Design of Stabilizing State Feedback for Delay Systems via Convex Optimization

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Abstract

For linear systems with delays, we define a new class of Lyapunov-like functionals that may be used to prove stability. We also show how we may design a stabilizing (delayed) state feedback for delay systems using these functionals and convex optimization techniques.

1 Introduction

We consider linear systems with delays, described by the state equation

\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i) + Bu(t), \]

where the state \( x(t) \in \mathbb{R}^n \), the input \( u(t) \in \mathbb{R}^p \), and \( 0 < \tau_1 < \tau_2 < \cdots < \tau_m \) are the delays in the system. We assume that the full state of the system is available with a delay \( \tau > 0 \). Our objective is to design a constant, delayed state feedback \( u(t) = -K x(t - \tau) \) that stabilizes the system. We remark that proving stability of system (1) (with \( u(t) = 0 \)) is in itself a hard problem. Our approach towards designing \( K \) combines a Lyapunov-like method with some recent advances in convex optimization.

Note that (1) is not a finite dimensional system, and therefore Lyapunov functionals rather than the more conventional Lyapunov functions are needed. In §2, we will describe one such functional, which we will call the Modified Lyapunov-Krasovskii (MLK) functional. We then show how we may pose the problem of design of a stabilizing (delayed) state-feedback as a convex feasibility problem.

2 Stabilizing state feedback

With the delayed state feedback \( u(t) = -K x(t - \tau) \), the state equation is

\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i) - BK x(t - \tau). \]

In the sequel, we assume that \( 0 < \tau < \tau_1 \); the case \( \tau_1 \leq \tau \) may be dealt with similarly.

Motivated by the work of Krasovskii [4] (see also [6]), we propose a class of functionals for system (2), which we will refer to as Modified Lyapunov-Krasovskii (MLK) functionals:

\[
V(x, t) = x(t)^T L_0 x(t) + \sum_{i=1}^{m} \int_{-\tau_i}^{0} x(t) x(t+s)^T L_i x(t+s) ds + \int_{-\tau}^{0} x(t+s)^T L x(t+s) ds,
\]

where \( L, L_0, \ldots, L_m \) are symmetric positive definite matrices and \( \tau_0 = \tau \). The derivative \( \frac{d}{dt} V(x, t) \), computed using (2) is

\[
\frac{d}{dt} V(x, t) = 2x(t)^T L_0 \left( A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i) - BK x(t - \tau) \right) + \sum_{i=1}^{m} \left( x(t - \tau_i)^T L_i x(t - \tau_i) - x(t - \tau_i)^T L x(t - \tau_i) \right) + \left( x(t)^T L x(t) - x(t - \tau)^T L x(t - \tau) \right).
\]

This can be rewritten as \( \frac{d}{dt} V(x, t) = y^T W y \),

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where $W$ and $y^T$ are given by

$$
\begin{bmatrix}
N & -L_0BK & L_0A_1 & \cdots & L_0A_m \\
-K^TB^TL_0 & L_1 - L & 0 & \cdots & 0 \\
A_1^T L_0 & 0 & L_2 - L_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_m^T L_0 & 0 & 0 & \cdots & -L_m
\end{bmatrix},
$$

and

$$[x(t)^T, x(t - \tau)^T, x(t - \tau_1)^T, \ldots, x(t - \tau_m)^T],$$

respectively, with $N = L_0A_0 + A_0^TL_0 + L$.

We then have:

*If there exist $L_0, L, L_1, \ldots, L_m$ and $K$ such that $W$ as above is negative definite, then system (2) is stable.*

The proof is along the lines of the one for Lyapunov-Krasovskii functionals in reference [4].

We now show that finding $L_0, L, L_1, \ldots, L_m$ and $K$ such that $W$ as above is negative definite can be posed as a convex feasibility problem. Our manipulations are based on a recent result on the parametrization of state-feedback controllers [3].

We multiply every block entry of $W$ on the left and on the right by $L_0^{-1}$ and set $M_0 = L_0^{-1}$, $M_i = L_0^{-1}L_iL_0^{-1}$, $i = 1, \ldots, m$, $M = L_0^{-1}LL_0^{-1}$ and $Y = KL_0^{-1}$, to obtain a new matrix $X$ given by

$$
\begin{bmatrix}
\tilde{N} & -BY & A_1M_0 & \cdots & A_mM_0 \\
-YB^T & M_1 - M & 0 & \cdots & 0 \\
M_0A_1^T & 0 & M_2 - M_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_0A_m^T & 0 & 0 & \cdots & -M_m
\end{bmatrix},
$$

where $\tilde{N} = A_0M_0 + M_0A_0^T + M$.

We then have: \textit{W} < 0 if and only if $X < 0$.

$X$ is a linear function of $M_0, M_1, \ldots, M_m$, $M$ and $Y$, and therefore therefore the set

$$\Psi = \{X \mid X < 0\}$$

is convex in these variables. Checking its non-emptiness can then be done via a convex feasibility program.

There exist several methods for solving this convex feasibility problem. In [6], Skorodinskii proposes the use of the ellipsoid algorithm [1]. There have been recent advances in convex programming which promise much faster algorithms [5, 2].

**References**


