/6) > P(Z > 5/3)$

so **choose Device 2**

Problem 54.
Let X be a Gaussian variable with $E(X) = 0$ and $P(|X| \leq 10) = 0.1$. What is the standard deviation of X? You may want to use the facts that $\Phi(-1.28) = 0.1$, $\Phi(-1.64) = 0.05$, and $\Phi(-1.96) = 0.025$.

Know $\mu = 0$. Need to find σ from available info.



$$0.1 = P(|X| \leq 10)$$

Let $Z \sim N(0,1)$

$$P(|X| \leq 10) = P(-10 \leq X \leq 10) = P\left(\frac{-10}{\sigma} \leq Z \leq \frac{10}{\sigma}\right)$$

$$= P\left(Z \leq \frac{10}{\sigma}\right) - P\left(Z \leq \frac{-10}{\sigma}\right)$$

$$= \Phi\left(\frac{10}{\sigma}\right) - \Phi\left(\frac{-10}{\sigma}\right) = 1 - \Phi\left(-\frac{10}{\sigma}\right) - \Phi\left(-\frac{10}{\sigma}\right)$$

$$= 0.1 \quad \Rightarrow \quad \Phi\left(-\frac{10}{\sigma}\right) = 0.45$$

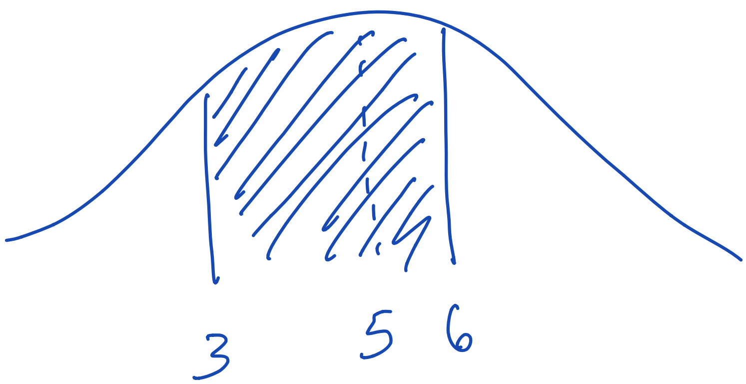
It turns out the provided values are not useful so we need to use the Φ-function table.

$$-\frac{10}{\sigma} = -0.125 \quad \text{so} \quad \sigma \approx \frac{10}{1/8} \approx \boxed{80}$$

Problem 55. (5 POINTS)

Let X be a Gaussian random variable with mean 5 and variance 4. Use the $\Phi(x)$ function to write an expression for the probability that X takes on a value between 3 and 6?

(It may be helpful to draw a **clear** picture of the PDF and how you compute the probability from it.)
You can leave your answer in terms of the Φ function (because no table is provided!).



$$P(3 \leq X \leq 6)$$

$$= P\left(\frac{3-5}{\sqrt{4}} \leq Z \leq \frac{6-5}{2} \right)$$

where $Z \sim N(0,1)$

so $P(3 \leq X \leq 6) = \Phi\left(\frac{6-5}{2}\right) - \Phi\left(\frac{3-5}{2}\right)$

$$= \boxed{\Phi\left(\tfrac{1}{2}\right) - \Phi(-1)}$$

Problem 56.
The peak temperature T in degrees Farenheit, on a July day in Antarctica is a Gaussian RV with variance 225. With probabilty 1/2, the peak temperature T exceeds 10 degrees. What is $P(T > 32)$, the probability the peak temperature is above freezing?

T is Gaussian w/ mean μ and variance 225

μ is not explicitly given, but $P(T > 10) = 1/2$

which means the median is 10 degrees,

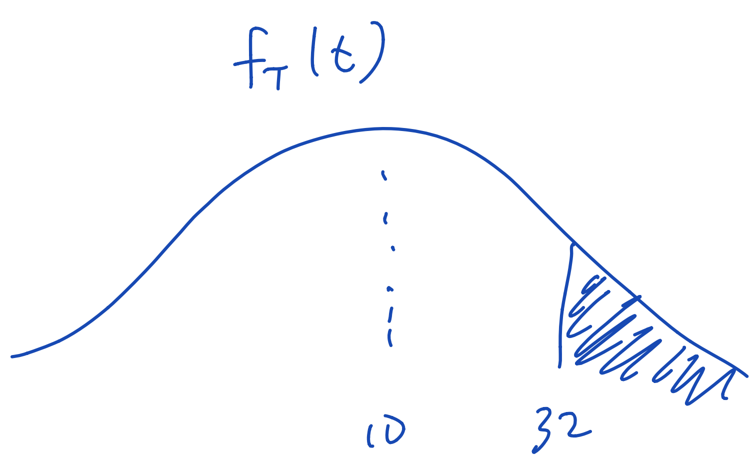
so $\mu = 10$

$$P(T > 32) = P\left(\frac{T-10}{15} > \frac{32-10}{15}\right)$$

$$= P\left(Z > \frac{22}{15}\right) \quad \text{where } Z \sim N(0,1)$$

$$= P(Z > 1.47)$$

$$= 0.071$$



Problem 57. (5 POINTS)

Let X be a Gaussian random variable with mean 52 and variance 4. What is the probability that $X > 57$ given that $X > 54$?

(You may leave the (approximate) answer in terms of a ratio of integers.)

$$X \sim N(52, 4)$$

$$Z \sim N(0,1) \text{ if } Z = \frac{X-52}{\sqrt{4}} = \frac{X-52}{2}$$

$$P(X > 57 \mid X > 54) = \frac{P(X > 57 \cap X > 54)}{P(X > 54)}$$

$$= \frac{P(X > 57)}{P(X > 54)} = \frac{1 - \Phi\left(\frac{57-52}{2}\right)}{1 - \Phi\left(\frac{54-52}{2}\right)}$$

$$= \frac{1 - \Phi(5/2)}{1 - \Phi(1)} = \frac{\Phi(-5/2)}{\Phi(-1)}$$

$$= \frac{0.0062}{0.1587} = \boxed{\frac{62}{1587}}$$

Problem 58. (15 POINTS)

Two manufacturers label resistors as 100-ohm resistors, but in reality the actual resistence is a random variable, X. For one company, X is Gaussian with mean 100 and variance 9. For the second company, X is Gaussian with mean 100 and variance 4. Equal numbers of resistors from each company are purchased and then mixed together.

(a) If you pick a resistor at random, what is the probability its resistance is between 97 and 100?

(b) If you pick a resistor with resistance $X = 97$, what is the probability it was made by Company 2?

Note: You may use (and detach) the Φ-function table on the last page of the exam.

a) we will use theorem of total probability to solve this.

$$P(97 < X < 100) = P(97 < X < 100 \mid C_1)\, P(C_1) + P(97 < X < 100 \mid C_2)\, P(C_2)$$

solve each piece separately

$$P(97 < X < 100 \mid C_1) = P\left(\frac{97-100}{\sqrt{9}} < Z < \frac{100-100}{3}\right)$$

$$= P(-1 < Z < 0) \quad \text{where } Z \sim N(0,1)$$

$$= \Phi(0) - \Phi(-1)$$

$$P(97 < X < 100 \mid C_2) = P\left(\frac{97-100}{\sqrt{4}} < Z < \frac{100-100}{2}\right)$$

$$= P(-3/2 < Z < 0) = \Phi(0) - \Phi(-3/2)$$

Combining, $P(97 < X < 100) = \Phi(0) - \frac{1}{2}\Phi(-1) - \frac{1}{2}\Phi(-3/2)$

$= 0.38725$

b) use Bayes Rule

$$P(C_2 \mid X = 97) = \frac{f_X(97 \mid C_2)\, P(C_2)}{f_X(97)} \quad \ldots$$

Table 1: Table of the Standard Normal Cumulative Distribution Function $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990