**ECE 302: Probabilistic Methods in Electrical and Computer Engineering**

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PURDUE UNIVERSITY

# Past Exam Questions
# (Fall 2015, Spring 2016, Fall 2016, Fall 2017)
# Chapters 3 and 4


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SOLUTIONS

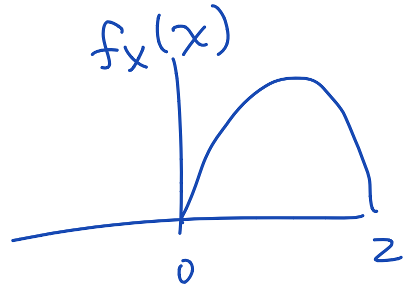
These form a collection of problems that have appeared in either Prof. Reibman's real exams or "sample exams." These can all be solved by applying the material we covered in class that appears in Chapters 3 and 4 of our textbook.

The last 3 pages of this document will be provided to you as the last pages of the exam. This will be all the formulas that will be available to you. The rest you must memorize.

Solutions to

# 42 - 49

**Problem 42.** (15 POINTS)




Let $X$ be a random variable with PDF

$$f_X(x) = c(2x - x^2), \text{for } 0 \le x \le 2$$

(a) Find $c$.

(b) What is $P(X < 1)$?

(c) Find $E(X|A)$ for the event $A = \{X < 1\}$.

a) Know $1 = \int_{-\infty}^{\infty} f_x(x)\,dx$

so $\int_0^2 c(2x - x^2)\,dx = c\left(\frac{2x^2}{2} - \frac{x^3}{3}\right)\Big|_0^2$

$= c\left(\frac{8}{2} - \frac{8}{3} - (0)\right) = \frac{8\cdot 3 - 8\cdot 2}{2\cdot 3} = \frac{8}{6}c = 1$

$\Rightarrow c = \frac{6}{8} = \boxed{3/4}$

b) $P(X<1) = \int_{-\infty}^{1} f_x(x)\,dx = \int_0^1 c(2x - x^2)\,dx$

$= c\left(\frac{2x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = c\left(\frac{2}{2} - \frac{1}{3}\right) = \frac{3}{4}\left(\frac{6-2}{6}\right) = \boxed{\frac{1}{2}}$

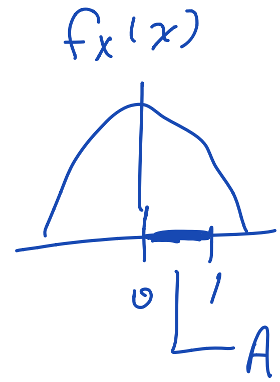
c) $f_X(x|A) = \begin{cases} \frac{f_x(x)}{P(A)} & \text{when } x \in A \\ 0 & \text{else} \end{cases} = \begin{cases} 3/2\,(2x - x^2) & \text{for } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

$E(X|A) = \int_0^1 x\, f_X(x|A)\,dx = \int_0^1 \frac{3}{2}(2x^2 - x^3)\,dx$

$= \frac{3}{2}\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{3}{2}\left(\frac{2}{3} - \frac{1}{4}\right) = \frac{3}{2}\left(\frac{8}{3\cdot 4} - \frac{3}{3\cdot 4}\right) = \frac{1}{2}\left(\frac{5}{4}\right) = \boxed{5/8}$

**Problem 43.** (15 POINTS)
Suppose $X$ is a continuous random variable with PDF

$$f_X(x) = \begin{cases} 3(4-x^2)/32 & \text{for } -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$




(a) What is the probability of event $A$, where $A = \{0 \le X \le 1\}$?

(b) Find the conditional PDF of $X$, conditioned on the event $A$.

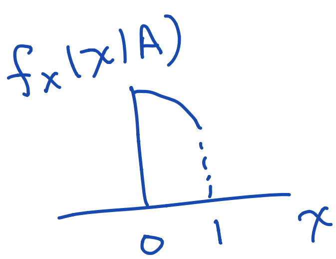
(c) What is $E(X|A)$?

a) $$P(0 \le X \le 1) = \int_0^1 f_X(x)\,dx$$

$$= \int_0^1 \frac{3}{32}(4-x^2)\,dx = \frac{3}{32}\left(4x - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{3}{32}\left(4 - \frac{1}{3} - 0\right) = \frac{3}{32}\left(\frac{11}{3}\right) = \boxed{\frac{11}{32}}$$

b) $$f_X(x|A) = \left\{\begin{array}{ll} \dfrac{f_X(x)}{P(A)} & \text{when } x \in A \\ 0 & \text{else} \end{array}\right\} = \begin{cases} \dfrac{3(4-x^2)}{11} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$




c) $$E(X|A) = \int x\, f_X(x|A)\,dx = \int_0^1 \frac{3}{11}(4x - x^3)\,dx$$

$$= \frac{12}{11}\,\frac{x^2}{2}\Big|_0^1 - \frac{3x^4}{44}\Big|_0^1 = \frac{12}{22} - \frac{3}{44} = \boxed{\frac{21}{44}}$$

**Problem 44.** (20 POINTS ((A) IS 4 POINTS; (B,C) ARE 8 POINTS EACH))
Consider the random variable $X$ with PDF given by

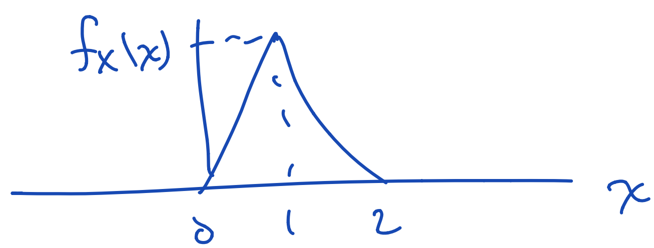
$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch $f_X(x)$. Label axes and relevant values.

(b) Find and sketch the conditional density $f_X(x|A)$ for the event $A = \{X < 1/4\}$.

(c) What is the conditional mean $E(X|B)$ for the event $B = \{2/3 < X < 4/3\}$?
(Hint: you do not need to find $P(B)$ to solve part (c)!)

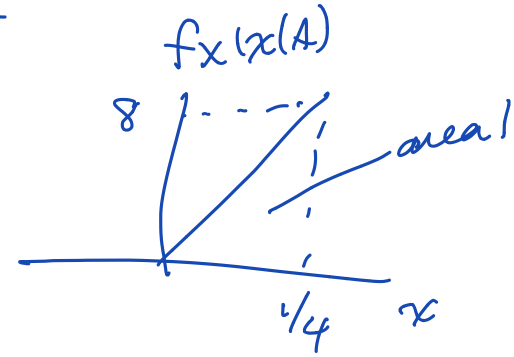
a)




b) because event A depends on the RV X, then we "chop and scale"

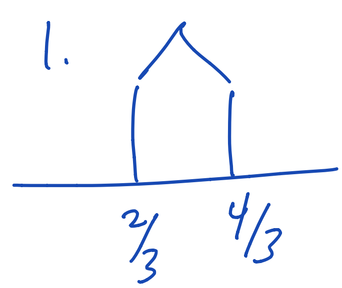
$$f_X(x|A) = \begin{cases} \dfrac{f_X(x)}{P(A)} & \text{when } x \in A \\ 0 & \text{else} \end{cases}$$

$$P(A) = \int_0^{1/4} x\,dx = \frac{x^2}{2}\Big|_0^{1/4} = \frac{1}{32}$$




$$f_X(x|A) = \begin{cases} 32x & x < 1/4 \\ 0 & \text{else} \end{cases}$$

c) $f_X(x|B)$ is symmetric about 1.




$\Rightarrow E(X|B) = 1$

**Problem 45.** (25 POINTS ((A) IS 7 POINTS; (B) IS 8 POINTS; (C) IS 10 POINTS))
Let $X$ be a discrete RV with sample space $S_X = \{1, 4\}$, each equally likely. Given that we know $X = x$, a second RV $Y$ is exponentially distributed with mean $1/x$.

(a) What is the conditional pdf of Y given X?

(b) What is the marginal pdf of Y?

(c) Find $E(Y)$.
(Hint: it will be faster to use the theorem of total expectations, but you may solve it any way you wish.)

X is discrete with 2 possible values: 1 and 4

a) from the back page, when an exponential RV Y has mean $1/x$, its pdf is $x \exp(-xy)$ for $y \geq 0$.

so $f_Y(y \mid X = x) = x \exp(-xy) \quad x > 0, \; y \geq 0$

b) $f_Y(y) = f_Y(y \mid X=1)\, P(X=1) + f_Y(y \mid X=4)\, P(X=4)$

by theorem of total probability

Problem statement says $P(X=1) = P(X=4) = 1/2$

so $f_Y(y) = \frac{1}{2}\left(\exp(-y) + 4\exp(-4y)\right] \quad y \geq 0$

c) Iterated expectations says $E(E(Y|X)) = E(Y)$

we know $E(Y|x) = 1/x$ from problem statement

$E(Y) = E(Y|X=1)\,P(X=1) + E(Y|X=4)\,P(X=4)$

$= \frac{1}{2}\left[1 + 1/4\right] = \boxed{5/8}$

**Problem 46.** (Multiple choice: 5 points)
Three engineers, Jan, Pat, and Rory, are processing work orders. The time it takes each to finish one work order is an exponential random variable. Jan takes an average of 3 hours; Pat takes an average of 1 hours, and Rory takes an average of 4 hours. Because of their speed, Pat processes 50% of the work orders, while Jan and Rory each process 25% of them.
What is the mean time (in hours) it takes any given work order to be completed?

(a) 2

(b) 9/4

(c) 8/3

(d) 8

(e) None of the above

(f) Too little information to solve.

$$E(T \mid Jan)\,P(Jan) + E(T \mid Pat)\,P(Pat) + E(T \mid Rory)\,P(Rory)$$

$$= (3)\left(\tfrac{1}{4}\right) + (1)\left(\tfrac{2}{4}\right) + (4)\left(\tfrac{1}{4}\right) = 9/4$$

**Problem 47.** (16 points)
A customer walks into a store and is equally likely to be served by one of three clerks. The time taken by the first clerk is an exponetial RV with mean 2; the time taken by the second clerk is an constant RV with mean 1; and the time taken by the third clerk is a uniform RV between zero and two.

(a) Express the PDF of $T$ the time to serve the customer.

(b) Find $E(T)$.

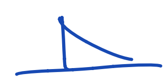
$C_i = \{\text{served by Clerk } i\}$

$P(C_i) = 1/3 \quad i = 1, 2, 3$

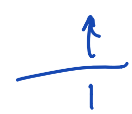
a) Theorem of Total Probability

$$f_T(t) = f_T(t \mid C_1)P(C_1) + f_T(t \mid C_2)P(C_2) + f_T(t \mid C_3)P(C_3)$$

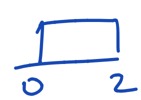
$$f_T(t \mid C_1) = 1/2 \exp(-t/2)\, u(t)$$



$$f_T(t \mid C_2) = \delta(t-1)$$




$$f_T(t \mid C_3) = 1/2\,[u(t) - u(t-2)]$$




so $f_T(t) = \frac{1}{3}\left[\frac{1}{2}e^{-t/2}u(t) + \delta(t-1) + \frac{1}{2}[u(t) - u(t-2)]\right]$

b) $E(T) = \sum_{i=1}^{3} E(T \mid C_i)P(C_i) = \frac{1}{3}[2+1+1] = 4/3$

**Problem 48.** (TRUE/FALSE: 5 POINTS EACH, TOTAL 20 POINTS)
Label each statement T or F to the left of the problem number.

F (a) $E(g(X)) = g(E(X))$ Not always. Example: $g(x) = X^2$

F (b) Let $X$ be a random variable and let $a$ be a constant. Then $P(X \geq a) = 1 - F_X(a)$.

T (c) A deck of 52 cards is fairly dealt to 2 hands, each with 26 cards. The probability that both hands get 2 aces is $\binom{4}{2}\binom{48}{24}/\binom{52}{26}$

T (d) Let $X$ be a random variable and let $Y = aX + b$, where $a, b$ are constants. Then VAR(Y)=$a^2$VAR(X).

b) $P(X > a) = 1 - F_X(a)$ There could be a jump at $x=a$

**Problem 49.** (15 POINTS)
The time it takes a computer program to execute is exponentially distributed with a mean of 5 minutes. Calculate the mean execution time, given that it is at least 4 minutes.

Know: $E(X) = 5$, $\lambda = 1/5$, $f_X(x) = \lambda e^{-\lambda x}$ $x > 0$

There are 3 steps to calculate $E(X \mid X \geq 4)$

① calculate $P(X \geq 4)$

② calculate $f_X(x \mid X \geq 4)$

③ calculate mean of $f_X(x \mid X \geq 4)$

① $P(X \geq 4) = \int_4^\infty f_X(x)\,dx = \int_4^\infty e^{-x/5}\,dx = -e^{-x/5}\Big|_4^\infty$

$= e^{-4/5}$

② $f_X(x \mid X \geq 4) = f_X(x) / P(X \geq 4)$ when $x > 4$ only

$$= \begin{cases} \dfrac{\exp(-x/5)}{5\exp(-4/5)} & \text{for } x \geq 4 \\ 0 & \text{else} \end{cases}$$

③ $E(X \mid X \geq 4) = \int_{-\infty}^{\infty} x f_X(x \mid X \geq 4)\,dx = \int_4^\infty \dfrac{x\exp(-x/5)}{5\exp(-4/5)}\,dx$

$= e^{4/5}\left((-x-5)e^{-x/5}\Big|_4^\infty\right) = e^{4/5}\left(9e^{-4/5}\right)$

$= \boxed{9}$

using integral table,

$\int x e^{ax}\,dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$

Here, $a = -1/5$

Note: X is a memoryless RV.

so $E(X \mid X \geq 4) = 4 + E(X)$

$= 4 + 5 = 9$. A quick and easy approach