On Computing the Worst-Case Peak Gain of Linear Systems*

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Abstract

We present simple upper and lower bounds for the \( \ell^\infty \)-gain of finite-dimensional discrete-time linear time-invariant systems. We use these bounds to derive upper and lower bounds for the worst-case \( \ell^\infty \)-gain of discrete-time systems with diagonal perturbations.

1 Bounds for the \( \ell^\infty \)-gain

We consider a stable, finite-dimensional discrete-time linear time-invariant (LTI) system, described by the state equations

\[
\begin{align*}
    x(k+1) &= Ax(k) + bu(k), \quad x(0) = 0, \\
    y(k) &= cx(k) + du(k),
\end{align*}
\]

(1)

where the input \( u(k) \in \mathbb{R} \), the output \( y(k) \in \mathbb{R} \), and the state \( x(k) \in \mathbb{R}^n \). We assume that \( \{A,b,c,d\} \) is minimal. The \( \ell^\infty \)-gain of system (1), which is the largest possible peak value of the output \( y \) over all possible inputs \( u \) with a peak value of at most one, is just the \( \ell^1 \)-norm of the impulse response:

\[
\sup\limits_{\|u\|_\infty > 0} \frac{\|y\|_\infty}{\|u\|_\infty} = \|h\|_1 = \sum_{k \geq 0} |h(k)|,
\]

where \( h(k) \) is the impulse response of system (1). \( \|u\|_\infty \) stands for the \( \ell^\infty \)-norm of the sequence \( u \), and equals \( \sup_{k \geq 0} |u(k)| \).

\( \|h\|_1 \) is usually approximated by summing only a finite, typically large (say \( N \)) number of terms:

\[
S_N = \sum_{k=0}^{N} |h(k)| \leq \|h\|_1.
\]

Obviously, \( S_N \) is a lower bound for \( \|h\|_1 \), and increases monotonically to \( \|h\|_1 \) with increasing \( N \). The ‘error’ \( \|h\|_1 - S_N \) is just the \( \ell^1 \) norm of the tail, \( \sum_{k>N} |h(k)| \). Many simple bounds on this error are possible.

The first purpose of this note is to present more sophisticated, and in many cases, substantially better bounds for the \( \ell^1 \)-norm of the tail. These bounds, based on Theorem 2 of [3], are given by

\[
\begin{align*}
\sigma_1(W_o^{1/2} A^N W_c^{1/2}) &\leq \sum_{k>N} |h(k)| \quad (2) \\
2 \sum_{i=1}^n \sigma_i(W_o^{1/2} A^N W_c^{1/2}) &\geq \sum_{k>N} |h(k)|
\end{align*}
\]

for all \( N \geq 0 \), where \( W_c \) and \( W_o \) are the controllability and observability Gramians respectively of system (1), \( \sigma_1(P), \sigma_2(P), \ldots, \sigma_n(P) \) are the singular values of \( P \) in decreasing order.) Based on (2), we may compute \( \|h\|_1 \) to within any desired accuracy. We refer the reader to [1] for details.

2 Bounds for the worst-case \( \ell^\infty \)-gain

We now combine the results of the previous section with results from [4] to derive bounds for the worst-case \( \ell^\infty \)-gain of discrete-time LTI systems with diagonal uncertainty. We consider the system shown in Figure 1: \( H \) is a stable discrete-time LTI plant. \( \Delta_1, \Delta_2, \ldots, \Delta_m \) are scalar LTI perturbations that act on the system.

We let \( \delta_i \) denote the impulse response of perturbation \( \Delta_i \), \( h_{00}, h_{10}, \) and \( h_{0i} \) denote the open-loop impulse responses from \( w \) to \( z \), \( w \) to \( y_1 \) and \( u_i \) to \( z \) respectively, and \( h_{cl}(\Delta) \), the closed-loop impulse response from \( w \) to \( z \).
section 1 about computing $\ell^\infty$-gains apply here as well. We may however use the fact that $M$ has nonnegative entries to derive bounds on $L_{wc}$ based on the bounds for the entries of $M$:

**Theorem 1** Let $\alpha_{ij}^N$ and $\beta_{ij}^N$ be lower and upper bounds for $\|h_{ij}\|_1$ computed using (2) for some $N > 0$. Let $M_{lb}^N$ and $M_{ub}^N$ be matrices with $(i, j)$-entry $\alpha_{ij}^N$ and $\beta_{ij}^N$ respectively $(i, j = 0, 1, \ldots, m)$. Then

$$L_{lb}^N = \Phi(M_{lb}^N) = \sup\{\gamma \mid \rho(D_\gamma M_{lb}^N D_\gamma) \geq 1\},$$

and

$$L_{ub}^N = \Phi(M_{ub}^N) = \sup\{\gamma \mid \rho(D_\gamma M_{ub}^N D_\gamma) \geq 1\},$$

are lower and upper bounds respectively for $L_{wc}$, i.e. $L_{lb}^N \leq L_{wc} \leq L_{ub}^N$.

$L_{lb}^N$ and $L_{ub}^N$ may be computed easily, using the following fact. For the $(m + 1) \times (m + 1)$ matrix $M$, with blocks as in equation (4), if $\rho(M^{(22)}) \geq 1$, we have $\Phi(M) = \infty$. Otherwise,

$$\Phi(M) = M^{(11)} + M^{(12)}(I - M^{(22)})^{-1} M^{(21)}.$$ 

We refer the reader to [1] for details.

**References**


