Constrained $H_\infty$-optimal
Controller Design via Interpolation Theory

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Summary

The set of achievable stable closed-loop maps $\mathcal{H}$ for a feedback control system can be expressed as an affine subset of the set of all stable closed-loop maps, parametrized by the familiar Youla parameter $Q$. This is the so-called “free-parameter” representation of $\mathcal{H}$. An alternate representation of $\mathcal{H}$ is via tangential interpolation conditions at points in the right half complex plane: Every element $H$ of $\mathcal{H}$ must satisfy conditions — in the simplest case — of the form $u^* H(\lambda) = v^*$ and/or $H(\lambda)u = v$, where $u$ and $v$ are complex vectors of appropriate sizes, $u^*$ is the complex conjugate transpose of $u$, and $\lambda$ lies in $\mathbb{C}_+$, the closed right half complex plane. The $H_\infty$ control problem is to find the element in $\mathcal{H}$ with the smallest $H_\infty$-norm.

The $H_\infty$ control problem has been solved in both the free-parameter setting as well as the interpolation setting. The former leads to the Nehari problem (one/two/four-block cases) and the latter to the Nevanlinna-Pick problem (or more generally, to the Hermite-Féjer problem).

The observation central to this report is this: Many design constraints on the closed-loop response are merely additional interpolation conditions on the elements of $\mathcal{H}$. Thus the machinery of $H_\infty$ controller design using interpolation theory may be employed to design $H_\infty$ optimal controllers that satisfy a class of design constraints.

This report will elaborate on the above ideas. Some new results in the theory of interpolation will also be presented.
1 Introduction

1.1 A Framework for Controller Design

A standard framework for modern controller design (see, for example, [BB91]) involves the partitioning of the various signals in the system as shown in the following block diagram: In the figure, $w$ consists of exogenous inputs such as disturbances, commands and noises, $z$ consists of regulated variables such as tracking errors, deviations and actuator outputs, $u$ consists of control inputs, i.e., actuator signals and $y$ consists of measured outputs such as sensor outputs and commands. The reader would already have noted that some signals may be regarded as part of more than one class of signals, so that the partitioning of the signals as above is not disjoint.

The set of all achievable stable closed-loop maps from $w$ to $z$ is then given by

$$
\mathcal{H} = \left\{ H \mid H = P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw}, \ K \text{ stabilizes } P \right\},
$$

where $P_{zw}$ is the open-loop transfer matrix from $w$ to $z$ and so on.

1.2 The Geometry of $\mathcal{H}$

The theory of Youla parametrization reveals a surprising fact about $\mathcal{H}$: $\mathcal{H}$ is an affine subspace of the set of all stable transfer matrices. Indeed, there exist stable transfer
matrices $T_1$, $T_2$ and $T_3$ of appropriate sizes such that

$$\mathcal{H} = \{ H | H = T_1 + T_2 QT_3, \text{ $Q$ is stable} \}.$$  

(For more details, see [YJB76, AS84, Fra87a].) Moreover, finding $T_1$, $T_2$ and $T_3$ is fairly straightforward. In fact, if the plant is described by the state-space equations

$$\begin{align*}
\dot{x} &= Ax + B_w w + B_u u, \\
z &= C_z z + D_{zw} w + D_{zu} u, \\
y &= C_y y + D_{yw} w + D_{yu} u,
\end{align*}$$

and $K_{sf}$ and $L_{est}$ are matrices state feedback and estimator gains such that $A - B_u K_{sf}$ and $A - L_{est} C_y$ are stable, then the transfer matrices $T_1$, $T_2$, and $T_3$ can be realized as

$$\begin{bmatrix} T_1(s) & T_2(s) \\ T_3(s) & 0 \end{bmatrix} = C_T(s I - A_T)^{-1} B_T + D_T,$$

where

$$\begin{align*}
A_T &= \begin{bmatrix} A & -B_u K_{sf} \\ L_{est} C_y & A - B_u K_{sf} - L_{est} C_y \end{bmatrix}, \\
B_T &= \begin{bmatrix} B_w & B_u \\ L_{est} D_{yw} & B_u \end{bmatrix}, \\
C_T &= \begin{bmatrix} C_z & -D_{zw} K_{sf} \\ C_y & -C_y \end{bmatrix}, \\
D_T &= \begin{bmatrix} D_{zw} & D_{zu} \\ D_{yw} & 0 \end{bmatrix}.
\end{align*}$$

A derivation of these formulae may be found, for example, in [BB91].

The above is the so-called free parameter representation of $\mathcal{H}$. As we will see in subsequent sections, there is an alternate representation of $\mathcal{H}$ as the set of stable transfer matrices that satisfy certain interpolation conditions.

### 1.3 The $\mathbf{H}_\infty$ controller design problem

The $\mathbf{H}_\infty$-norm of a stable transfer matrix $H(s)$ is denoted by $\| H \|_\infty$ and defined as

$$\| H \|_\infty = \sup_{\Re s > 0} \sigma_{\text{max}}(H(s)),$$
where $\sigma_{\text{max}}(M)$ stands for the maximum singular value of the matrix $M$. ($\|H\|_\infty \overset{\Delta}{=} \infty$ for unstable $H$.)

The $H_\infty$-norm of $H(s)$ has the interpretation of the $L_2(-\infty, \infty)$ gain of $H$:

$$\|H\|_\infty = \sup \left\{ \frac{\|Hu\|_2}{\|u\|_2} \mid u \in L_2, \ u \neq 0 \right\},$$

where $\|u\|_2$ is the $L_2$-norm of $u$. This interpretation has two immediate and important consequences [DV75, FD87, Fra87b, BB91]:

- For the system\(^1\) $y = Hu$ with a (random) input $u$ satisfying $E u^T u \leq 1$, the expected value of the $L_2$-norm of the output $E y^T y$ is guaranteed to be not larger than $\|H\|_\infty$.

- A ‘small’ $\|H\|_\infty$ implies that the (stable) system $y = Hu$ is \textit{robustly stable} to ‘large’ perturbations appearing between its output and input: From the Small

\begin{center}
\begin{tikzpicture}
\node (sum) at (0,0) {$\Delta$};
\node (u) at (-1,1) {$u$};
\node (H) at (0,1) {$H$};
\node (y) at (1,1) {$y$};
\path [->] (u) edge (H); 
\path [->] (H) edge (y); 
\path [->] (y) edge (sum); 
\path [->] (sum) edge (H); 
\end{tikzpicture}
\end{center}

Gain Theorem, the closed-loop system is stable for all perturbations $\Delta$ satisfying $\|\Delta\|_\infty < 1/\|H\|_\infty$.

In the context of controller design, the above facts mean that it is of interest to design controllers to minimize the $H_\infty$-norm of certain closed-loop transfer matrices. This leads us to the $H_\infty$ control problem: Find

$$\gamma_{\text{opt}} = \inf \{\|H\|_\infty \mid H \in \mathcal{H}\}, \quad (1)$$

\(^1\text{Abusing notation, we use the symbol } u \text{ to denote both the signal } u(t) \text{ and its Laplace transform; the intended meaning should be clear from context.}
and the corresponding closed-loop map $H_{\text{opt}}$ and the associated controller $K_{\text{opt}}$. Under many circumstances, the infimum in (1) is not achieved. Therefore, problem that is usually tackled is the following: Given $\epsilon > 0$, find the set of all $H \in \mathcal{H}$ such that $\|H\|_{\infty} < \gamma_{\text{opt}} + \epsilon$, and the corresponding set of controllers. For a nice discussion on the $H_{\infty}$ control problem, see [Fra87a].

In the next few subsections, we will review some of the established methods for solving the above problem, present some new theoretical results, and finally indicate the application of these results to the design of controllers for the RTP system.

## 2 Solving the $H_{\infty}$ Control Problem: The SISO Case

The SISO case means that the signals $w$, $u$, $z$ and $y$ are all scalar so that the $T_i$s and $Q$ are transfer functions. Then the set of achievable stable closed-loop transfer functions $\mathcal{H}$ may be written as

$$ \mathcal{H} = \{ H \mid H = T_1 + T_2 Q, Q \text{ stable} \}, $$

with $T_3 = 1$ assumed without loss of generality.

### 2.1 An alternate formulation via interpolation conditions

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the zeros of $T_2$ in the closed right half complex plane (assumed to be isolated for simplicity). Then, since $Q$ is stable, every element $H$ of $\mathcal{H}$ must satisfy

$$ H(\lambda_i) = T_1(\lambda_i), \ i = 1, 2, \ldots, n. $$

For convenience, we let $T_1(\lambda_i) = f_i, \ i = 1, 2, \ldots, n$. Thus the set of stable transfer matrices

$$ \tilde{\mathcal{H}} \triangleq \{ H \mid H(\lambda_i) = f_i, \ i = 1, 2, \ldots, n, \ H \text{ stable} \} $$

(2) contains $\mathcal{H}$. It can also be shown that $\tilde{\mathcal{H}} \subseteq \mathcal{H}$, so that $\mathcal{H} = \tilde{\mathcal{H}}$. Thus (2) gives an alternate characterization of the set of achievable stable transfer functions for the
control system. The characterization of $\mathcal{H}$ via interpolation conditions has a long history; some references are [Ber56, RF58, Big58].

The $H_\infty$ control problem may now be restated as

$$\text{Find } \gamma_{\text{opt}} = \inf \{ \| H \|_\infty \mid H(\lambda_i) = f_i, \ i = 1, 2, \ldots, n \},$$

and the optimal interpolant $H_{\text{opt}}$ (if it exists).

### 2.2 Solution of the $H_\infty$ problem via Nevanlinna-Pick Theory

Assuming that the $\lambda_i$ are bounded, we have the following theorem.

**Theorem 1** The infimum in equation (3) is given by

$$\gamma_{\text{opt}} = \min \{ \gamma \mid N_\gamma \geq 0 \},$$

where $N_\gamma$ is a Pick matrix with $(i, j)$-entry $\gamma^2 - f_i f_j^* \overline{\lambda_i + \lambda_j^*}$.

Theorem (1) is the centerpiece of the celebrated Nevanlinna-Pick theory of interpolation [Nev19, Pic16, Akh65]. However, it appears to be less well-known that the optimal interpolant $H_{\text{opt}}$ can be constructed from the eigenvectors of the matrix $N_{\gamma_{\text{opt}}}$:

**Theorem 2** Let $a \in \mathbb{R}^n$ satisfy $N_{\gamma_{\text{opt}}} a = 0$. Then

$$H_{\text{opt}}(s) = \frac{\sum_{i=1}^{n} a_i \frac{f_i}{s - \lambda_i}}{\sum_{i=1}^{n} \frac{a_i}{s - \lambda_i}}$$

satisfies:

1. $H_{\text{opt}}(\lambda_i) = f_i, \ i = 1, 2, \ldots, n.$

2. $H_{\text{opt}}(s)$ is stable with $\| H_{\text{opt}} \|_\infty = \gamma_{\text{opt}}$, indeed, $H_{\text{opt}}(-s^*) H_{\text{opt}}(s) = \gamma_{\text{opt}}^2$ for all $s \in \mathbb{C}$.

Theorem 2 was first stated by S. Boyd at the Pre-conference Tutorial Workshop on $H_\infty$ control theory at the 28th CDC [FBSM87]. We first give a simple energy interpretation of the theorem (due to Boyd) and then give a formal proof.
2.3 An energy interpretation of the Nevanlinna-Pick solution [FBSM87]

We seek a stable function \( H(s) \) which satisfies

\[
H(\lambda_i) = f_i, \ i = 1, 2, \ldots, n.
\]

Interpreting \( H(s) \) as the transfer function of a stable linear system, an input \( e^{\lambda t}1(-t) \) to the system must yield an output \( f_i e^{\lambda t} \) for \( t \in (-\infty, 0] \) (where \( 1(t) \) is the standard Heaviside (unit step) function). Then, from linearity, the input \( u(t) = \sum_{i=1}^{n} a_i e^{\lambda_i t} 1(-t) \) must yield an output \( y(t) = \sum_{i=1}^{n} a_i f_i e^{\lambda_i t} \) for \( t \in (-\infty, 0] \) (where \( a_i \in \mathbb{R} \)). Let us now compute the \( \mathbf{L}_2 \)-norms of the input and the output:

\[
\|u\|_2 = a^T G_i a, \text{ where } \Lambda G_i + G_i \Lambda^* - \varepsilon e^T = 0,
\]

\[
\|y\|_2 \geq a^T G_o a, \text{ where } \Lambda G_o + G_o \Lambda^* - f f^* = 0,
\]

where

\[
\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),
\]

\[
e^T = [1 \ 1 \ \cdots \ 1],
\]

\[
f^T = [f_1 \ f_2 \ \cdots \ f_n].
\]

The second inequality is a consequence of the fact that the output \( y(t) \) may be nonzero over \([0, \infty)\).

Suppose that for some \( \gamma > 0 \) and some \( a \in \mathbb{R}^n \),

\[
a^T G_o a > \gamma^2 a^T G_i a.
\]
Then, from the interpretation of the $\mathbf{H}_\infty$-norm as the $\mathbf{L}_2$-gain, we must have $\|H\|_\infty > \gamma$ for every interpolant $H$. Thus $\gamma_{\text{opt}} > \gamma$.

Noting that $\gamma^2 G_i - G_o$ is just $N_\gamma$, we conclude that

$$\gamma \geq \gamma_{\text{opt}} \implies N_\gamma \geq 0$$

$N_\gamma$ is the Pick matrix of Theorem 1.

Of course, Nevanlinna-Pick theory shows that the reverse implication is also true. Indeed, $\gamma_{\text{opt}}$ is equal to the square root of the maximum generalized eigenvalue of the pair $(G_o, G_i)$. We will not attempt to prove this here.

**Proof of Theorem 1:**

1. To show that $H_{\text{opt}}(s)$ is an interpolant is straightforward. For $j = 1, 2, \ldots, m$,

$$H_{\text{opt}}(\lambda_j) = \lim_{s \to \lambda_j} \frac{\sum_{i=1}^n a_i f_i}{s - \lambda_i} = \frac{1}{\sum_{i=1}^n a_i} f_j.$$ 

2. Starting with

$$\Lambda N_{\gamma_{\text{opt}}} + N_{\gamma_{\text{opt}}} \Lambda^* - (\gamma_{\text{opt}}^2 e e^* - f f^*) = 0,$$

we obtain

$$N_{\gamma_{\text{opt}}} (s I - \Lambda)^{-1} + (-s I - \Lambda^*)^{-1} N_{\gamma_{\text{opt}}}$$

$$+ (-s I - \Lambda^*)^{-1} (\gamma_{\text{opt}}^2 e e^* - f f^*) (s I - \Lambda)^{-1} = 0.$$

Thus, for every $s \in \mathbb{C}$,

$$\gamma_{\text{opt}}^2 a^T (-s I - \Lambda^*)^{-1} e e^* (s I - \Lambda)^{-1} a = a^T (-s I - \Lambda^*)^{-1} f f^* (s I - \Lambda)^{-1} a.$$

This immediately means that $H_{\text{opt}}^*(-s) H_{\text{opt}}(s)$ equals $\gamma_{\text{opt}}^2$ for every $s \in \mathbb{C}$. In particular,

$$|H_{\text{opt}}(j\omega)| = \gamma_{\text{opt}} \text{ for every } \omega \in \mathbb{R}.$$
Thus $\|H_{\text{opt}}\|_\infty = \gamma_{\text{opt}}$.

Finally we show that $H_{\text{opt}}$ is stable. We return to the Lyapunov equation

$$\Lambda N_{\gamma_{\text{opt}}}^* + N_{\gamma_{\text{opt}}} \Lambda^* - \left( \gamma_{\text{opt}}^2 ee^* - ff^* \right) = 0;$$

Adding $-(s + s^*) N_{\gamma_{\text{opt}}}$ to the left hand side of the above equation, and noting that $N_{\gamma_{\text{opt}}} \geq 0$, we get

$$\gamma_{\text{opt}}^2 a^T (s^* I - \Lambda^*)^{-1} ee^* (s I - \Lambda)^{-1} a$$

$$\geq a^T (s^* I - \Lambda^*)^{-1} ff^* (s I - \Lambda)^{-1} a,$$

for every $s \in C_+$. Thus $|H_{\text{opt}}(s)|$ is bounded by $\gamma_{\text{opt}}^2$ in $C_+$, which means that $H_{\text{opt}}(s)$ is analytic in $C_+$.

\[\blacksquare\]

3 Solving the $H_\infty$ Control Problem: The MIMO Case

In the MIMO case, $w, u, z$ and $y$ are vector signals of dimensions $n_w, n_u, n_z$ and $n_y$ respectively. Then the set $\mathcal{H}$ of achievable stable closed-loop transfer matrices is

$$\mathcal{H} = \{ H | H = T_1 + T_2 Q T_3, \ H \text{ is stable} \}.$$ 

For the remainder of this report, we will assume that $n_w = n_u = n_z = n_y = p$ (say), so that $T_1, T_2$ and $T_3$ are square transfer matrices. This assumption is made for purposes of exposition; removing this assumption involves routine, albeit cumbersome manipulations [Fra87a, SCK92].
3.1 An interpolation characterization of \( \mathcal{H} \)

As in the SISO case, we may characterize \( \mathcal{H} \) via the so-called \textit{tangential interpolation conditions} [SWG+81, AS81, AS84]. The derivation of the interpolation conditions proceeds in the same spirit as in the SISO case: Let \( \omega_i, \ i = 1, 2, \ldots, m \) and \( \lambda_i, \ i = 1, 2, \ldots, n \) be the distinct (assumed, for simplicity) transmission zeros of \( T_2 \) and \( T_3 \) respectively. Let \( v_i \) and \( u_i \) be complex vectors of appropriate sizes such that
\[
 v_i^* T_2(\omega_i) = 0 \quad \text{and} \quad T_3(\lambda_i) u_i = 0.
\]
Then, since \( Q \) is constrained to be stable, every \( H \in \mathcal{H} \) must satisfy
\[
 v_i^* H(\omega_i) = v_i^* T_1(\omega_i), \ i = 1, 2, \ldots, m, \\
 H(\lambda_i) u_i = T_1(\lambda_i) u_i, \ i = 1, 2, \ldots, n.
\]

For convenience, we let \( y_i^* = v_i^* T_1(\omega_i) \) and \( x_i = T_1(\lambda_i) u_i \). It can then be shown that
\[
 \mathcal{H} = \left\{ H \left| \begin{array}{c}
 v_i^* H(\omega_i) = y_i^*, \ i = 1, 2, \ldots, m, \\
 H(\lambda_i) u_i = x_i, \ i = 1, 2, \ldots, n,
 \end{array} \right. \right\}.
\]

For simplicity, we assume that \( T_2 = I \), so that \( \mathcal{H} \) is characterized by one-sided interpolation conditions only. With this assumption, the MIMO \( \mathcal{H}_\infty \) control problem is
\[
 \text{Find}\quad \gamma_{\text{opt}} = \inf \left\{ \left. \|H\|_\infty \right| H(\lambda_i) u_i = x_i, \ i = 1, 2, \ldots, n \right\}, \quad (5)
\]
and the optimal interpolant \( H_{\text{opt}} \) (if it exists).

3.2 Solution from Nevanlinna-Pick Theory

As with the SISO case, \( \gamma_{\text{opt}} \) can be written as the square root of the maximum generalized eigenvalue of a pair of input and output Gramians. Let
\[
 \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \\
 U = [u_1 \ u_2 \ \cdots \ u_n], \\
 X = [x_1 \ x_2 \ \cdots \ x_n].
\]
Let $G_i$ and $G_o$ be input and output Gramians given by
\[
\Lambda^* G_i + G_i \Lambda - U^* U = 0,
\]
\[
\Lambda^* G_o + G_o \Lambda - X^* X = 0.
\]

Now define
\[
N_\gamma = \gamma^2 G_i - G_o.
\]

Then it is well known [Fed75, BGR90, Kim87, LA88] that the infimum $\gamma_{\text{opt}}$ in the interpolation problem (5) is given by
\[
\gamma_{\text{opt}} = \min \{ \gamma | N_\gamma \geq 0 \}.
\]

However, it seems less well-known that we may construct the optimal interpolant $H_{\text{opt}}(s)$ from the eigenvectors of the Pick matrix. We present the construction of the optimal interpolant in a special case.

### 3.3 Optimal Interpolant: Special Case

**Theorem 3** For the problem (5), suppose that $N_{\gamma_{\text{opt}}}$ has a nullspace of dimension $p$ with a basis $\{a^{(1)}, a^{(2)}, \ldots, a^{(p)}\}$. Let
\[
\mathcal{A} = [a^{(1)}, a^{(2)}, \ldots, a^{(p)}].
\]

Then
\[
H_{\text{opt}}(s) = \mathcal{X}(s) \mathcal{U}^{-1}(s),
\]

where
\[
\mathcal{U}(s) = U (sI - \Lambda)^{-1} \mathcal{A},
\]
\[
\mathcal{X}(s) = X (sI - \Lambda)^{-1} \mathcal{A}.
\]

Moreover, $H_{\text{opt}}$ satisfies:

1. $H_{\text{opt}}(\lambda_i) u_i = x_i \ i = 1, 2, \ldots, n$. 

2. $H_{\text{opt}}(s)$ is stable, with $\|H_{\text{opt}}\|_\infty = \gamma_{\text{opt}}$. Indeed $H_{\text{opt}}(-s^*)^*H_{\text{opt}}(s) = \gamma_{\text{opt}}^2$ for all $s \in \mathbb{C}$.

Proof.

1. To show that $H_{\text{opt}}(s)$ is an interpolant is a little tricky. For $j = 1, 2, \ldots, n$,

$$H_{\text{opt}}(\lambda_j)u_j = \lim_{s \to \lambda_j} X(s)U^{-1}(-s)u_j$$

$$= \lim_{s \to \lambda_j} \left( X\left(\frac{sI - \Lambda}{s - \lambda_j}\right)^{-1} A \right) \left( U\left(\frac{sI - \Lambda}{s - \lambda_j}\right)^{-1} A \right)^{-1} u_j.$$

Now let

$$t_j = \lim_{s \to \lambda_j} \left( U\left(\frac{sI - \Lambda}{s - \lambda_j}\right)^{-1} A \right)^{-1} u_j.$$

Then,

$$u_j = \lim_{s \to \lambda_j} \left( U\left(\frac{sI - \Lambda}{s - \lambda_j}\right)^{-1} A \right) t_j.$$

Therefore, we must have

$$\lim_{s \to \lambda_j} \left( X\left(\frac{sI - \Lambda}{s - \lambda_j}\right)^{-1} A \right) t_j = x_j.$$

2. Starting with

$$\Lambda^* N_{\gamma_{\text{opt}}} + N_{\gamma_{\text{opt}}} \Lambda - \left( \gamma_{\text{opt}}^2 U^* U - X^* X \right) = 0,$$

we obtain

$$N_{\gamma_{\text{opt}}} (sI - \Lambda)^{-1} + (-sI - \Lambda^*)^{-1} N_{\gamma_{\text{opt}}}$$

$$+ (-sI - \Lambda^*)^{-1} \left( \gamma_{\text{opt}}^2 U^* U - X^* X \right) (sI - \Lambda)^{-1} = 0.$$

Thus, for every $s \in \mathbb{C}$,

$$\gamma_{\text{opt}}^2 A^T (-sI - \Lambda^*)^{-1} U^* (sI - \Lambda)^{-1} A$$
\[= \mathcal{A}^T (-s I - \Lambda^*)^{-1} X^* X (s I - \Lambda)^{-1} \mathcal{A},\]

or \( H_{\text{opt}}^*(-s) H_{\text{opt}}(s) = \gamma_{\text{opt}}^2 I \) for every \( s \in \mathbb{C} \). In particular,

\[\sigma_{\text{max}}(H_{\text{opt}}(j\omega)) = \gamma_{\text{opt}} \text{ for every } \omega \in \mathbb{R}.\]

Thus \( \|H_{\text{opt}}\|_\infty = \gamma_{\text{opt}} \).

Finally we show that \( H_{\text{opt}} \) is stable. We return to the Lyapunov equation

\[\Lambda^* N_{\gamma_{\text{opt}}} + N_{\gamma_{\text{opt}}} \Lambda - \left( \gamma_{\text{opt}}^2 U^* U - X^* X \right) = 0;\]

Adding \(-(s + s^*) N_{\gamma_{\text{opt}}} \) to the left hand side of the above equation, and noting that \( N_{\gamma_{\text{opt}}} \geq 0 \), we get

\[\gamma_{\text{opt}}^2 \mathcal{A}^T (s^* I - \Lambda^*)^{-1} U^* U (s I - \Lambda)^{-1} \mathcal{A} \geq \mathcal{A}^T (s^* I - \Lambda^*)^{-1} X^* X (s I - \Lambda)^{-1} \mathcal{A}, \quad (6)\]

for every \( s \in \mathbb{C}_+ \).

Equation (6) means that the maximum singular value of \( H_{\text{opt}}(s) \) is bounded by \( \gamma_{\text{opt}}^2 \) in \( \mathbb{C}_+ \), which means that \( H_{\text{opt}}(s) \) is analytic in \( \mathbb{C}_+ \).

\[\blacksquare\]

4 Controller Design via Interpolation

We now describe briefly a few controller design problems that can posed as interpolation problems; therefore the techniques described in the previous sections may be used to synthesize \( H_\infty \)-optimal controllers for these problems. For more along these lines, see J. W. Helton [Hel87].

4.1 \( H_\infty \)-Optimal Controllers with Tracking Constraints

Recall that \( \mathcal{H} \) denotes the set of all achievable stable closed-loop maps for a control system. A constraint of the form “Reject constant disturbances along the direction
$u$ is merely an additional interpolation condition: The subspace $\mathcal{H}_{\text{rej}}$ of closed-loop maps that satisfies the disturbance rejection condition is just

$$\mathcal{H}_{\text{rej}} = \{ H \in \mathcal{H} \mid H(0)u = 0 \}$$

Here are some tracking constraints that can be posed as interpolation conditions:

- Rejecting/Tracking modes in the right half plane (sinusoids, growing exponentials etc.)
- Asymptotic decoupling (“The steady state response from $w_i$ to signal $z_j$ ($i \neq j$) is zero”)

4.2 $H_\infty$-Optimal Controllers with Constraints on the Decay Rate

The reader may have noted that steady-state tracking constraints lead to interpolation conditions “on the boundary” of $\mathbb{C}_+$ and therefore cannot be handled directly in the framework of sections 1 and 2. However, we may use the standard trick of exponential time weighting: For instance, instead of solving

$$\text{Find } \gamma_{\text{opt}} = \inf \left\{ \| H \|_\infty \left| \begin{array}{c} H(\lambda_i)u_i = x_i, \ i = 1, 2, \ldots, n \\ H(0)u = 0 \end{array} \right. \right\}, \quad (7)$$

we solve the modified problem

$$\text{Find } \gamma_{\text{opt}} = \inf \left\{ \| H \|_\infty \left| \begin{array}{c} H(\epsilon + \lambda_i)u_i = x_i, \ i = 1, 2, \ldots, n \\ H(\epsilon)u = 0 \end{array} \right. \right\}, \quad (8)$$

for some suitable $\epsilon > 0$. Let $H_{\text{opt}}(s)$ be the solution of the modified problem (8). Then $\hat{H}_{\text{opt}}(s) = H_{\text{opt}}(s + \epsilon)$ is a “suboptimal interpolant” of the original problem (7) in the following sense. $\hat{H}_{\text{opt}}(s)$ is an interpolant that is analytic and has the smallest norm over the larger region $\{ s \in \mathbb{C} \mid \text{Re}s > -\epsilon \}$. In other words, $\hat{H}_{\text{opt}}(s)$ is an “optimal” interpolant with a guaranteed decay rate of $\epsilon$. 
5 Extensions and Future Work

5.1 Theory of Interpolation

- Recall that we assumed only one-sided interpolation conditions for the MIMO problem. Moreover we assumed that the zeros of \( T_2 \) and \( T_3 \) were simple. Eliminating these assumptions is fairly straightforward.

- We have a simple formula for the optimal interpolant via the eigenvectors of the Pick matrix for the SISO problem and a special MIMO problem. Of immediate interest is the characterization of the set of all “suboptimal” solutions in terms of the Pick matrix: Given a \( \gamma \) such that \( N_\gamma > 0 \), find the set of all interpolating functions with \( H_\infty \)-norm less than \( \gamma \). Note that this problem (as well as the construction of the optimal interpolant itself) has been extensively studied. Thus, the thrust of the new research is to rederive these already existing results in terms of the eigenvalues and eigenvectors of the Pick matrix.

- Constructing the optimal interpolant for the general MIMO case is currently being investigated.

- The interpolant was formed from the generalized eigenvectors corresponding to the maximum generalized eigenvalue of \( (G_o, G_i) \). The interpolants formed from the other generalized eigenvectors also have a constant norm on the imaginary axis, but have different inertia. (This observation has connections with meromorphic interpolation.) A formal study of these remains.

- Given a state-space description of \( T_2 \) and \( T_3 \), we need to be able to determine the transmission zeros and the corresponding left and right null-chains (since these determine the tangential interpolation conditions). Devising a reliable numerical procedure for this will be of some importance.
5.2 Relevance to the RTP project

For the RTP system, S. Norman has computed numerically the optimal steady-state closed-loop response that would give the best spatial temperature uniformity across the wafer. Synthesizing a dynamic controller that would ensure this steady state response has been a problem of some interest.

By recognizing that the constraint that the closed-loop map have a certain steady-state response is merely an interpolation condition, we may synthesize $\mathbf{H}_{\infty}$-optimal controllers for the RTP system with desired steady-state performance. How well these $\mathbf{H}_{\infty}$-optimal controllers perform from an engineering point of view (do they yield unrealistic actuator signals, etc.) remains to be seen.

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References


