ECE 302: Probabilistic Methods in Electrical and Computer Engineering

Instructor: Prof. A. R. Reibman

Previous Exam Questions, Chapter 2

Reibman
(compiled September 2018)

SOLUTIONS

These form a collection of 36 **or more** problems that have appeared in either Prof. Reibman's real exams or "sample exams." The material to solve these problems is in Chapter 2 of the course textbook. ~~These can all be solved by applying the material we covered in class up until September 13, 2018 (including the video about Counting).~~

Problem 1. (True/False: 5 points each, total 20 points; Exam 1 Fall 2015)
For each of the following relations, determine which is valid for arbitrary events A, B, and C. (Note: to be true "for arbitrary events", it must be true for any such event. Use a Venn diagram if it is helpful.) Label each statement T or F to the left of the problem number.

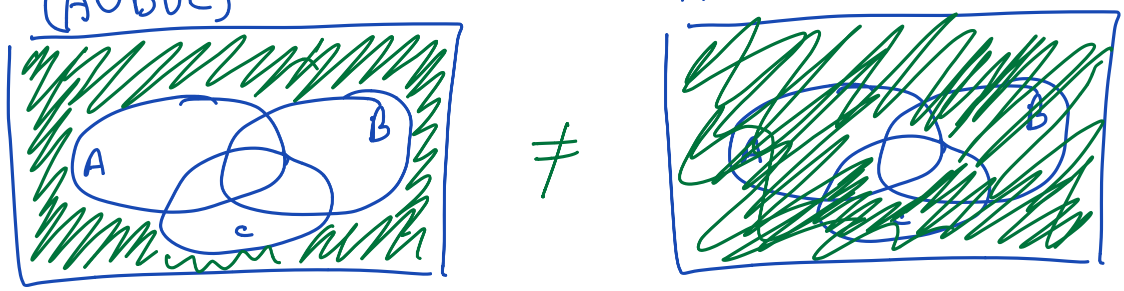
F (a) $(A \cup B \cup C)^c = A^c \cup B^c \cup C^c$

T (b) $(A \cup B) \cap (A^c \cup B^c) = (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B \cap C^c)$

T (c) $(A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) = (A^c \cap B^c)^c$

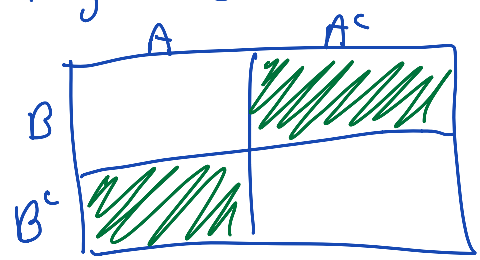
F (d) $(A - B) - C = A - (B - C)$.

a) $(A \cup B \cup C)^c \neq A^c \cup B^c \cup C^c$

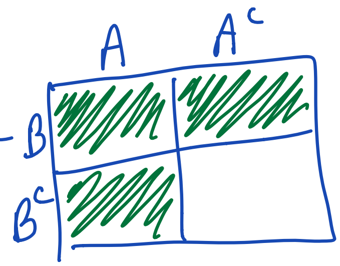


b) Note that $(A^c \cap B \cap C^c)$ is already contained inside $(A^c \cap B^c)$, so this final term on the right can be dropped.

Then both become



c) both are



d) counterexample : let $B = C$. then $(A-B)-C = A-B$

but $A-(B-C) = A - \phi = A$

Problem 2. (5 POINTS, MULTIPLE CHOICE)
Let $A = [0, 3]$, $B = [-1, 1]$, and $C = (2, 4)$ be sets of real numbers. Simplify $A \cap (B \cup C)$.
(If you show your work, you may receive partial credit.)

(a) $[0, 1] \cup [2, 3]$

(b) $[0, 3]$

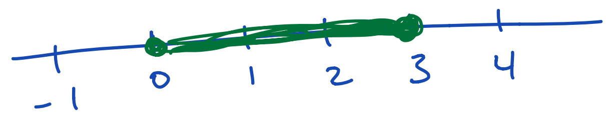
(c) $[0, 1] \cup (2, 3]$

(d) $[-1, 4)$

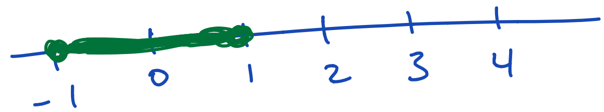
(e) $[-1, 0] \cup (3, 4)$

(f) None of the above

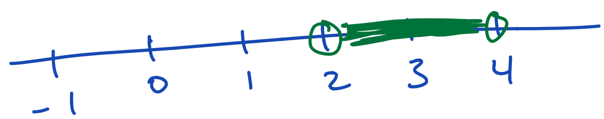
set A



set B



set C



$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$= [0, 1] \cup (2, 3]$$

Problem 3. (TRUE/FALSE: 5 POINTS EACH, TOTAL 30 POINTS)
(Note: If a statement is not true for all events, then it is FALSE. Use a Venn diagram if it is helpful.) Label each statement T or F to the left of the problem number.

F (a) If two events A and B are independent, then $P(A|B) = P(A)/P(B)$. $P(A|B) = P(A)$ if independent

T (b) If two events A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$

F (c) If two events A and B are disjoint, they must also be independent. only if $P(A)=0$ or $P(B)=0$

F (d) If two events A and B are collectively exhaustive then $P(A) + P(B) = 1$. Not if $A \cap B \neq \emptyset$

T (e) If two events A and B are collectively exhaustive then $P(A \cup B) = 1$.

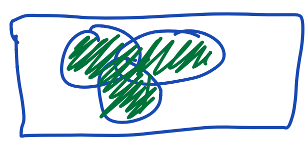
T (f) If two evens A and B are independent, then A^c and B^c are also independent.

(f) proof: Know $P(A \cap B) = P(A)P(B)$. want $P(A^c \cap B^c) = P(A^c)P(B^c)$.

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$$
$$= 1 - P(A) - P(B) - P(A)P(B) = [1-P(A)][1-P(B)] = P(A^c)P(B^c)$$

Problem 4. (MULTIPLE CHOICE: 5 POINTS; EXAM 1 FALL 2015)
Consider three events, A, B, and C, with sample space S. Which of the following are **always** correct ways to compute $P(A \cup B \cup C)$? Hint: Draw a Venn diagram to help you visualize this.

(a) $P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$



$A \cup B \cup C$

(b) $1 - P(A^c \cap B^c \cap C^c)$

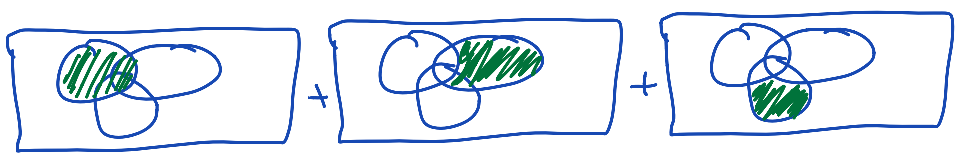
(c) $P(A) + P(B) + P(C)$

(d) Both (b) and (a).

(e) None of the above.

(f) All of the above.

a)



c) Note if $A \cap B \neq \emptyset$, or $A \cap C \neq \emptyset$ or $B \cap C \neq \emptyset$ then (c) is not true

Problem 5. (YES/NO: 4 POINTS)
For any events A and B, it is always true that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Problem 6. (YES/NO: 4 POINTS)
For any events A and B with $P(A) > 0$ and $P(B) > 0$, $P(A|B) + P(B|A) = 1$.

But it is true that $P(A|B) + P(A^c|B) = 1$

Problem 7. (True/False: 5 points each, total 15 points)
For each of the following statements, determine which is valid for events A, B, and C. If you show your reasoning you might get partial credit. Use a Venn diagram only if it is helpful. Finding a counter-example might be helpful if the answer is FALSE.

Clearly label each statement T or F to the left of the problem number.

T (a) $P(A^c \cup B) \leq 1 - P(A) + P(B)$ for any A and any B.

F (b) If A and B are independent, and B and C are independent, then A and C are also independent.

F (c) If A, B, and C are mutually exclusive and collectively exhaustive, then A^c, B^c, and C^c are also mutually exclusive and collectively exhaustive.

a) $$P(A^c \cup B) = P(A^c) + P(B) - P(A^c \cap B)$$
$$= 1 - P(A) + P(B) \underbrace{- P(A^c \cap B)}$$
$$\leq 1 - P(A) + P(B)$$

this term is always ≥ 0, making the sum smaller

b) A simple counter example: Let $A = C$.
A and C are NOT independent

Problem 8. (Always/Sometimes/Never (5 points each, 20 points total))
Events E, F, and G form a list of mutually exclusive and collectively exhaustive events with $P(E) \neq 0$, $P(F) \neq 0$, and $P(G) \neq 0$. Determine for each of the following statements, whether it must be true (always), it might be true (sometimes), or it cannot be true (never). Mark your answer clearly to the left of each question.

(a) E^c, F^c and G^c are mutually exclusive.

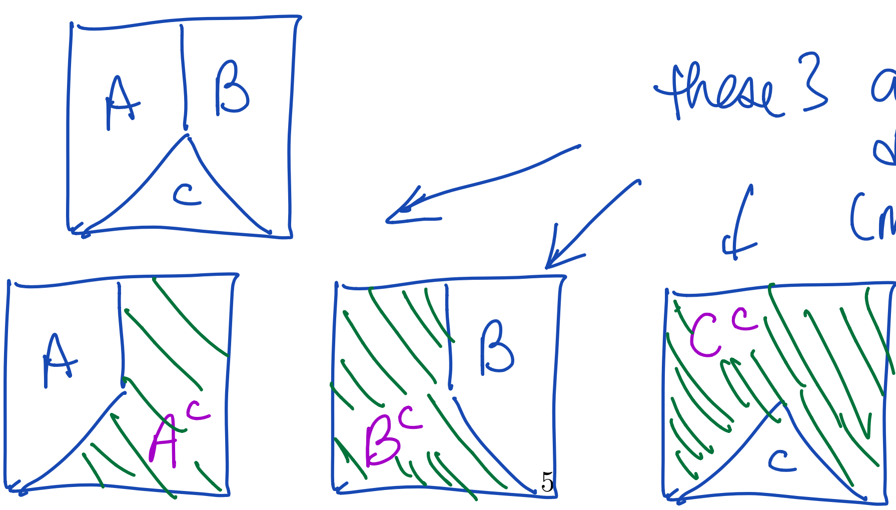
(b) E^c, F^c and G^c are collectively exhaustive.

(c) $P(E^c) + P(F^c) > 1$

(d) $P(E^c \cup (E \cap F^c) \cup (E \cap F \cap G^c)) = 1$

(see next page for solution to #8)

7c)



these 3 are NOT disjoint (mutually exclusive) although they are collectively exhaustive

Problem 8. (ALWAYS/SOMETIMES/NEVER (5 POINTS EACH, 20 POINTS TOTAL))
Events E, F, and G form a list of mutually exclusive and collectively exhaustive events with $P(E) \neq 0$, $P(F) \neq 0$, and $P(G) \neq 0$. Determine for each of the following statements, whether it must be true (always), it might be true (sometimes), or it cannot be true (never). Mark your answer clearly to the left of each question.

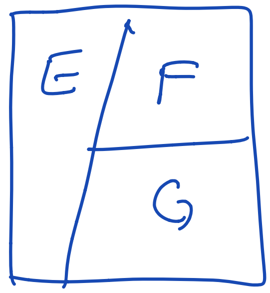
N (a) E^c, F^c and G^c are mutually exclusive.

A (b) E^c, F^c and G^c are collectively exhaustive.

A (c) $P(E^c) + P(F^c) > 1$

A (d) $P(E^c \cup (E \cap F^c) \cup (E \cap F \cap G^c)) = 1$

Example sketch:



Note:

$E^c = F \cup G$

$F^c = E \cup G$

$G^c = E \cup F$

a) $E^c \cap F^c = G \neq \phi$

therefore $\boxed{\text{NEVER}}$

b) $E^c \cup F^c = E \cup F \cup G = S$

$\Rightarrow \boxed{\text{ALWAYS}}$

c) $P(E^c) + P(F^c)$

$= P(F \cup G) + P(E \cup G)$

$= P(F) + P(G) + P(E) + P(G)$

$= 1 + P(G)$ and $P(G) > 0$ so

> 1 $\Rightarrow \boxed{\text{ALWAYS}}$

d) Start at the end and simplify: $E \cap F \cap G^c = \phi$ since $E \cap F = \phi$

$E \cap F^c = E \cap (E \cup G) = E$

so $P(E^c \cup E \cup \phi) = P(S) = 1$ $\Rightarrow \boxed{\text{ALWAYS}}$

Problem 9. (15 POINTS)
Let A and B be events and let $P(A) = 1/4$ and $P(B) = 1/3$.

(a) Compute $P(A \cup B)$ assuming $A \cap B = \phi$.

(b) Compute $P(A \cup B)$ assuming A and B are independent.

(c) Compute $P(A \cup B)$ assuming $P(B|A) = 2/3$.

In all cases,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$, but the last term differs.

a) $P(A \cap B) = 0 \Rightarrow P(A \cup B) = 1/4 + 1/3 = \boxed{\frac{7}{12}}$

b) $P(A \cap B) = P(A)P(B) = \frac{1}{12} \Rightarrow P(A \cup B) = 1/4 + 1/3 - 1/12 = \boxed{6/12}$

c) $P(A \cap B) = P(B|A)P(A) = \frac{2}{12} \Rightarrow P(A \cup B = 1/4 + 1/3 - \frac{2}{12} = \boxed{5/12}$

Problem 10. (MULTIPLE CHOICE: 5 POINTS)
Two fair dice are rolled. Let X be the absolute value of the difference between the numbers on each die. What is $P(X \geq 3)$?

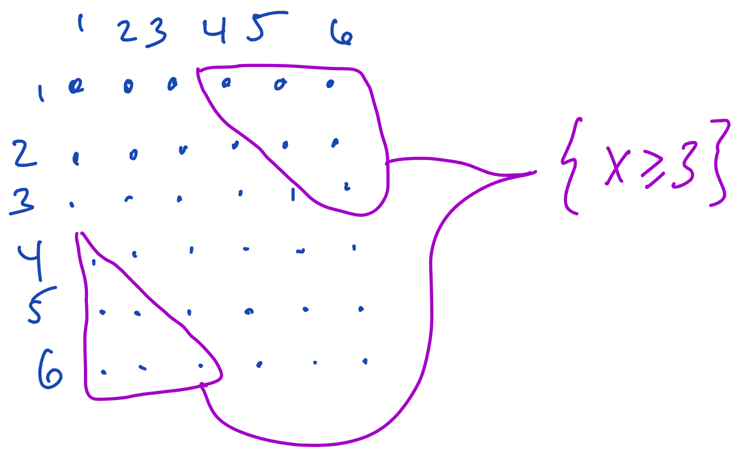
(a) 7/9

(b) 2/3

(c) 5/9

(d) 4/9

(e) 1/3



$$P(X \geq 3) = \frac{\text{size}(\{X \geq 3\})}{36}$$

$$= 12/36 = 1/3$$

Problem 11. (Multiple choice: 5 points)
You roll two fair dice, and record the number of dots facing upward on each die. What is the probability the sum of the two is even and at least one of the dice shows 5.

(a) 1/2

(b) 11/36

(c) 2/3

(d) 5/36

(e) Impossible to determine based on the information given.

(f) None of the above.

event $A = \{\text{even sum}\}$

$B = \{\text{at least one } 5\}$

$B = \{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5),$
$(5,1),(5,2),(5,3),(5,4),(5,6)\}$

$B \cap A = \{(1,5),(3,5),(5,5),(5,1),(5,3)\}$

S contains 36 outcomes, all equally likely.

$B \cap A$ contains 5 outcomes.

$\Rightarrow P(B \cap A) = 5/36$

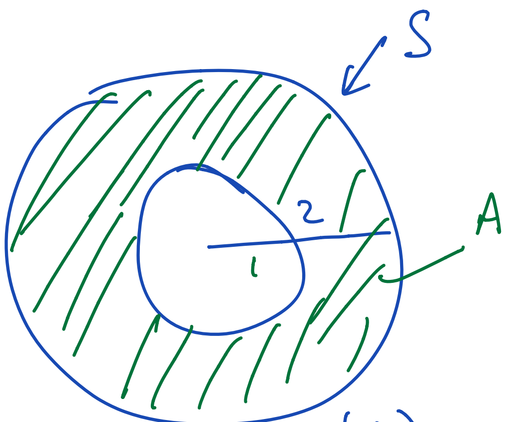
Problem 12. (5 POINTS EACH, 20 POINTS TOTAL)
Consider the x-y plane. Consider a disk (circle) of radius 2 on the plane, which are those points satisfying $x^2 + y^2 \leq 2$. We throw a dart at the disk and we know that the dart will land uniformly likely on the disk. Let X and Y denote the x and y coordinates of the landing location of the dart. Answer the following questions:

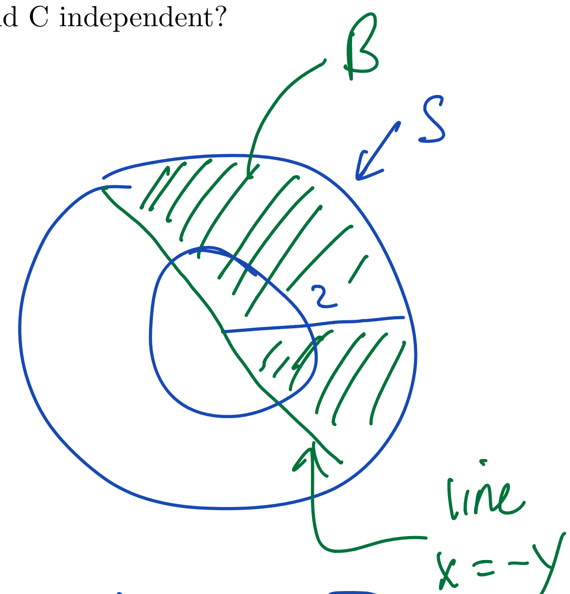
(a) Consider an event $A = \{X^2 + Y^2 \geq 1\}$. What is the probability of the event A?

(b) Consider an event $B = \{X + Y > 0\}$. What is the probability of the event B?

(c) Are events A and B independent?

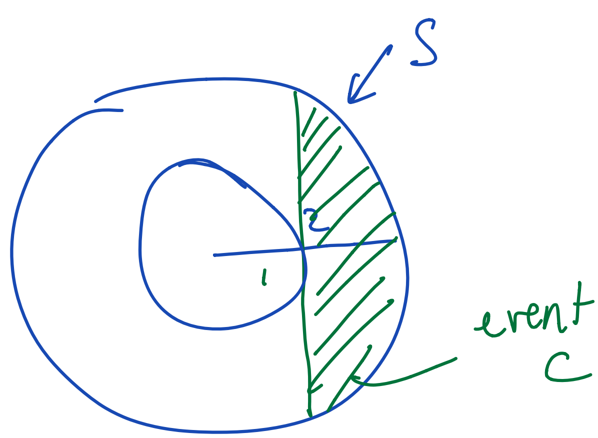
(d) Consider an event $C = \{X > 1\}$. Are events A and C independent?





a) $P(A) = \dfrac{\text{area}(A)}{\text{area}(S)} = \dfrac{\pi(2^2) - \pi(1^2)}{\pi(2^2)} = \boxed{\dfrac{3}{4}}$

b) $P(B) = \boxed{\dfrac{1}{2}}$



c) $P(A \cap B) = \frac{1}{2} P(A)$ so yes

d) $P(A \cap C) = P(C) \neq P(A)P(C)$

$\Rightarrow$ NO they are not independent

Problem 13. (Multiple choice: 5 points)
Bill is a physicist with probability 0.6, an amateur jazz musician with probability 0.7, and is neither with probability 0.25. Determine the probability that he is a physicist or an amateur jazz musician but not both.

(a) 0.1

(b) 0.2

(c) 0.3

(d) 0.5

(e) Impossible to determine based on the information given.

(f) None of the above.

Let $E = \{\text{Bill is a physicist}\}$

$m = \{\text{Bill is an amateur musician}\}$

we know $P(E) = 0.6$, $P(m) = 0.7$

and $P(E^c \cap m^c) = 0.25$

So we can start the table:

	E	E^c	
m			0.7
m^c		0.25	
	0.6		

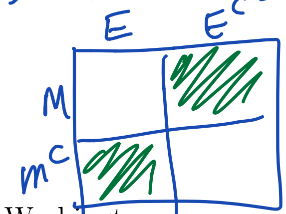
and complete it!

	E	E^c	
m	0.55	0.15	0.7
m^c	0.05	0.25	0.3
	0.6	0.4	

we want

$P((E \cap m^c) \cup (E^c \cap m))$

$= P(E \cap m^c) + P(E^c \cap m)$



Problem 14. (5 points)
Suppose a packet from NYC to Indianapolis is routed through either Philadelpia or Washington DC. Let W be the event a given packet is routed through Washington, and let L be the event that it is lost along the way and doesn't arrive at its destination.

Suppose twice as many packets are routed through Washington as Philadelpha, and the probability a packet is lost is 1/4. In addition, given that a packet is lost, it is equally likely to have been routed through either Washington or Philadelphia.

What is the probability that a packet is routed through Philadelphia and not lost?

$L = \{\text{lost}\}$
$W = \{\text{Washington}\}$

given: $P(W) = 2/3$. $P(W^c) = 1/3$. $P(L) = 1/4$.

$P(W|L) = P(W^c|L) = 1/2$ (equally likely, given L)

want: $P(W^c \cap L^c) = 1 - P(W \cup L)$ by DeMorgan's Rule and $P(A) = 1 - P(A^c)$

$= 1 - [P(W) + P(L) - P(W \cap L)]$

$= 1 - (2/3 + 1/4 - P(W|L)P(L))$

$= 1 - \left(\frac{8+3}{12} - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\right) = 1 - \frac{19}{24} = \boxed{\frac{5}{24}}$

Problem 15. (7 POINTS)
Computers purchased from the surplus store will experience hard drive failures with probablitiy 0.4, memory failures with probability 0.2, and experience both types at the same time with probability 0.1.

What is the probability that there will be either a hard drive failure or a memory failure, but not both.

$D = \{\text{disk fail}\}$

$M = \{\text{memory fail}\}$

Given:

	M	M^c	
D	0.1		0.4
D^c			
	0.2		

Complete

	M	M^c	
D	0.1	0.3	0.4
D^c	0.1	0.5	0.6
	0.2	0.8	

want

$P((D \cap M^c) \cup (D^c \cap M))$

$= 0.1 + 0.3 = \boxed{0.4}$

Problem 16. (10 POINTS)
Among the Purdue students taking ECE 302 this semester, some like dogs, some like cats, some like both, and some like neither. Let D be the set of Purdue ECE302 students who like dogs, and C be the set who like cats.
A study shows that 22% like both cats and dogs, and 12% like neither. The probability a student likes dogs exceeds the probability a student likes cats by 0.14. What is the probability a randomly chosen student likes cats?

$D = \{\text{likes dogs}\}$

$C = \{\text{likes cats}\}$

$P(C \cap D) = 0.22$

$P(C^c \cap D^c) = 0.12$

	C	C^c	
D	0.22		$x + 0.14$
D^c		0.12	$1 - x - 0.14$
	x	$1 - x$	

value to find

$$P(C \cup D) = 1 - P(C^c \cap D^c)$$
$$= 1 - 0.12 = 0.88$$
$$= P(C) + P(D) - P(C \cap D)$$
$$= x + (x + 0.14) - 0.22$$

$$\Rightarrow 0.88 + 0.22 = 2x + 0.14$$
$$x = \tfrac{1}{2}(0.96) = \boxed{0.48}$$

Problem 17. (5 POINTS)
On Tuesday next week, the probability that a Purdue student exercises is 0.7, while the probability a student eats broccoli is 0.2. Of those students who exercise next Tuesday, 15% will eat broccoli. What is the probability that a randomly selected student doesn't exercise and doesn't eat broccoli next Tuesday.

Let $A = \{\text{exercise}\}$

$B = \{\text{broccoli}\}$

Given: $P(A) = 0.7$

$P(B) = 0.2$

$P(B|A) = 0.15$ (conditional probability)

want $P(A^c \cap B^c)$

	A	A^c	
B	0.105	0.095	0.2
B^c		0.205	0.8
	0.7	0.3	

Read answer from table:

$P(A^c \cap B^c) = \boxed{0.205}$

$P(A \cap B)$

$= P(B|A)P(A)$

$= (0.15)(0.7)$

$= 0.105$

$P(A^c \cap B)$

$= P(B) - P(A \cap B)$

$= 0.2 - 0.105$

$= 0.095$

$P(A^c \cap B^c) =$

$P(A^c) - P(A^c \cap B)$

$= (1 - 0.7) - 0.095$

$= \boxed{0.205}$

Problem 18. (10 POINTS)
Alice and Bob are good friends, who both work at a security company. The company has a design group and an installation group; Alice works in design, Bob in installation. There are 20 people in the design group and 80 people in the installation group. For each project, the company sends 2 from the design group and 5 from the installation group. If each person has the same probability to be selected to a specific project, what is the probability that Alice and Bob will be both be sent to the same project?

ways to form design group $= \binom{20}{2}$

ways to form design group that includes Alice $= \binom{19}{1}$

ways to form installation group $= \binom{80}{5}$

ways to form installation group that includes Bob $= \binom{79}{4}$

$$P(\text{Alice and Bob}) = P(\text{Alice})P(\text{Bob}) = \frac{\binom{19}{1}}{\binom{20}{2}} \cdot \frac{\binom{79}{4}}{\binom{80}{5}}$$

$$= \frac{1}{10} \cdot \frac{1}{16} = \boxed{\frac{1}{160}}$$

Problem 19. (YES/NO: 5 POINTS)
A deck of cards is shuffled and fairly dealt into 4 hands, each with 13 cards. You pick one of the hands at random. The probability your hand has one (and only one) ace is $\binom{4}{1}\binom{48}{12}/\binom{52}{13}$.

ways to get Ace of hearts (or any specific Ace)

$$= \frac{\binom{48}{12}}{\binom{52}{13}}$$

But there are $\binom{4}{1}$ possible aces, so

ways to get one ace $= \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}$

(Don't need to worry for this problem where the other aces go)

Problem 20. (10 points each, 30 points total)
Zeros and ones are sent over a noisy communication channel, where the transmission of each bit can be considered to be independent sequential experiements. The probability that each 0 is correctly sent is 0.9, while the probability that each 1 is correctly sent is 0.85. The digit 0 is sent with probability 0.6.

(a) Find the probability that an error occurs, for each bit sent.

(b) Given that you detect a 1, what is the probability that a 1 had been sent.

(c) If the string 0010 is sent, what is the probability the string is correctly received.

Let $R_0 = \{0 \text{ received}\}$
$R_1 = \{1 \text{ received}\}$
$S_0 = \{0 \text{ sent}\}$
$S_1 = \{1 \text{ sent}\}$

we know
$P(R_0|S_0) = 0.9 \Rightarrow P(R_1|S_0) = 0.1$
$P(R_1|S_1) = 0.85 \Rightarrow P(R_0|S_1) = 0.15$
$P(S_0) = 0.6 \Rightarrow P(S_1) = 0.4$

a) $P(\text{error}) = P(R_1 \cap S_0) + P(R_0 \cap S_1)$ (because these are disjoint)
$= P(R_1|S_0)P(S_0) + P(R_0|S_1)P(S_1)$
$= (0.1)(0.6) + (0.15)(0.4)$
$= 0.06 + 0.06 = \boxed{0.12}$

b) $P(S_1|R_1) = \dfrac{P(R_1|S_1)P(S_1)}{P(R_1)}$ by Bayes Rule

and
$P(R_1) = P(R_1|S_0)P(S_0) + P(R_1|S_1)P(S_1)$ by theorem of total probability

so $P(S_1|R_1) = \dfrac{(0.85)(0.4)}{(0.85)(0.4) + (0.1)(0.6)} = \dfrac{.34}{.34 + .06} = \boxed{\dfrac{34}{40}}$

c) Independent sequential experiments, all have to be correctly received $\Rightarrow (0.9)(0.9)(0.85)(0.9)$

Problem 21. (15 POINTS) (A IS 10 POINTS; B IS 5 POINTS)
Among the workstations available to a group of students in a computing course, let C be the set of workstations with 4 or more cores, and M be the set of workstations with 16 GBytes or more of memory.
40% of workstations have four or more cores, while 60% have 16 GBytes or more of memory. Of those workstations with more than 16 GBytes or more of memory, five eighths (5/8-ths) of them also have 4 or more cores.

(a) What is the probability a randomly chosen workstation is in both sets C and M, or is in neither?

(b) Are the events C and M independent? Prove your answer with adequate justification.

$C = \{$ workstation has 4 or more cores $\}$

$M = \{$ workstation has 16 GBytes or more memory $\}$

$P(M) = 0.6$

$P(C) = 0.4$

$P(C|M) = 5/8$

a) find $P((M \cap C) \cup (M^c \cap C^c)) = P(M \cap C) + P(M^c \cap C^c)$ (because disjoint)

$= P(M \cap C) + 1 - P(M \cup C)$

$= P(M \cap C) + 1 - [P(M) + P(C) - P(M \cap C)]$

$= 1 - P(M) - P(C) + 2P(M \cap C)$

$= 1 - 0.6 - 0.4 + 2P(C|M)P(M) = 2\left(\frac{5}{8}\right)\left(\frac{6}{10}\right)$

$= \boxed{3/4}$

b) If independent, then $P(M \cap C) = P(M)P(C)$

But $P(M \cap C) = P(C|M)P(M) = \left(\frac{5}{8}\right)\left(\frac{6}{10}\right) = \frac{3}{8}$

and $P(M)P(C) = \left(\frac{6}{10}\right)\left(\frac{4}{10}\right) = \frac{24}{100} = \frac{6}{25}$ which are not equal

$\boxed{\text{So } M \text{ and } C \text{ are NOT independent}}$

Problem 22. (8 POINTS EACH, TOTAL 40 POINTS; EXAM 1 FALL 2015)
Roll 2 fair 6-sided dice.

(a) What is the probability that you roll doubles (both die show the same number).

(b) Given the sum is less than 6, what is the probability of doubles?

(c) Find the probability at least one die is 5.

(d) Given that each die shows a different number, find the conditional probability that at least one die is 5.

(e) If we are told that both the product and sum of the face values are less than 7 and that at least one of the face values is a two, determine the probability the other face value is a one.

a) 6 possible doubles, 36 possible rolls

$\Rightarrow P(\text{doubles}) = 6/36 = \boxed{1/6}$

b) $A = \{\text{sum less than } 6\} = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)\}$

10 outcomes

2 doubles in this set

so $P(\text{doubles} \mid A) = P(\text{doubles} \cap A)/P(A) = \dfrac{2/36}{10/36} = \boxed{\dfrac{1}{5}}$

c) 11 possible outcomes w/ at least one 5

$P(\text{at least one } 5) = \boxed{11/36}$

d) 30 possible outcomes w/ different numbers.
of these, 10 have at least one 5.
$P(\text{at least one } 5 \mid \text{diff \#'s}) = 10/30 = \boxed{1/3}$

e) Given set: $\{(1,2), (2,1), (2,2), (2,3), (3,2)\}$
there are 2 cases of these 5 outcomes where one die shows a 1, so $\Rightarrow \boxed{2/5}$

Problem 23. (20 POINTS; EXAM 1 FALL 2015)
You have 4 otherwise identical cans of soda (also known as pop or cola), except you know that 1 was shaken up about 10 minutes ago, while the other 3 have been stable for hours. (You have lost track of which can is which.) The probability of the shaken can splattering when opened is 4/5, and the probability of a stable can splattering when opened is 1/3.

(a) If you choose one can at random and open it, what is the probability of it splattering?

(b) If you open a can and it splatters, what is the probability that it was the shaken can?

(You may leave your answers in fractional form.)

Know: $P(\text{splat} \mid \text{shaken}) = 4/5$

$P(\text{splat} \mid \text{stable}) = 1/3$

$P(\text{shaken}) = 1/4$

$(\text{shaken})^c = \text{stable}$

a) $P(\text{splat}) = P(\text{splat} \mid \text{shaken})\,P(\text{shaken}) + P(\text{splat} \mid \text{stable})\,P(\text{stable})$

$$= \frac{4}{5} \cdot \frac{1}{4} + \frac{1}{3}\left(1 - 1/4\right)$$

$$= 1/5 + 1/4 = \boxed{9/20}$$

b) $$P(\text{shaken} \mid \text{splat}) = \frac{P(\text{splat} \mid \text{shaken})\,P(\text{shaken})}{P(\text{splat})}$$

$$= \frac{\frac{4}{5}\ \frac{1}{4}}{9/20} = \frac{1/5}{9/20} = \boxed{\frac{4}{9}}$$

Problem 24. (20 POINTS (A AND B ARE 7 POINTS; C IS 6 POINTS))
In a given binary communcation channel, it is equally likely to send either a 1 or a 0. The probability of error given that a 1 was sent is 2/9, while the probability of an error given that a 0 was sent is 1/9.

(a) What is the probability that a 1 is received?

(b) What is the probability that an error occurs?

(c) What is the probability that a 1 was sent, given that a 1 was received?

$S_1 = \{\text{send } 1\}$ $\quad S_0 = \{\text{send } 0\}$ $\quad P(S_1) = P(S_0) = 1/2$

$R_1 = \{\text{receive } 1\}$ $\quad R_0 = \{\text{receive } 0\}$ $\quad P(R_1 \mid S_0) = 1/9$

$P(R_0 \mid S_1) = 2/9$

a) $P(R_1) = P(R_1 \mid S_1)P(S_1) + P(R_1 \mid S_0)P(S_0)$ thm total probability

$= \left(1 - \frac{2}{9}\right)\left(\frac{1}{2}\right) + \frac{1}{9}\,\frac{1}{2} = \frac{7+1}{18} = \boxed{4/9}$

b) $P(\text{error}) = P(R_1 \cap S_0) + P(R_0 \cap S_1)$

$= P(R_1 \mid S_0)P(S_0) + P(R_0 \mid S_1)P(S_1)$

$= \frac{1}{9}\,\frac{1}{2} + \frac{2}{9}\,\frac{1}{2} = \frac{3}{18} = \boxed{1/6}$

c) $P(S_1 \mid R_1) = \dfrac{P(R_1 \mid S_1)P(S_1)}{P(R_1)}$ use answer from part (a)

$= \dfrac{\left(1 - \frac{2}{9}\right)\frac{1}{2}}{4/9} = \dfrac{7/18}{4/9} = \boxed{\frac{7}{8}}$

Problem 25. (20 POINTS (A AND C ARE 7 POINTS; B IS 6 POINTS))
Online video streaming is achieved by attempting to match the video bandwidth to fit the bandwidth available between the video server and the viewer's computer. When more bandwidth is available, higher quality video can be achieved. However, not all movies require the same bandwidth for "acceptable" quality. (For example, action movies typically require more bandwidth than romantic comedies, for the same video quality.)

Suppose video bandwidth is divided into 3 categories: low-, medium- and high bandwidth. Further, suppose studies of viewer satisfaction show that medium-bandwidth videos are twice as likely as high-bandwidth videos to be rated as "unacceptable", but only half as likely as low-bandwidth videos to be rated "unacceptable".

(a) Suppose a company designs their system such that 50% of videos are delivered at low bandwidth, 30% at medium bandwidth, and 20% at high bandwidth.

If the probability a medium-bandwidth video is "unacceptable" is 2/7, compute the probability any video is "unacceptable".

(b) If a delivered video is rated by viewers as "unacceptable", what is the probability it was sent at high bandwidth?

(c) The company decides their system performs too poorly. They realize they can only increase the fraction of high-bandwidth videos to 0.5. What is the smallest possible probability of "unacceptable" videos they can hope to achieve given this constraint?

(You may leave all answers in fractional form.)

$U = \{\text{video is unacceptable}\}$

$L = \{\text{low bandwidth}\}$
$M = \{\text{medium bandwidth}\}$
$H = \{\text{high bandwidth}\}$

these 3 form a partition

a) Theorem of total probability

$$P(U) = P(U|L)P(L) + P(U|M)P(M) + P(U|H)P(H)$$

and $P(U|L) = 2P(U|M) = 4/7$

and $P(U|H) = P(U|M)/2 = 1/7$

so $P(U) = \left(\frac{4}{7}\right)\left(\frac{5}{10}\right) + \left(\frac{2}{7}\right)\left(\frac{3}{10}\right) + \left(\frac{1}{7}\right)\left(\frac{2}{10}\right) = \frac{20+6+2}{70} = \frac{28}{70} = \boxed{\frac{2}{5}}$

b) Bayes Rule: $P(H|U) = \frac{P(U|H)P(H)}{P(U)} = \frac{(1/7)(2/10)}{2/5} = \boxed{1/14}$

c) Set $P(H) = 1/2$, with no other constraint on $P(M)$ or $P(L)$.
So set $P(M) = 1/2$ and $P(L) = 0$.
Then use theorem of total probability again.

$$P(U) = \left(\frac{4}{7}\right)(0) + \left(\frac{2}{7}\right)\frac{1}{2} + \frac{1}{7}\left(\frac{1}{2}\right) = \boxed{\frac{3}{14}}$$

Problem 26. (15 POINTS (A IS 8 POINTS AND AND B IS 7 POINTS))
Pallavi, Varsha, and Smita repair amplifiers at a local music shop. Pallavi is still in training, so she repairs 20% of the amplifiers and has a 60% success rate (meaning that 60% of her repairs are effective). Varsha and Smita, experienced pros, each repair 40% of the amplifiers and each have a 90% success rate.

(b) What is the probability that an amplifier does not get properly repaired?

(a) Given that an amplifier was defective after being repaired, what is the probability that Pallavi did the repair?

Let $N = \{$ Novice does repair $\}$ $\qquad P(N) = \frac{2}{10}$

Let $F = \{$ amplifier is Fixed $\}$ $\qquad P(N^c) = 8/10$

$P(F|N) = \frac{6}{10}$

$P(F|N^c) = 9/10$

a) want $P(F^c) = P(\{\text{amplifier is not fixed}\})$

$= P(F^c|N)P(N) + P(F^c|N^c)P(N^c)$ (by the theorem of total probability)

$= [1 - P(F|N)]P(N) + [1 - P(F|N^c)]P(N^c)$

$= \frac{4}{10}\frac{2}{10} + \frac{1}{10}\frac{8}{10} = \frac{8}{100} + \frac{8}{100} = \frac{16}{100} = \boxed{\frac{4}{25}}$

b) $P(N|F^c) = \dfrac{P(F^c|N)P(N)}{P(F^c)}$ (by Bayes Rule)

$= \dfrac{\frac{4}{10}\frac{2}{10}}{\frac{16}{100}} = \boxed{\frac{1}{2}}$

Problem 27. (32 POINTS)

Suppose 3 boxes contain Red, Green, and Blue marbles, denoted R, G, B, respectively.
Box 1 has 3 Red, 4 Green, and 3 Blue.
Box 2 has 8 Red, 1 Green, and 1 Blue.
Box 3 has 0 Red, 4 Green, and 1 Blue.

a sequential experiment!

Suppose a box is chosen at random, and then a marble is selected from the box.

(a) If box 1 is selected, what is the probability a Green marble is drawn?

(b) What is the probability a Blue marble is drawn from Box 3?

(c) What is the probability a Red marble is drawn?

(d) Suppose a Red marble is drawn. What is the probability it came from Box 2?

First, pick a box at random $\Rightarrow$ each box has probability $1/3$.

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = 1/3.$$

Either a table or a tree can be used here.

	R	G	B	
$\{1\}$	$\frac{1}{3}\cdot\frac{3}{10}$	$\frac{1}{3}\cdot\frac{4}{10}$	$\frac{1}{3}\cdot\frac{3}{10}$	$1/3$
$\{2\}$	$\frac{1}{3}\cdot\frac{8}{10}$	$\frac{1}{3}\cdot\frac{1}{10}$	$\frac{1}{3}\cdot\frac{1}{10}$	$1/3$
$\{3\}$	$\frac{1}{3}\cdot 0$	$\frac{1}{3}\cdot\frac{4}{5}$	$\frac{1}{3}\cdot\frac{1}{5}$	$1/3$

If box 1 is picked,

$$P(R \mid \{1\}) = \frac{3}{10}$$
$$P(G \mid \{1\}) = \frac{4}{10} \quad \text{etc.}$$
$$P(B \mid \{1\}) = 3/10$$

a) $P(G \mid \{1\}) = \boxed{\frac{4}{10}}$

b) $P(B \cap \{3\}) = P(B \mid \{3\})\,P(\{3\}) = \frac{1}{5}\cdot\frac{1}{3} = \boxed{\frac{1}{15}}$

c) $P(R) = P(R \mid \{1\})\,P(\{1\}) + P(R \mid \{2\})\,P(\{2\}) + P(R \mid \{3\})\,P(\{3\})$

$$= \frac{1}{3}\left(\frac{3}{10} + \frac{8}{10} + 0\right) = \boxed{\frac{11}{30}}$$

d) $P(\{2\} \mid R) = \dfrac{P(R \mid \{2\})}{P(R)} = \dfrac{\frac{8}{10}\cdot\frac{1}{3}}{\frac{11}{30}} = \boxed{\frac{8}{11}}$

Problem 28. (5 POINTS)

The probability that a randomly selected tablet (like an iPad or a Microsoft Surface) has cellular connectivity is 0.25. A tablet with cellular is twice as likely to have a USB connector as a tablet that does not have cellular.

Compute the probability that a tablet has cellular connectivity given that it has a USB connector.

$A = \{\text{tablet has cellular}\}$

$B = \{\text{tablet has USB}\}$

Given: $P(A) = 1/4$

$P(B|A) = 2P(B|A^c)$

Want: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$ (Bayes Rule)

$= \dfrac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(B|A^c)}$ by thm total prob

Let

$P(B|A^c) = x$

$P(B|A) = 2x$

$P(A^c) = 1 - P(A)$

$= 3/4$

$= \dfrac{2x\,P(A)}{2x\,P(A) + x\,P(A^c)}$

cancel x's from numerator and denominator

$= \dfrac{2\left(\frac{1}{4}\right)}{2\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)} = \boxed{\dfrac{2}{5}}$

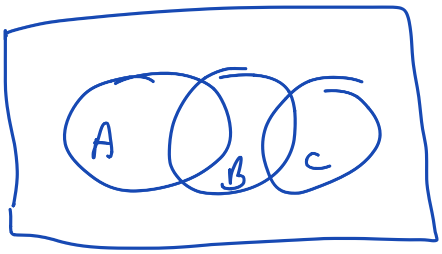
Problem 29. (20 POINTS (A AND B ARE EACH 5 POINTS; C IS 10 POINTS))
Let A, B, and C be events in a sample space, where $P(A) = 0.2$, $P(B) = 0.1$, and $P(C) = 0.3$ In addition, A and B are pairwise independent, B and C are pairwise independent, and A and C are disjoint.

(a) Draw a Venn diagram indicating a possible configuration of these three events.

(b) Find $P(A \cap B)$.

(c) Find $P(A \cup B \cup C)$

a)



A and B must intersect because they're independent and $P(A) \neq 0$ and $P(B) \neq 0$

B and C must intersect for similar reasons.

A and C are disjoint and do not intersect.

b) $P(A \cap B) = P(A)P(B)$ because they're independent

$= (0.2)(0.1) = \boxed{0.02}$

c) $P(A \cup B \cup C) = P((A \cup B) \cup C)$

$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$

$= [P(A) + P(B) - P(A \cap B)] + P(C)$

$- P((A \cap C) \cup (B \cup C))$

$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C)$

$- P(A \cap B \cap C)]$

$= 0.2 + 0.1 + 0.3 - 0.02 - 0 - (0.1)(0.3) + 0$

$= \boxed{0.55}$

Problem 30. (YES/NO: 4 POINTS)
If events A and B are independent, then they are also disjoint.

Problem 31. (YES/NO: 4 POINTS)
If events A and B are disjoint, then they are also independent.

Problem 32. (MULTIPLE CHOICE: 5 POINTS)

$A = \{(2,*),(4,*),(6,*)\}$

$B = \{(*,2),(*,4),(*,6)\}$

Two fair dice – one Red and one Blue – are rolled.
Let A be the event that the Red die shows an even number.
Let B be the event that the Blue die shows an even number.
Let C be the event that the sum of the numbers shown on the two dice is even.

Which of the following is true? (If you show your work you may receive partial credit.)

(a) Each pair of events is independent, but A, B, and C are not mutually independent.

(b) A, B, and C are mutually independent.

X (c) Exactly one pair of the three events is independent.

X (d) Exactly two of the three pairs of events are independent.

X (e) There is no pair among the three events that is independent.

$C = \{(1,1),(3,1),(5,1),$ .... $(5,1)(5,3),(5,5),$ $(6,2),(6,4),(6,6)\}$

A and B and C each have 18 items, so $P(A) = P(B) = P(C) = 1/2$.

$A \cap B = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

9 items, so $P(A \cap B) = 1/4$ and $P(A)P(B) = (\frac{1}{2})(\frac{1}{2}) = 1/4$

so A and B are independent.

$A \cap C$ is actually the same set as $A \cap B$, and $P(A \cap C) = 1/4 = P(A)P(C)$ so A and C are also independent.

And we can do the same for $B \cap C$, to show their independent.

So all are pairwise independent.

$A \cap B \cap C = \ldots$ You can show this is exactly $A \cap B$.

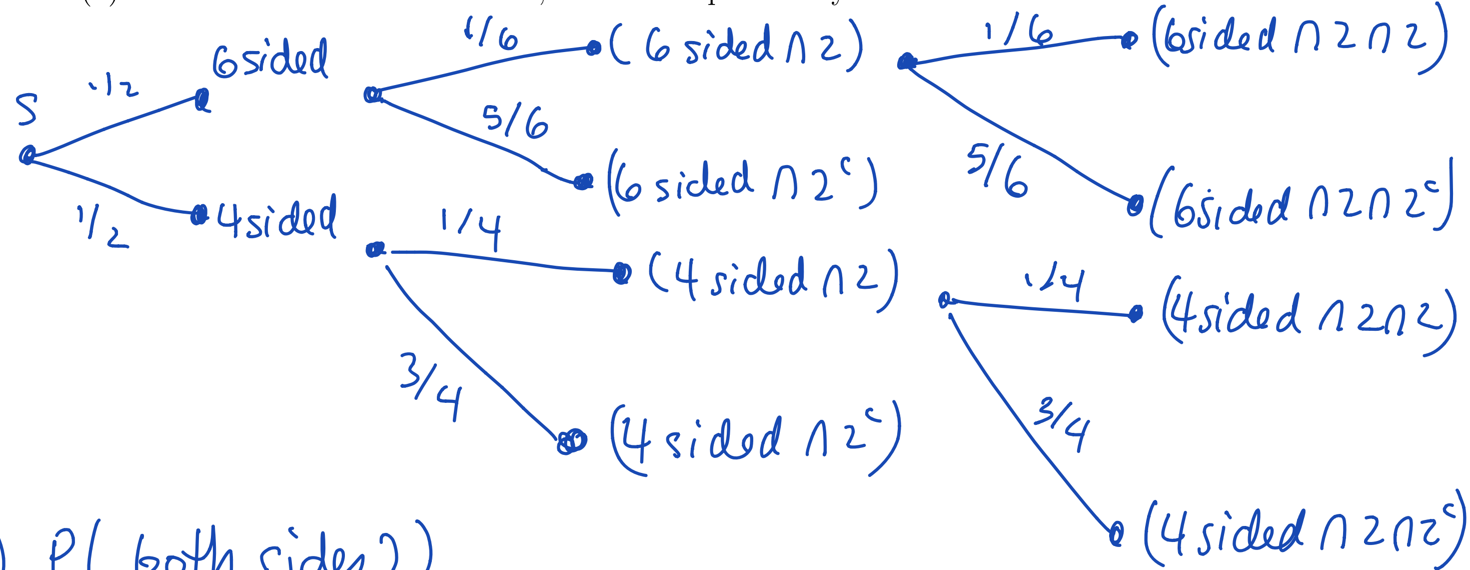
so $P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$

So all 3 are NOT mutually independent $\Rightarrow$ answer (a)

Problem 33. (15 POINTS (A IS 8 POINTS, B IS 7 POINTS))
A box contains 2 dice: a fair die with 6 sides (labelled 1,2,3,4,5,6) and a fair die with 4 sides (labelled 1,2,3,4). You select one object at random, then roll it twice.

(a) What is the probability both rolls show a 2?

(b) Given that the first roll was a 3, what is the probability the chosen die is the 6-sided die?



a) P(both sides 2)

$= P(6 \text{ sides}) P(\text{both } 2 | 6 \text{sides})$

$+ P(4 \text{ sides}) P(\text{both } 2 | 4 \text{ sides})$

$$= \frac{1}{2}\frac{1}{6}\frac{1}{6} + \frac{1}{2}\frac{1}{4}\frac{1}{4} = \frac{1}{2}\frac{1}{4}\left(\frac{1}{9}+\frac{1}{4}\right)$$

$$= \frac{13}{8 \cdot 36} = \boxed{\frac{13}{288}}$$

b) $P(6 \text{ side} | \text{first } 3) = \dfrac{P(\text{first } 3 | 6 \text{sides}) P(6 \text{sides})}{P(\text{first } 3)}$

$P(\text{first } 3) = P(\text{first } 3 | 6 \text{sides}) P(6 \text{ sides}) + P(\text{first } 3 | 4 \text{sides}) P(4 \text{sides})$

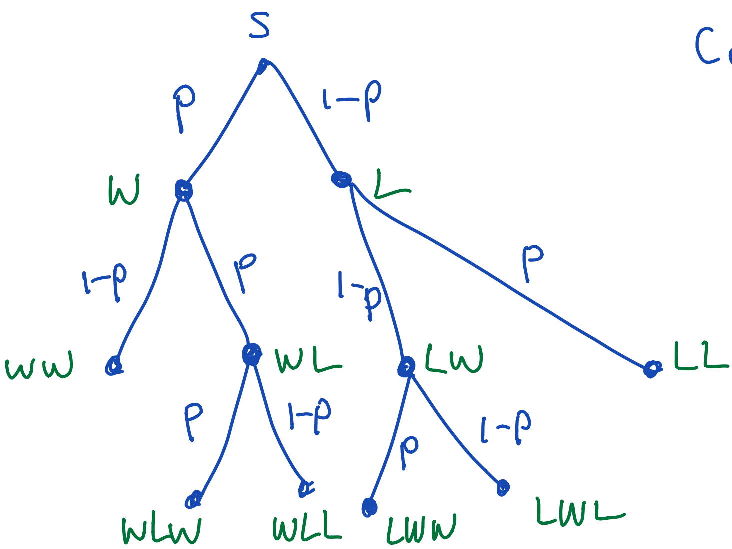
$$= \frac{1}{6}\frac{1}{2} + \frac{1}{4}\frac{1}{2} = \frac{1}{4}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5}{24}$$

$$P(6 \text{sides} | \text{first } 3) = \frac{\frac{1}{6}\frac{1}{2}}{5/24} = \frac{24}{5}\frac{1}{12} = \boxed{\frac{2}{5}}$$

Problem 34. (15 POINTS)
When two teams of equal strength play each other, the home team has a probability p of winning, where $p > 1/2$. Suppose two teams play a three-game series, where the team with home-field advantage plays first at home, then away, and then, if necessary, at home. (There is no third game if one team wins the first two games.)

What is the probability the team with home-field advantage wins the series?
(Hint: a tree diagram can be helpful here.)

BONUS 5 points: Does the home team have a greater, equal, or lesser advantage of winning a 3-game series rather than just one game? Explain why.



Consider this from the perspective of the team that starts with a home game, and draw one tree of the possible outcomes.

On the 1st split, prob W is p.
On the 2nd split $1-p$
On the 3rd split p.

There are 3 outcomes where this team wins the series:

$\{WW, WLW, LWW\}$

$P(\text{1st home team wins series}) = P(WW) + P(WLW) + P(LWW)$

$= p(1-p) + p^3 + (1-p)^3 = \boxed{p(2p^2 - 3p + 2)}$

Bonus: is $P(\text{1st hometeam wins series}) > P(\text{1st team wins 1st game})$?

Is $p(2p^2 - 3p + 2) > p$?

$(2p^2 - 3p + 2) > 1$

$2p^2 - 3p + 1 > 0$

$(2p-1)(p-1) > 0$?

Two possibilities:

1. $2p-1 > 0$ and $p-1 > 0$
2. $2p-1 < 0$ and $p-1 < 0$

Case 1 can never happen.
Case 2 happens if $2p < 1$ or $p < 1/2$. But $p > 1/2$, so NO.

swapping home/away evens out the series.

Problem 35. (15 POINTS (A IS 8 POINTS, B IS 7 POINTS))
You flip a unfair, weighted coin, that has $P(head) = 1/4$. If the result is a heads, you then roll a fair 4-sided die. If the result is a tails, you then roll a fair 6-sided die.

(a) What is the probability your die-roll results in a 2?

(b) Given that the die-roll results in a 2, what is the probability you first got a tail?

A sequential experiment.

$P(H) = 1/4 \qquad P(T) = 3/4$

Let $R_2 = \{\text{roll a } 2\}$ $\qquad P(R_2 | H) = 1/4 \qquad P(R_2 | T) = \frac{1}{6}$

a) $P(R_2) = P(R_2 | H) P(H) + P(R_2 | T) P(T)$

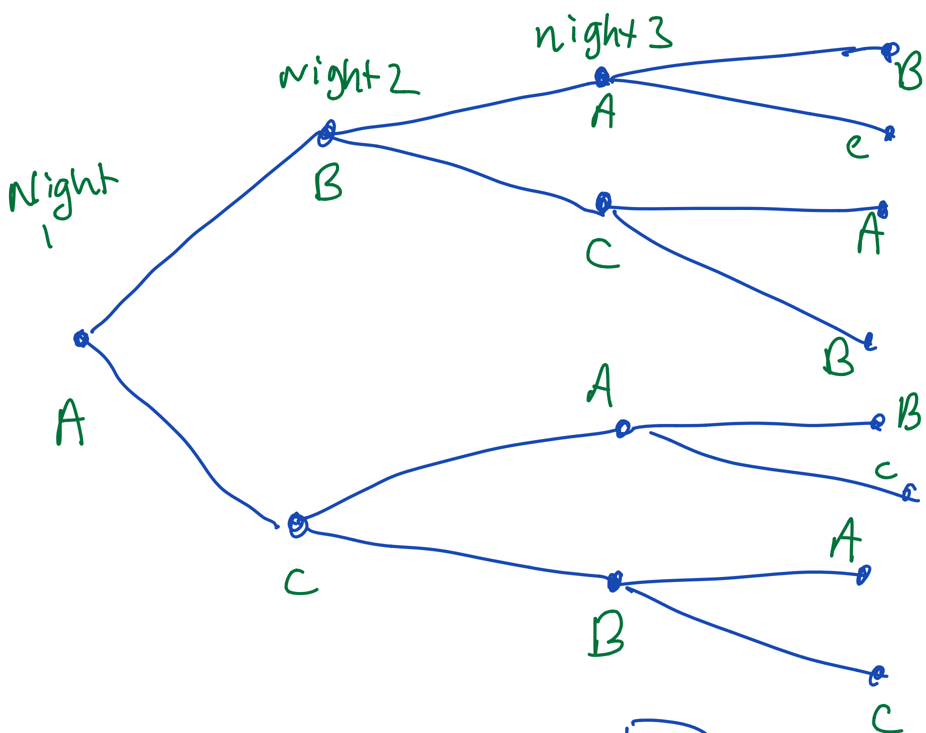
$$= \frac{1}{4}\frac{1}{4} + \frac{1}{6}\frac{3}{4} = \frac{1}{4}\left(\frac{1}{4} + \frac{1}{2}\right) = \boxed{\frac{3}{16}}$$

b) $$P(T | R_2) = \frac{P(R_2 | T) P(T)}{P(R_2)} = \frac{\frac{1}{6}\frac{3}{4}}{3/16} = \frac{16}{24} = \boxed{2/3}$$

Problem 36. (16 POINTS)
A salesperson travels between three cities, labeled A, B, and C, on 4 consecutive nights. Each night, she stays in only one city. On day 1 she starts at City A and stays there the first night. For the next 3 days, she chooses one of the *other* two cities at random, travels there, and stays the night. For example, if she stays in City C on night 3, on night 4 she can only stay in either City A or City B.

(a) What is the probability the salesperson visits all three cities?

(b) Given she visits all 3 cities, what is the probability she visits City A twice?



	all 3?	A twice?
ABAB	X	✓
ABAC	✓	✓
ABCA	✓	✓
ABCB	✓	X
ACAB	✓	✓
ACAC	X	✓
ACBA	✓	✓
ACBC	✓	X

a) $P(\text{all } 3) = \frac{6}{8} = \boxed{\frac{3}{4}}$

all outcomes are equally likely.

b) $P(A \text{ twice} \mid \text{all } 3)$

$$= \frac{P(A \text{ twice} \cap \text{all } 3)}{P(\text{all } 3)} = \frac{4/8}{6/8} = \frac{4}{6} = \boxed{\frac{2}{3}}$$