1. Problem Statement

We consider the problem of transmitting $N$ signals of bounded amplitude through a discrete-time, linear, shift-invariant channel with finite impulse response, so that the corresponding output signals are maximally separated in $l_\infty$ norm. The problem arises naturally in digital communications if we assume the channel is corrupted by noise limited in amplitude by $d$. Error-free transmission can be achieved if the minimum pairwise $l_\infty$ separation between channel outputs is greater than $2d$.

We assume that each channel input sequence is zero outside of the time interval $[0, K-1]$, so that we can represent the set of inputs as a vector $y$ with $K$ dimensions, $y = (y[0], \ldots, y[K-1])$. Letting $y_i$ denote the output vector corresponding to $u_i$, then

$$y_i = H u_i$$

where $H$ is the appropriate $(K + m - 1) \times K$ lower-triangular Toeplitz convolution matrix, and $m$ is the length of the channel impulse response. The problem of maximizing the minimum pairwise separation in $l_\infty$ can be expressed as finding values for the $N$-component vector $u$, $i = 1, \ldots, N$, to

\[
\max_{u} \quad d_{\text{max}} = \min_{i \neq j} \max_{k=0}^{K-1} |y_i[k] - y_j[k]| \quad \text{(P)}
\]

subject to

\[
\forall i \neq j: |u_i[k]| \leq d \quad \text{(C)}
\]

for all $i$ and $k$, where $u_i[k]$ is the $k$th component of $u_i$. The objective function $d_{\text{max}}$ is a piecewise linear function defined over the unit cube in $N$-dimensional space. In general this is a nonconvex optimization problem, and may have local optima that are not globally optimal.

2. Bounds on $d_{\text{max}}$

The continuous-time version of problem (P-C) was considered in [1], where upper and lower bounds on $d_{\text{max}}$, given $N$ and $K$, are presented for any piecewise-continuous impulse response $h(t)$. Although these bounds are tight when $h(t) = e^{-t}$, the upper bound is especially loose when $h(t)$ is highly oscillatory. The lower bound given in [1] can be improved upon, for discrete-time channels, by the computational approach to be described. Before describing this approach we first present a conjectured upper bound for $d_{\text{max}}$ that is always less than or equal to the upper bound given in [1].

To state the bound in the discrete-time case, assume that the rate $R = (\log_2 N)/K = 2^r$, where $r$ is a finite integer can be positive or negative. Generalization to arbitrary $R$ is straightforward, but is omitted here. Let the sequence $h[k]$, $k = 0, \ldots, m-1$, be the magnitudes of the channel impulse response coefficients arranged in nonincreasing order. That is, $\hat{h}[k] = \max \{h[k], 0\}$, and

\[
\hat{h}[0] \geq \hat{h}[1] \geq \cdots \geq \hat{h}[m-1],
\]

where $\hat{h}[k]$ is the channel impulse response, and $\hat{h}[k] = 0, k \notin \{0, m-1\}$. Then we conjecture that

\[
d_{\text{max}} \leq 2 \sum_{i=0}^{R} \hat{h}[i]
\]

if $R \leq 1$, and

\[
d_{\text{max}} \leq 2d(\Omega - 2^R + 1) \quad \text{if } R > 1.
\]

3. Computational Algorithms

We now describe a heuristic computational approach to obtaining solutions to (P-C). The algorithms to be described do not guarantee global optimality, but appear to yield very good, if not optimal, solutions for the examples tried.

We need a few definitions. The extremal dimension between an ordered pair of output vectors $y_i$ and $y_j$ is defined to be one of the dimensions $k$ where the $l_\infty$ norm $f_i[k] - f_j[k]$ is achieved, and the extremal sense is the sign $(\pm 1)$ of $y_i[k] - y_j[k]$. The shape of a set of $N$ vectors $y_i, i = 1, \ldots, N$, is the extremal dimension and sense for each pair $i, j$. For $\Delta$, loosely speaking, for each pair of output vectors the shape tells the dimension in which they are maximally separated, and the sense in which they are ordered along that dimension.

Once the shape of a solution is known, the optimal input constraint can be determined by solving a linear program. Specifically, for each pair of signals $y_i$ and $y_j$, and $d_{\text{max}}$, we enforce the constraint $d(y_i[k] - y_j[k]) \geq \Delta$, where $k$ is the extremal dimension between $y_i$ and $y_j$, and $\Delta$ is the corresponding extremal sense. These $N(N - 1)/2$ constraints imply that $d_{\text{max}} \geq \Delta$. We therefore wish to maximize $\Delta$, subject to these constraints and the amplitude constraints on the input signals $u$.

The general approach consists of two stages:

A. Search for a reasonably good solution by using an iterative ascent algorithm, based on the steepest ascent or random perturbations.

B. Find the shape of that solution, and use the linear program to find the optimal solution with that shape.

In practice the algorithm was run many times with different initial inputs chosen at random, and the search in step A was terminated well before convergence to go on to step B. The idea is that the shape of the solution can be found relatively quickly via an iterative ascent method, and that the optimal solution (for that shape) can be computed quickly from the linear program. The simplex method was used to solve the linear program. The two iterative ascent methods tried in step A yielded similar results. The first searched along the direction of steepest ascent, and the second searched randomly. The random search method requires many more steps to achieve the same increase in objective function as the steepest ascent method, but each step is much less expensive computationally. The practical limits of the algorithm are reached, not because of time but memory, because the linear program has $N(N - 1)/2 + 4N$ constraints. Computational results will be presented for some commonly encountered channels, including the $(1 - D)$ partial response channel.