

Automatic Differentiation of Functional Programs and its use for Probabilistic Programming

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Purdue University

TTI-C
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Part of this talk covers joint work with Barak A. Pearlmutter.

The Essence

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(define (f x) 2x3)
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`(define (f x) 2x3)` $\rightsquigarrow$ `(define (f' x) 6x2)`

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need flow analysis

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need reflective transformation of closure bodies
want transformation done at compile time
need **polyvariant** flow analysis

Nesting

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(sqrt (sqrt x))
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$$\max_x \min_y f(x, y)$$

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Stuart Issett for The New York Times



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The Essence of Forward-Mode AD

$$f(c + \varepsilon) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}\varepsilon + \frac{f''(c)}{2!}\varepsilon^2 + \dots + \frac{f^{(i)}(c)}{i!}\varepsilon^i + \dots$$

Taylor, B. (1715). *Methodus Incrementorum Directa et Inversa*. London.

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To compute $\mathcal{D} f c$:

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To compute $\mathcal{D} f c$:

- evaluate f

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$(\mathcal{D} f)$ is $\mathcal{O}(1)$ relative to f (both space and time).

Arithmetic on Complex Numbers

$$a + bi$$

Hamilton, W. R. (1837). *Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time*. Transactions of the Royal Irish Academy, **17**(1):293–422.

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Clifford, W. K. (1873). *Preliminary Sketch of Bi-quaternions*. Proceedings of the London Mathematical Society, **4**:381–95.

Arithmetic on Dual Numbers

$$x + x'\varepsilon$$

$$\varepsilon^2 = 0, \text{ but } \varepsilon \neq 0$$

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$$\langle x, x' \rangle$$

$$\langle x_1, x'_1 \rangle + \langle x_2, x'_2 \rangle = \langle (x_1 + x_2), (x'_1 + x'_2) \rangle$$

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Dynamic Overloading: SCMUTILS

```
(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define +
  (let ((+ +))
    (lambda (x1 x2)
      (make-bundle (+ (primal x1) (primal x2))
                    (+ (tangent x1) (tangent x2))))))

(define *
  (let ((+ +) (* *))
    (lambda (x1 x2)
      (make-bundle (* (primal x1) (primal x2))
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(define ((D f) x) (tangent (f (make-bundle x 1))))
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Convenient

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Convenient but **slow**

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(define (f x) (* 2 (* x (* x x))))

(D f)
(D (D f))
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```

Convenient but **slow**

Dynamic Overloading: SCMUTILS

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(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define ((D f) x)
  (fluid-let ((+ (lambda (x1 x2)
                    (make-bundle (+ (primal x1) (primal x2))
                                   (+ (tangent x1) (tangent x2))))))
    (* (lambda (x1 x2)
        (make-bundle (* (primal x1) (primal x2))
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                        (* (tangent x1) (primal x2))))))
      (tangent (f (make-bundle x 1)))))

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( $\mathcal{D}$  ( $\mathcal{D}$  f))
( $\mathcal{D}$  (lambda (x) ... ( $\mathcal{D}$  (lambda (y) ...) ...) ...))
```

Convenient but **slow**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Preprocessor: ADIFOR and TAPENADE

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Fast

Preprocessor: ADIFOR and TAPENADE

```
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f = 2.0d0*x*x*x
end
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```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast but **inconvenient**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

AD_TOP = f

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast but **inconvenient**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
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function f(x)
double precision x, f
f = 2.0d0*x*x*x
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AD_TOP = f
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Preprocessor: ADIFOR and TAPENADE

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function f(x)
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AD_TOP = f
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AD_TOP = gf
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Fast but **inconvenient**

Preprocessor: ADIFOR and TAPENADE

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function f(x)
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```

```
AD_TOP = gf
AD_IVARS = x, gx
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```

```
function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
gresult = 6.0d0*x*x*gx
ggresult = 6.0d0*x*x*ggx+12.0d0*x*gx*gx
end
```

Fast but **inconvenient**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
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AD_TOP = f
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function gf(x, gx, gresult)
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gf = 2.0d0*x*x*x
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AD_TOP = gf
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function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
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ggf = 2.0d0*x*x*x
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function f(x)
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AD_TOP = f
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AD_TOP = gf
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function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
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ggf = 2.0d0*x*x*x
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```

Fast but **inconvenient**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
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```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = gf
AD_IVARS = x, gx
AD_DVARS = gf, gresult
AD_PREFIX = h
```

```
function hgf(x, hx, gx, hgx, gresult, hgresult, hresult)
double precision x, hx, gx, hgx, hgf, hresult, gresult, hgresult
hgf = 2.0d0*x*x*x
hresult = 6.0d0*x*x*hx
gresult = 6.0d0*x*x*gx
hgresult = 6.0d0*x*x*hgx+12.0d0*x*gx*hx
end
```

Fast but **inconvenient**

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}  
double x;  
... f(x) ...
```

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double f(double x) {return 2*x*x*x;}  
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```
F<double> f(F<double> x) {return 2*x*x*x;}  
F<double> x;  
x.diff(0, 1);  
... f(x).d(0) ...
```

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```
F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
F<F<double> > x;  
x.diff(0, 1);  
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double x;  
... f(x) ...
```

```
F<double> f(F<double> x) {return 2*x*x*x;}  
F<double> x;  
x.diff(0, 1);  
... f(x).d(0) ...
```

```
F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
F<F<double> > x;  
x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
... f(x).d(0).d(0) ...
```

Slow

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}  
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F<double> f(F<double> x) {return 2*x*x*x;}  
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```

Slow and **inconvenient**

Static Overloading: FADBAD++

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Slow and inconvenient

Static Overloading: FADBAD++

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Slow and **inconvenient**

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Slow and inconvenient

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Slow and inconvenient

Static Overloading: FADBAD++

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Slow and **inconvenient**

Static Overloading: FADBAD++

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F<F<double> > x;  
x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
... f(x).d(0).d(0) ...
```

```
template <typename T>  
T f(T x) {return 2*x*x*x;}  
T x;
```

Slow and **inconvenient**

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}  
double x;  
... f(x) ...
```

```
F<double> f(F<double> x) {return 2*x*x*x;}  
F<double> x;  
x.diff(0, 1);  
... f(x).d(0) ...
```

```
F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
F<F<double> > x;  
x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
... f(x).d(0).d(0) ...
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template <typename T>  
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T x;
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Slow and **inconvenient**

Static Overloading: FADBAD++

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double f(double x) {return 2*x*x*x;}  
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... f(x).d(0).d(0) ...
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```
template <typename T>  
T f(T x) {return 2*x*x*x;}  
T x;
```

Slow and **inconvenient**

Our API for Functional Forward AD

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$

Our API for Functional Forward AD

$$\text{bundle} : \mathbb{R}^n \times \overline{\mathbb{R}^n} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^n})$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

Our API for Functional Forward AD

`bundle` : $\mathbb{R}^n \times \overline{\mathbb{R}^h} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$

`primal` : $(\textcolor{red}{\mathbb{R}^n} \triangleright \overline{\mathbb{R}^h}) \rightarrow \textcolor{red}{\mathbb{R}^n}$

`tangent` : $(\mathbb{R}^n \triangleright \textcolor{red}{\overline{\mathbb{R}^h}}) \rightarrow \textcolor{red}{\overline{\mathbb{R}^h}}$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$

Our API for Functional Forward AD

$$\text{bundle} : \mathbb{R}^n \times \overline{\mathbb{R}^h} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$$

$$\text{primal} : (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow \mathbb{R}^n$$

$$\text{tangent} : (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow \overline{\mathbb{R}^h}$$

$$j^* : (\mathbb{R}^n \rightarrow \mathbb{R}^m) \rightarrow ((\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow (\mathbb{R}^m \triangleright \overline{\mathbb{R}^h}))$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$
 j^* maps a **function** to its *push forward*

Our API for Functional Forward AD

$$\text{bundle} : \mathbb{R}^n \times \overline{\mathbb{R}^h} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$$

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Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$
 $\mathbf{j}^*$ maps a function to its *push forward*

Our API for Functional Forward AD

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Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$
 $\mathbf{j}^*$ maps a function to its *push forward*

Our API for Functional Forward AD

`bundle` : $\tau \times \overline{\tau'} \rightarrow (\tau \triangleright \overline{\tau'})$
`primal` : $(\tau \triangleright \overline{\tau'}) \rightarrow \tau$
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`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \triangleright \overline{\tau'_1}) \rightarrow (\tau_2 \triangleright \overline{\tau'_2}))$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$
`j*` maps a function to its *push forward*

Generalize to arbitrary types

Our API for Functional Forward AD

`bundle` : $\tau \times \overline{\tau} \rightarrow (\tau \triangleright \overline{\tau})$
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`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \triangleright \overline{\tau}_1) \rightarrow (\tau_2 \triangleright \overline{\tau}_2))$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Our API for Functional Forward AD

$$\begin{aligned}\text{bundle} &: \tau \times \overline{\tau} \rightarrow (\tau \triangleright \overline{\tau}) \\ \text{primal} &: (\tau \triangleright \overline{\tau}) \rightarrow \tau \\ \text{tangent} &: (\tau \triangleright \overline{\tau}) \rightarrow \overline{\tau} \\ \text{j}^* &: (\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \triangleright \overline{\tau}_1') \rightarrow (\tau_2 \triangleright \overline{\tau}_2'))\end{aligned}$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

j^* maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overline{\tau}$ as $\overrightarrow{\tau}$

Our API for Functional Forward AD

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Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

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Sometimes write $\mathbf{j}^*$ as $\overrightarrow{\mathcal{J}}$

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$\text{bundle} : \tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$
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 $\text{j}^* : (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

j^* maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write j^* as $\mathcal{J}$

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

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(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
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Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

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Generalize to arbitrary types

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Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

Convenient

Our API for Functional Forward AD

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```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
(D f)  
(D (D f))
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

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Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

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Our API for Functional Forward AD

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`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
(D f)  
(D (D f))  
(D (lambda (x) ... (D (lambda (y) ...) ...) ...) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as `J`

Convenient

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Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

What is `(j* j*)`?

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

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Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

What is $(j^* \ j^*)$?

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`primal` : $\overrightarrow{\tau} \rightarrow \tau$

`tangent` : $\overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
(D f)  
(D (D f))  
(D (lambda (x) ... (D (lambda (y) ...) ...) ...)) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as `J`

What is $(j^* \ j^*)$?

Convenient and **fast**

A property

A property

$x : \mathbb{R}^n$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$((\mathcal{J} f) x)[i, j] = \frac{\partial f(x)[i]}{\partial x[j]}$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$((\mathcal{J} f) x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$$

$$((\mathcal{J} f) x) : \mathbb{R}^{m \times n}$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$((\mathcal{J} f) x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$$

$$((\mathcal{J} f) x) : \mathbb{R}^{m \times n}$$

$$((\mathcal{J} f) x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$$

A property

$$\begin{aligned} x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\ ((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\ (\text{primal } ((j^* f) (\text{bundle } x \ \overline{x})))) &= (f \ x) \end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'}\end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'})\end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x)))\end{aligned}$$

A property

$x : \mathbb{R}^n$ $\overline{x'} : \mathbb{R}^n$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J} f) x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $((\mathcal{J} f) x) : \mathbb{R}^{m \times n}$ $((\mathcal{J} f) x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) = (f x)$

$(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) = ((\mathcal{J} f) x) \times \overline{x'}$

$(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) = (((\mathcal{J} f) x) \overline{x'})$

$((j^* f) x) = (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x)))$

rearrangement function: $(\forall i)(\exists j)(f x)[i] = x[j]$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j] \\f &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m\end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j] \\f &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\0/1 \text{ matrix, every row has exactly one } &1\end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= ((\mathcal{J} f) x) \times \overline{x} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= (((\mathcal{J} f) x) \overline{x}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j]\end{aligned}$$

$$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$$

0/1 matrix, every row has exactly one 1

$$((\mathcal{J} f) x)$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= ((\mathcal{J} f) x) \times \overline{x} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= (((\mathcal{J} f) x) \overline{x}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j]\end{aligned}$$

$$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$$

0/1 matrix, every row has exactly one 1

$$((\mathcal{J} f) x) = \frac{\partial f(x)[i]}{\partial x[j]}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= ((\mathcal{J} f) x) \times \overline{x} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= (((\mathcal{J} f) x) \overline{x}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j]\end{aligned}$$

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$$((\mathcal{J} f) x) = \frac{\partial f(x)[i]}{\partial x[j]} = \begin{cases} 1 & \text{when } (f x)[i] = x[j] \\ 0 & \text{otherwise} \end{cases}$$

A property

$$\begin{aligned}
 x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\
 ((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\
 (\text{primal } ((j^* f) (\text{bundle } x \overline{x}))) &= (f x) \\
 (\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= ((\mathcal{J} f) x) \times \overline{x} \\
 (\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= (((\mathcal{J} f) x) \overline{x}) \\
 ((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\
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when f is a rearrangement function

$$((j^* f) x) = (\text{bundle } (f (\text{primal } x)) (f (\text{tangent } x)))$$

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A property

$$\begin{aligned}x &: \tau_1 & \overline{x} &: \tau_1 & f &: \tau_1 \rightarrow \tau_2 \\((\mathcal{J} f) x)[i, j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \tau_1 \xrightarrow{L} \tau_2 \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= ((\mathcal{J} f) x) \times \overline{x} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= (((\mathcal{J} f) x) \overline{x}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j]\end{aligned}$$

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What is the tangent of $\#t$?

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What if we take $\overline{\#t} = \#f$?

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when f is a rearrangement function

$((j * f) \ x) = (\text{bundle } (f \ (\text{primal } x)) \ (f \ (\text{tangent } x)))$

$f : (\#t \ x \ y) \mapsto (\#t \ x \ y) \quad \text{but} \quad f : (\#f \ x \ y) \mapsto (\#f \ y \ x)$

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$$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$$

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$$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$$

$$= (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{y} \ \overline{x}))$$

What is the tangent of $\#t$?

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when f is a rearrangement function

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f is a rearrangement function

$$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$$

$$= (\text{bundle } (\#t \ \textcolor{red}{x} \ y) \ (\#f \ \textcolor{red}{\overline{y}} \ \overline{x}))$$

What is the tangent of $\#t$?

What if we take $\overline{\#t} = \#f$?

when f is a rearrangement function

$((j * f) \ x) = (\text{bundle } (f \ (\text{primal } x)) \ (f \ (\text{tangent } x)))$

$f : (\#t \ x \ y) \mapsto (\#t \ x \ y) \quad \text{but} \quad f : (\#f \ x \ y) \mapsto (\#f \ y \ x)$

f is a rearrangement function

$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$

$= (\text{bundle } (\#t \ x \ \textcolor{red}{y}) \ (\#f \ \overline{y} \ \textcolor{red}{\overline{x}}))$

What is the tangent of $\#t$?

What if we take $\overline{\#t} = \#f$?

when f is a rearrangement function

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$$f : (\#t \ x \ y) \mapsto (\#t \ x \ y) \quad \text{but} \quad f : (\#f \ x \ y) \mapsto (\#f \ y \ x)$$

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$$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$$

$$= (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{y} \ \overline{x}))$$

Problem avoided if we take $\overline{\#t} = \#t$

What is $(j^* \ j^*)$?

What is $(j^* \ j^*)$?

when f is a rearrangement function

$$((j^* \ f) \ x) = (\text{bundle } (f \ (\text{primal } x)) \ (f \ (\text{tangent } x)))$$

What is $(j^* \circ j^*)$?

when f is a rearrangement function

$$((j^* \circ f) \circ x) = (\text{bundle } (f \circ (\text{primal } x)) \circ (f \circ (\text{tangent } x)))$$

bundle , primal , tangent , and j^* are rearrangement functions

What is $(j^* \ j^*)$?

when f is a rearrangement function

$$((j^* \ f) \ x) = (\text{bundle } (f \ (\text{primal } x)) \ (f \ (\text{tangent } x)))$$

bundle , primal , tangent , and j^* are rearrangement functions

$$((j^* \ \text{bundle}) \ x) = (\text{bundle } (\text{bundle } (\text{primal } x)) \ (\text{bundle } (\text{tangent } x)))$$

$$((j^* \ \text{primal}) \ x) = (\text{bundle } (\text{primal } (\text{primal } x)) \ (\text{primal } (\text{tangent } x)))$$

$$((j^* \ \text{tangent}) \ x) = (\text{bundle } (\text{tangent } (\text{primal } x)) \ (\text{tangent } (\text{tangent } x)))$$

$$((j^* \ j^*) \ x) = (\text{bundle } (j^* \ (\text{primal } x)) \ (j^* \ (\text{tangent } x)))$$

A (Not So) Brief Tutorial on AD

$$z = g(f(x))$$

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$$\begin{aligned} z &= g(f(x)) \\ &= (f \circ g)(x) \end{aligned}$$

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$$\begin{aligned} z &= g(f(x)) \\ &= (f \circ g)(x) \end{aligned}$$

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$$\begin{aligned} z &= g(f(x)) \\ &= (f \circ g)(x) \end{aligned}$$

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$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$$

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$$\begin{aligned} z &= g(f(x)) \\ &= (f \circ g)(x) \end{aligned}$$

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$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$$

$$\mathcal{D} (f \circ g) x = (\mathcal{D} g y) \times (\mathcal{D} f x)$$

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$$f = f_1 \circ \cdots \circ f_n$$

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$$f = f_1 \circ \cdots \circ f_n$$

$$\mathbf{x}_1 = f_1 \mathbf{x}_0$$

$$\vdots$$

$$\mathbf{x}_n = f_n \mathbf{x}_{n-1}$$

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$$f = f_1 \circ \cdots \circ f_n$$

$$\mathbf{x}_1 = f_1 \mathbf{x}_0$$

$$\vdots$$

$$\mathbf{x}_n = f_n \mathbf{x}_{n-1}$$

$$\mathcal{J} f \mathbf{x}_0 = (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_1 \mathbf{x}_0)$$

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$$f = f_1 \circ \cdots \circ f_n$$

$$\mathbf{x}_1 = f_1 \mathbf{x}_0$$

$$\vdots$$

$$\mathbf{x}_n = f_n \mathbf{x}_{n-1}$$

$$\begin{aligned}\mathcal{J} f \mathbf{x}_0 &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_1 \mathbf{x}_0) \\ (\mathcal{J} f \mathbf{x}_0)^\top &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J} f_n \mathbf{x}_{n-1})^\top\end{aligned}$$

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$$\overline{\mathbf{X}}'_n = \mathcal{J} f \mathbf{x}_0$$

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$$\begin{aligned}\overline{\mathbf{X}}'_n &= \mathcal{J} f \mathbf{x}_0 \\ &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_2 \mathbf{x}_1) \times (\mathcal{J} f_1 \mathbf{x}_0)\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{X}}_n &= \mathcal{J} f \mathbf{x}_0 \\ &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_2 \mathbf{x}_1) \times (\mathcal{J} f_1 \mathbf{x}_0)\end{aligned}$$

$$\overline{\mathbf{X}}_1 = (\mathcal{J} f_1 \mathbf{x}_0)$$

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$$\begin{aligned}\overline{\mathbf{X}}_n &= \mathcal{J} f \mathbf{x}_0 \\ &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_2 \mathbf{x}_1) \times (\mathcal{J} f_1 \mathbf{x}_0)\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{X}}_1 &= (\mathcal{J} f_1 \mathbf{x}_0) \\ \overline{\mathbf{X}}_2 &= (\mathcal{J} f_2 \mathbf{x}_1) \times \overline{\mathbf{X}}_1\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{X}}_n' &= \mathcal{J} f \mathbf{x}_0 \\ &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_2 \mathbf{x}_1) \times (\mathcal{J} f_1 \mathbf{x}_0)\end{aligned}$$

$$\overline{\mathbf{X}}_1' = (\mathcal{J} f_1 \mathbf{x}_0)$$

$$\overline{\mathbf{X}}_2' = (\mathcal{J} f_2 \mathbf{x}_1) \times \overline{\mathbf{X}}_1'$$

$$\vdots$$

$$\overline{\mathbf{X}}_n' = (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \overline{\mathbf{X}}_{n-1}'$$

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$$\overline{\mathbf{X}_0} = (\mathcal{J} f \mathbf{x}_0)^\top$$

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$$\begin{aligned}\overline{\mathbf{X}_0} &= (\mathcal{J} f \mathbf{x}_0)^\top \\ &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times (\mathcal{J} f_n \mathbf{x}_{n-1})^\top\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{X}}_0 &= (\mathcal{J} f \mathbf{x}_0)^\top \\ &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times (\mathcal{J} f_n \mathbf{x}_{n-1})^\top\end{aligned}$$

$$\overline{\mathbf{X}_{n-1}} = (\mathcal{J} f_n \mathbf{x}_{n-1})^\top$$

A (Not So) Brief Tutorial on AD

$$\begin{aligned}\overline{\mathbf{X}}_0 &= (\mathcal{J} f \mathbf{x}_0)^\top \\ &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times (\mathcal{J} f_n \mathbf{x}_{n-1})^\top\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{X}}_{n-1} &= (\mathcal{J} f_n \mathbf{x}_{n-1})^\top \\ \overline{\mathbf{X}}_{n-2} &= (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times \overline{\mathbf{X}}_{n-1}\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{X}}_0 &= (\mathcal{J} f \mathbf{x}_0)^\top \\ &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times (\mathcal{J} f_n \mathbf{x}_{n-1})^\top\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{X}}_{n-1} &= (\mathcal{J} f_n \mathbf{x}_{n-1})^\top \\ \overline{\mathbf{X}}_{n-2} &= (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times \overline{\mathbf{X}}_{n-1} \\ &\vdots \\ \overline{\mathbf{X}}_0 &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \overline{\mathbf{X}}_1\end{aligned}$$

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$$\begin{array}{ll} \overline{\mathbf{X}}_1 &= (\mathcal{J} f_1 \mathbf{x}_0) \\ \overline{\mathbf{X}}_2 &= (\mathcal{J} f_2 \mathbf{x}_1) \times \overline{\mathbf{X}}_1 \\ &\vdots \\ \overline{\mathbf{X}}_n &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \overline{\mathbf{X}}_{n-1} \end{array} \qquad \begin{array}{ll} \overline{\mathbf{X}}_{n-1} &= (\mathcal{J} f_n \mathbf{x}_{n-1})^\top \\ \overline{\mathbf{X}}_{n-2} &= (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times \overline{\mathbf{X}}_{n-1} \\ &\vdots \\ \overline{\mathbf{X}}_0 &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \overline{\mathbf{X}}_1 \end{array}$$

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$$\begin{array}{ll} \overline{\mathbf{X}}_1 &= (\mathcal{J} f_1 \mathbf{x}_0) \\ \overline{\mathbf{X}}_2 &= (\mathcal{J} f_2 \mathbf{x}_1) \times \overline{\mathbf{X}}_1 \\ &\vdots \\ \overline{\mathbf{X}}_n &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \overline{\mathbf{X}}_{n-1} \end{array} \qquad \begin{array}{ll} \overline{\mathbf{X}}_{n-1} &= (\mathcal{J} f_n \mathbf{x}_{n-1})^\top \\ \overline{\mathbf{X}}_{n-2} &= (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times \overline{\mathbf{X}}_{n-1} \\ &\vdots \\ \overline{\mathbf{X}}_0 &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \overline{\mathbf{X}}_1 \end{array}$$

A (Not So) Brief Tutorial on AD

$$\begin{array}{ll} \overline{\mathbf{X}}_1 &= (\mathcal{J} f_1 \mathbf{x}_0) \\ \overline{\mathbf{X}}_2 &= (\mathcal{J} f_2 \mathbf{x}_1) \times \overline{\mathbf{X}}_1 \\ &\vdots \\ \overline{\mathbf{X}}_n &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \overline{\mathbf{X}}_{n-1} \end{array} \qquad \begin{array}{ll} \overline{\mathbf{X}}_{n-1} &= (\mathcal{J} f_n \mathbf{x}_{n-1})^\top \\ \overline{\mathbf{X}}_{n-2} &= (\mathcal{J} f_{n-1} \mathbf{x}_{n-2})^\top \times \overline{\mathbf{X}}_{n-1} \\ &\vdots \\ \overline{\mathbf{X}}_0 &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \overline{\mathbf{X}}_1 \end{array}$$

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$$\overline{\mathbf{x}}_n' = (\mathcal{J} f \mathbf{x}_0) \times \overline{\mathbf{x}}_0'$$

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$$\begin{aligned}\overline{\mathbf{x}}_n' &= (\mathcal{J} f \mathbf{x}_0) \times \overline{\mathbf{x}}_0' \\ &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_1 \mathbf{x}_0) \times \overline{\mathbf{x}}_0'\end{aligned}$$

A (Not So) Brief Tutorial on AD

$$\begin{aligned}\overline{\mathbf{x}}'_n &= (\mathcal{J} f \mathbf{x}_0) \times \overline{\mathbf{x}}'_0 \\ &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_1 \mathbf{x}_0) \times \overline{\mathbf{x}}'_0\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{x}}'_1 &= (\mathcal{J} f_1 \mathbf{x}_0) \times \overline{\mathbf{x}}'_0 \\ &\vdots \\ \overline{\mathbf{x}}'_n &= (\mathcal{J} f_n \mathbf{x}_{n-1}) \times \overline{\mathbf{x}}'_{n-1}\end{aligned}$$

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$$\overline{\mathbf{x}}_0 = (\mathcal{J} f \mathbf{x}_0)^\top \times \overline{\mathbf{x}}_n$$

A (Not So) Brief Tutorial on AD

$$\begin{aligned}\overline{\mathbf{x}}_0 &= (\mathcal{J} f \mathbf{x}_0)^\top \times \overline{\mathbf{x}}_n \\ &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J} f_n \mathbf{x}_{n-1})^\top \times \overline{\mathbf{x}}_n\end{aligned}$$

A (Not So) Brief Tutorial on AD

$$\begin{aligned}\overline{\mathbf{x}}_0 &= (\mathcal{J} f \mathbf{x}_0)^\top \times \overline{\mathbf{x}}_n \\ &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J} f_n \mathbf{x}_{n-1})^\top \times \overline{\mathbf{x}}_n\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{x}}_{n-1} &= (\mathcal{J} f_n \mathbf{x}_{n-1})^\top \times \overline{\mathbf{x}}_n \\ &\vdots \\ \overline{\mathbf{x}}_0 &= (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \overline{\mathbf{x}}_1\end{aligned}$$

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$$\mathbf{y} = \mathbf{A} \times \mathbf{x}$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{A} \times \mathbf{x} \\ &= f(\mathbf{x})\end{aligned}$$

A (Not So) Brief Tutorial on AD

$$\begin{aligned}\mathbf{y} &= \mathbf{A} \times \mathbf{x} \\ &= f(\mathbf{x})\end{aligned}$$

$$\overline{\mathbf{x}}'_n = (\mathcal{J} f \mathbf{x}_0) \times \overline{\mathbf{x}}'_0$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{A} \times \mathbf{x} \\ &= f(\mathbf{x})\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{x}}'_n &= (\mathcal{J} f \mathbf{x}_0) \times \overline{\mathbf{x}}'_0 \\ &= \overline{f'} \mathbf{x}_0 \overline{\mathbf{x}}'_0\end{aligned}$$

A (Not So) Brief Tutorial on AD

$$\begin{aligned}\mathbf{y} &= \mathbf{A} \times \mathbf{x} \\ &= f(\mathbf{x})\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{x}}'_n &= (\mathcal{J} f \mathbf{x}_0) \times \overline{\mathbf{x}}'_0 \\ &= \overline{f'} \mathbf{x}_0 \overline{\mathbf{x}}'_0\end{aligned}$$

$$\overline{\mathbf{x}}_0 = (\mathcal{J} f \mathbf{x}_0)^\top \times \overline{\mathbf{x}}_n$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{A} \times \mathbf{x} \\ &= f(\mathbf{x})\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{x}}'_n &= (\mathcal{J} f \mathbf{x}_0) \times \overline{\mathbf{x}}'_0 \\ &= \overline{f'} \mathbf{x}_0 \overline{\mathbf{x}}'_0\end{aligned}$$

$$\begin{aligned}\overleftarrow{\mathbf{x}}_0 &= (\mathcal{J} f \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{x}}_n \\ &= \overleftarrow{f} \mathbf{x}_0 \overleftarrow{\mathbf{x}}_n\end{aligned}$$

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$$\overline{\mathbf{X}}_n[:,j] = \overline{f'} \mathbf{x}_0 \overline{\mathbf{e}}_j$$

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$$\begin{aligned}\overline{\mathbf{X}}_n[:,j] &= \overline{f'} \mathbf{x}_0 \overline{\mathbf{e}}_j \\ &= (\mathcal{J} f \mathbf{x}_0)[:,j]\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{X}}_n[j] &= \overline{f'} \mathbf{x}_0 \overline{\mathbf{e}}_j \\ &= (\mathcal{J} f \mathbf{x}_0)[j]\end{aligned}$$

$$\overleftarrow{\mathbf{X}}_0[i] = \overleftarrow{f'} \mathbf{x}_0 \overleftarrow{\mathbf{e}}_i$$

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$$\begin{aligned}\overline{\mathbf{X}}_n[j] &= \overline{f} \mathbf{x}_0 \overline{\mathbf{e}}_j \\ &= (\mathcal{J} f \mathbf{x}_0)[j]\end{aligned}$$

$$\begin{aligned}\overleftarrow{\mathbf{X}}_0[i] &= \overleftarrow{f} \mathbf{x}_0 \overleftarrow{\mathbf{e}}_i \\ &= (\mathcal{J} f \mathbf{x}_0)^\top[i]\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{X}}_n[j] &= \overline{f} \mathbf{x}_0 \overline{\mathbf{e}}_j \\ &= (\mathcal{J} f \mathbf{x}_0)[j]\end{aligned}$$

$$\begin{aligned}\overleftarrow{\mathbf{X}}_0[i] &= \overleftarrow{f} \mathbf{x}_0 \overleftarrow{\mathbf{e}}_i \\ &= (\mathcal{J} f \mathbf{x}_0)^\top[i] \\ &= (\mathcal{J} f \mathbf{x}_0)[i;]\end{aligned}$$

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$$\mathbf{y} = \mathbf{B} \times (\mathbf{A} \times \mathbf{x})$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x}) \\ &= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x}\end{aligned}$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x}) \\ &= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x} \\ &= g(f(\mathbf{x}))\end{aligned}$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x}) \\ &= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x} \\ &= g(f(\mathbf{x})) \\ &= (f \circ g)(\mathbf{x})\end{aligned}$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x}) \\ &= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x} \\ &= g(f(\mathbf{x})) \\ &= (f \circ g)(\mathbf{x})\end{aligned}$$

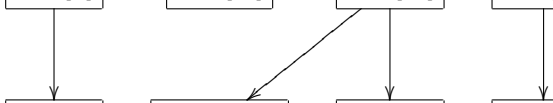
$$(\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_1 \mathbf{x}_0) = (\overline{f_1} \mathbf{x}_0) \circ \cdots \circ (\overline{f_n} \mathbf{x}_{n-1})$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x}) \\ &= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x} \\ &= g(f(\mathbf{x})) \\ &= (f \circ g)(\mathbf{x})\end{aligned}$$

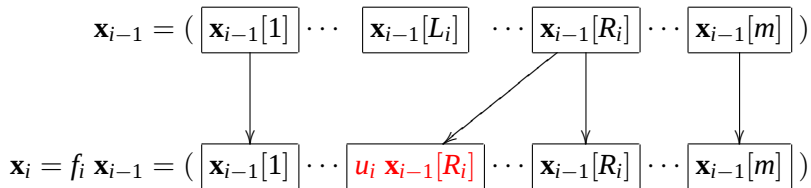
$$\begin{aligned}(\mathcal{J} f_n \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J} f_1 \mathbf{x}_0) &= (\overline{f_1} \mathbf{x}_0) \circ \cdots \circ (\overline{f_n} \mathbf{x}_{n-1}) \\ (\mathcal{J} f_1 \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J} f_n \mathbf{x}_{n-1})^\top &= (\overline{f_n} \mathbf{x}_{n-1}) \circ \cdots \circ (\overline{f_1} \mathbf{x}_0)\end{aligned}$$

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$$\mathbf{x}_{i-1} = (\boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{\mathbf{x}_{i-1}[L_i]} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]})$$

$$\mathbf{x}_i = f_i \mathbf{x}_{i-1} = (\boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{u_i \mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]})$$

$$\mathbf{x}[L_i] := u_i \mathbf{x}[R_i]$$

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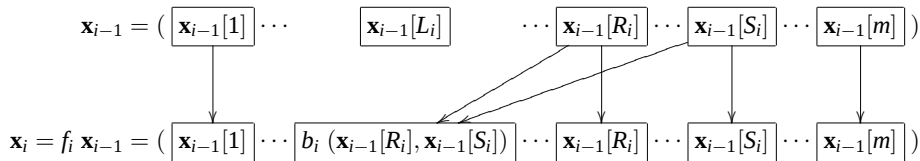
$$\mathbf{x}[L_i] := u_i \mathbf{x}[R_i]$$

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$$\mathbf{x}_{i-1} = (\boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{\mathbf{x}_{i-1}[L_i]} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]})$$
$$\mathbf{x}_i = f_i \mathbf{x}_{i-1} = (\boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{u_i \mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]})$$

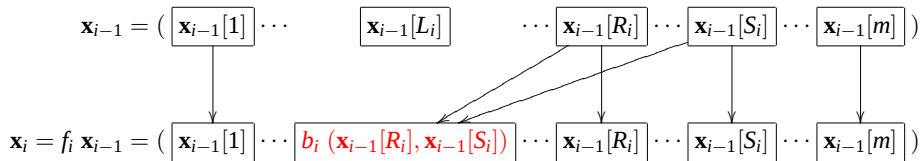
$$\mathbf{x}[L_i] := u_i \mathbf{x}[R_i]$$

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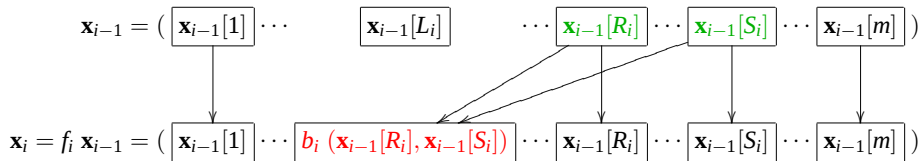
$$\mathbf{x}[L_i] := b(\mathbf{x}[R_i], \mathbf{x}[S_i])$$

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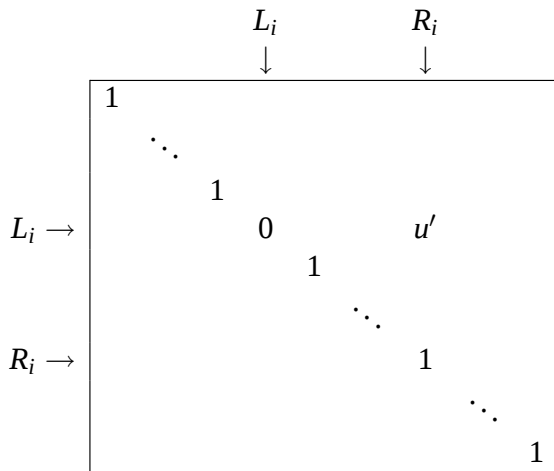
$$\mathbf{x}[L_i] := b(\mathbf{x}[R_i], \mathbf{x}[S_i])$$

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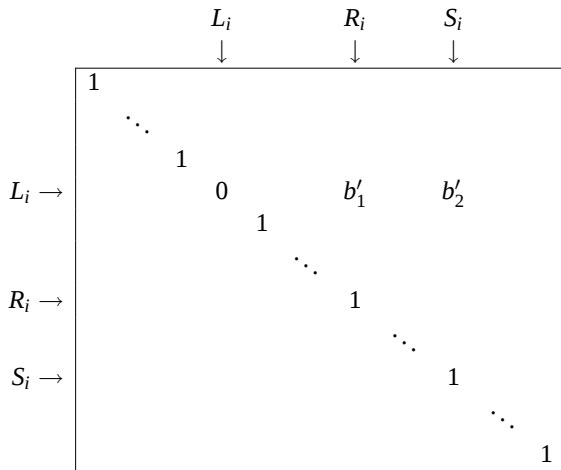
$$\mathbf{x}[L_i] := b(\mathbf{x}[R_i], \mathbf{x}[S_i])$$

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$$u' = \mathcal{D} u_i \mathbf{x}_{i-1}[R_i]$$

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$$b'_1 = \mathcal{D}_1 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])$$

$$b'_2 = \mathcal{D}_2 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])$$

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$$\overline{\mathbf{x}}'_i = \overline{f}_i \mathbf{x}_{i-1} \overline{\mathbf{x}}'_{i-1}$$

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$$\begin{aligned}\overline{\mathbf{x}}'_i &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}'_{i-1}} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1}) \times \overline{\mathbf{x}'_{i-1}}\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{x}}'_i &= \overline{f}_i \mathbf{x}_{i-1} \overline{\mathbf{x}}'_{i-1} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1}) \times \overline{\mathbf{x}}'_{i-1}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}}'_{i-1}[1] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[L_i - 1] \\ u' \times \overline{\mathbf{x}}'_{i-1}[R_i] \\ \overline{\mathbf{x}}'_{i-1}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[R_i] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & u' & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}}'_{i-1}[1] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[L_i - 1] \\ \overline{\mathbf{x}}'_{i-1}[L_i] \\ \overline{\mathbf{x}}'_{i-1}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[R_i] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[m] \end{pmatrix}$$

$$u' = \mathcal{D} u_i \mathbf{x}_{i-1}[R_i]$$

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$$\begin{aligned}\overline{\mathbf{x}}'_i &= \overline{f}_i \mathbf{x}_{i-1} \overline{\mathbf{x}}'_{i-1} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1}) \times \overline{\mathbf{x}}'_{i-1}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}}'_{i-1}[1] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[L_i - 1] \\ \textcolor{red}{u' \times \overline{\mathbf{x}}'_{i-1}[R_i]} \\ \overline{\mathbf{x}}'_{i-1}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[R_i] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & u' & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}}'_{i-1}[1] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[L_i - 1] \\ \overline{\mathbf{x}}'_{i-1}[L_i] \\ \overline{\mathbf{x}}'_{i-1}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[R_i] \\ \vdots \\ \overline{\mathbf{x}}'_{i-1}[m] \end{pmatrix}$$

$$u' = \mathcal{D} u_i \mathbf{x}_{i-1}[R_i]$$

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$$\begin{aligned}\overline{\mathbf{x}}_i' &= \overline{f}_i' \mathbf{x}_{i-1} \overline{\mathbf{x}}_{i-1}' \\ &= (\mathcal{J} \overline{f}_i' \mathbf{x}_{i-1}) \times \overline{\mathbf{x}}_{i-1}'\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}}_{i-1}'[1] \\ \vdots \\ \overline{\mathbf{x}}_{i-1}'[L_i - 1] \\ \textcolor{red}{u' \times \overline{\mathbf{x}}_{i-1}'[R_i]} \\ \overline{\mathbf{x}}_{i-1}'[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}}_{i-1}'[R_i] \\ \vdots \\ \overline{\mathbf{x}}_{i-1}'[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & u' & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}}_{i-1}'[1] \\ \vdots \\ \overline{\mathbf{x}}_{i-1}'[L_i - 1] \\ \overline{\mathbf{x}}_{i-1}'[L_i] \\ \overline{\mathbf{x}}_{i-1}'[L_i + 1] \\ \vdots \\ \textcolor{green}{\overline{\mathbf{x}}_{i-1}'[R_i]} \\ \vdots \\ \overline{\mathbf{x}}_{i-1}'[m] \end{pmatrix}$$

$$u' = \mathcal{D} u_i \mathbf{x}_{i-1}[R_i]$$

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$$\overline{\mathbf{x}}_i' = \overline{f}_i' \mathbf{x}_{i-1} \overline{\mathbf{x}}_{i-1}'$$

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$$\begin{aligned}\overline{\mathbf{x}}_i &= \overline{f_i'} \mathbf{x}_{i-1} \overline{\mathbf{x}_{i-1}} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}}\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{x}}_i &= \overline{f}_i' \mathbf{x}_{i-1} \overline{\mathbf{x}_{i-1}} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_{i-1}}[1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[L_i - 1] \\ b'_1 \times \overline{\mathbf{x}_{i-1}}[R_i] + b'_2 \times \overline{\mathbf{x}_{i-1}}[S_i] \\ \overline{\mathbf{x}_{i-1}}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[R_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[S_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & b'_1 & b'_2 \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_{i-1}}[1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[L_i - 1] \\ \overline{\mathbf{x}_{i-1}}[L_i] \\ \overline{\mathbf{x}_{i-1}}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[R_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[S_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[m] \end{pmatrix}$$

$$\begin{aligned}b'_1 &= \mathcal{D}_1 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\ b'_2 &= \mathcal{D}_2 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{x}}_i &= \overline{f}_i' \mathbf{x}_{i-1} \overline{\mathbf{x}_{i-1}} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_{i-1}}[1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[L_i - 1] \\ \textcolor{red}{b'_1 \times \overline{\mathbf{x}_{i-1}}[R_i] + b'_2 \times \overline{\mathbf{x}_{i-1}}[S_i]} \\ \overline{\mathbf{x}_{i-1}}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[R_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[S_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & b'_1 & b'_2 \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_{i-1}}[1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[L_i - 1] \\ \overline{\mathbf{x}_{i-1}}[L_i] \\ \overline{\mathbf{x}_{i-1}}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[R_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[S_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[m] \end{pmatrix}$$

$$b'_1 = \mathcal{D}_1 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])$$

$$b'_2 = \mathcal{D}_2 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])$$

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$$\begin{aligned}\overline{\mathbf{x}}_i &= \overline{f_i} \ \mathbf{x}_{i-1} \ \overline{\mathbf{x}_{i-1}} \\ &= (\mathcal{J} f_i \ \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_{i-1}}[1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[L_i - 1] \\ \textcolor{red}{b'_1 \times \overline{\mathbf{x}_{i-1}}[R_i] + b'_2 \times \overline{\mathbf{x}_{i-1}}[S_i]} \\ \overline{\mathbf{x}_{i-1}}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[R_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[S_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & b'_1 & b'_2 \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_{i-1}}[1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[L_i - 1] \\ \overline{\mathbf{x}_{i-1}}[L_i] \\ \overline{\mathbf{x}_{i-1}}[L_i + 1] \\ \vdots \\ \textcolor{green}{\overline{\mathbf{x}_{i-1}}[R_i]} \\ \vdots \\ \textcolor{green}{\overline{\mathbf{x}_{i-1}}[S_i]} \\ \vdots \\ \overline{\mathbf{x}_{i-1}}[m] \end{pmatrix}$$

$$b'_1 = \mathcal{D}_1 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])$$

$$b'_2 = \mathcal{D}_2 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])$$

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$$\overline{\mathbf{x}_{i-1}} = \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i}$$

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$$\begin{aligned}\overline{\mathbf{x}_{i-1}} &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1})^\top \times \overline{\mathbf{x}_i}\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{x}_{i-1}} &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1})^\top \times \overline{\mathbf{x}_i}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ 0 \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ u' \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & u' & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ \overline{\mathbf{x}_i}[L_i] \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix}$$

$$u' = \mathcal{D} u_i \mathbf{x}_{i-1}[R_i]$$

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$$\begin{aligned}\overline{\mathbf{x}_{i-1}} &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1})^\top \times \overline{\mathbf{x}_i}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ \textcolor{red}{0} \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ \textcolor{red}{u' \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[R_i]} \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & u' & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ \overline{\mathbf{x}_i}[L_i] \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix}$$

$$u' = \mathcal{D} u_i \mathbf{x}_{i-1}[R_i]$$

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$$\begin{aligned}\overline{\mathbf{x}_{i-1}} &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1})^\top \times \overline{\mathbf{x}_i}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ \textcolor{red}{0} \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ \textcolor{red}{u' \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[R_i]} \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & u' & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ \textcolor{green}{\overline{\mathbf{x}_i}[L_i]} \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ \textcolor{green}{\overline{\mathbf{x}_i}[R_i]} \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix}$$

$$u' = \mathcal{D} u_i \mathbf{x}_{i-1}[R_i]$$

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$$\overline{\mathbf{x}_{i-1}} = \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i}$$

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$$\begin{aligned}\overline{\mathbf{x}_{i-1}} &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1})^\top \times \overline{\mathbf{x}_i}\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{x}_{i-1}} &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1})^\top \times \overline{\mathbf{x}_i}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ 0 \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ b'_1 \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ b'_2 \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[S_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & b'_1 & & & & 1 & & \\ & & & & & & & \ddots & \\ & & b'_2 & & & & & & 1 \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ \overline{\mathbf{x}_i}[L_i] \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ \overline{\mathbf{x}_i}[S_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix}$$

$$\begin{aligned}b'_1 &= \mathcal{D}_1 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\ b'_2 &= \mathcal{D}_2 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])\end{aligned}$$

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$$\begin{aligned}\overline{\mathbf{x}_{i-1}} &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1})^\top \times \overline{\mathbf{x}_i}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ 0 \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ b'_1 \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ b'_2 \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[S_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & b'_1 & & & & 1 & & \\ & & & & & & & \ddots & \\ & b'_2 & & & & & & & 1 \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ \overline{\mathbf{x}_i}[L_i] \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ \overline{\mathbf{x}_i}[S_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix}$$

$$\begin{aligned}b'_1 &= \mathcal{D}_1 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\ b'_2 &= \mathcal{D}_2 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])\end{aligned}$$

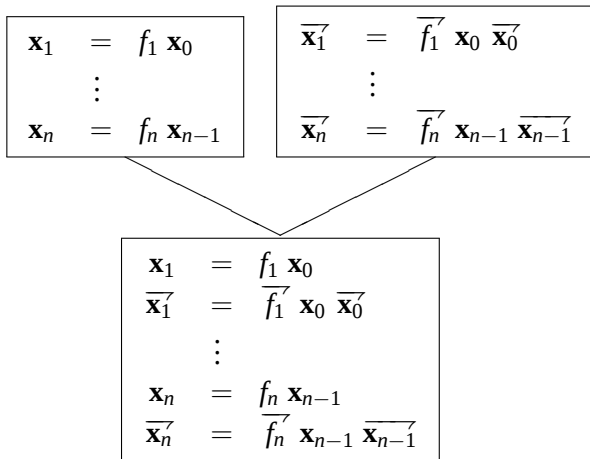
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$$\begin{aligned}\overline{\mathbf{x}_{i-1}} &= \overline{f_i} \mathbf{x}_{i-1} \overline{\mathbf{x}_i} \\ &= (\mathcal{J} f_i \mathbf{x}_{i-1})^\top \times \overline{\mathbf{x}_i}\end{aligned}$$

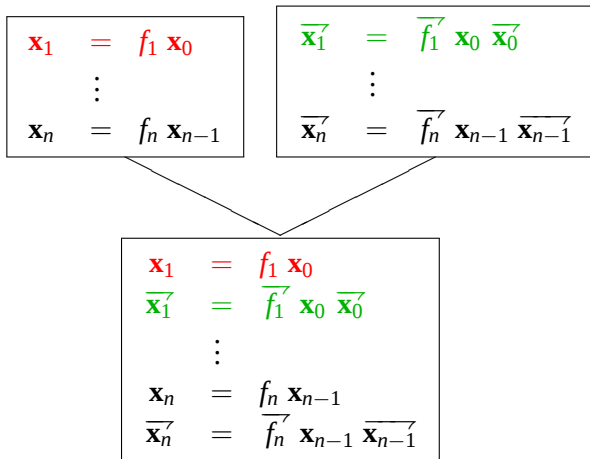
$$\begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ 0 \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ b'_1 \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ b'_2 \times \overline{\mathbf{x}_i}[L_i] + \overline{\mathbf{x}_i}[S_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & b'_1 & & & & 1 & & \\ & & & & & & & \ddots & \\ & b'_2 & & & & & & & 1 \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_i}[1] \\ \vdots \\ \overline{\mathbf{x}_i}[L_i - 1] \\ \overline{\mathbf{x}_i}[L_i] \\ \overline{\mathbf{x}_i}[L_i + 1] \\ \vdots \\ \overline{\mathbf{x}_i}[R_i] \\ \vdots \\ \overline{\mathbf{x}_i}[S_i] \\ \vdots \\ \overline{\mathbf{x}_i}[m] \end{pmatrix}$$

$$\begin{aligned}b'_1 &= \mathcal{D}_1 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\ b'_2 &= \mathcal{D}_2 b_i (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])\end{aligned}$$

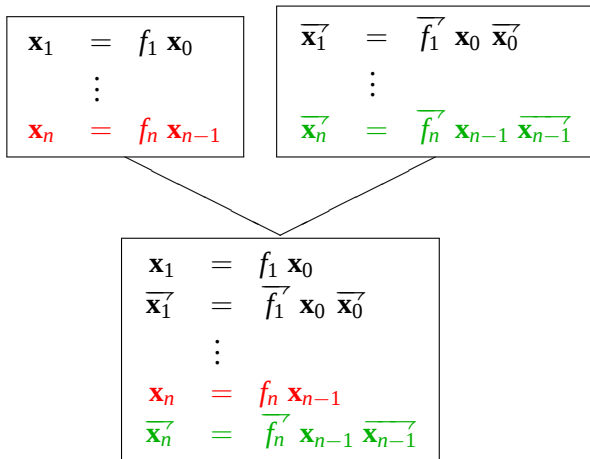
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$$\mathbf{x}_1 = f_1 \mathbf{x}_0$$

$$\overline{\mathbf{x}}_1' = \overline{f_1'} \mathbf{x}_0 \overline{\mathbf{x}}_0'$$

$$\vdots$$

$$\mathbf{x}_n = f_n \mathbf{x}_{n-1}$$

$$\overline{\mathbf{x}}_n' = \overline{f_n'} \mathbf{x}_{n-1} \overline{\mathbf{x}}_{n-1}'$$

$$(\mathbf{x}_1, \overline{\mathbf{x}}_1') = ((f_1 \mathbf{x}_0), (\overline{f_1'} \mathbf{x}_0 \overline{\mathbf{x}}_0'))$$

$$\vdots$$

$$(\mathbf{x}_n, \overline{\mathbf{x}}_n') = ((f_n \mathbf{x}_{n-1}), (\overline{f_n'} \mathbf{x}_{n-1} \overline{\mathbf{x}}_{n-1}'))$$

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$$\left. \begin{array}{l} \mathbf{x}_1 = f_1 \mathbf{x}_0 \\ \vdots \\ \mathbf{x}_n = f_n \mathbf{x}_{n-1} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{l} \overrightarrow{\mathbf{x}}_1 = \overrightarrow{f}_1 \overrightarrow{\mathbf{x}}_0 \\ \vdots \\ \overrightarrow{\mathbf{x}}_n = \overrightarrow{f}_n \overrightarrow{\mathbf{x}}_{n-1} \end{array} \right.$$

$$\begin{aligned} \overrightarrow{\mathbf{x}} &\equiv (\mathbf{x}, \overline{\mathbf{x}}) \\ \overrightarrow{f} \overrightarrow{\mathbf{x}} &\equiv ((f \mathbf{x}), (\overline{f} \mathbf{x} \overline{\mathbf{x}})) \end{aligned}$$

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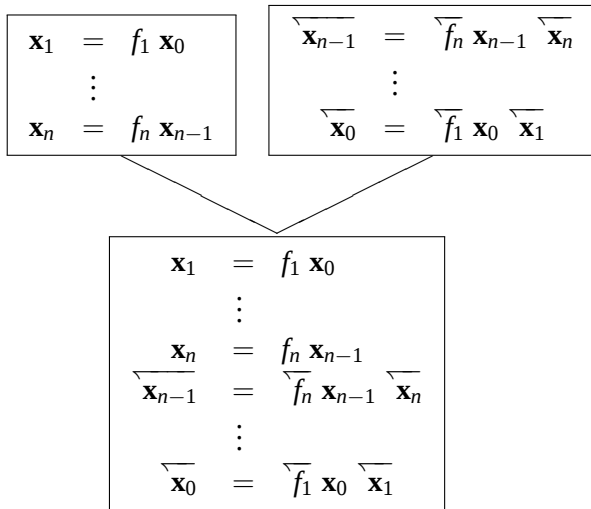
$$\begin{aligned} x_{L_i} &:= u_i \ x_{R_i} & \rightsquigarrow & \quad \overrightarrow{x_{L_i}} := \overrightarrow{u_i} \ \overrightarrow{x_{R_i}} \\ x_{L_i} &:= b_i (x_{R_i}, x_{S_i}) & \rightsquigarrow & \quad \overrightarrow{x_{L_i}} := \overrightarrow{b_i} (\overrightarrow{x_{R_i}}, \overrightarrow{x_{S_i}}) \end{aligned}$$

$$\overrightarrow{x} \equiv (x, \overline{x'})$$

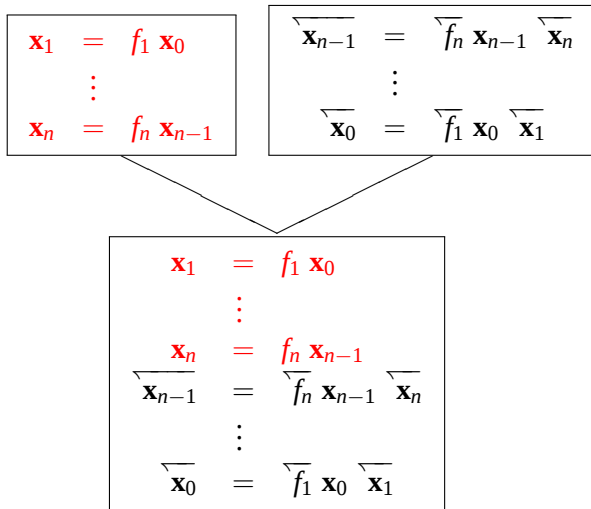
$$\overrightarrow{u} \ \overrightarrow{x} \equiv ((u \ x), ((\mathcal{D} \ u \ x) \times \overline{x'}))$$

$$\overrightarrow{b} (\overrightarrow{x_1}, \overrightarrow{x_2}) \equiv ((b(x_1, x_2)), ((\mathcal{D}_1 \ b(x_1, x_2)) \times \overline{x'_1}) + ((\mathcal{D}_2 \ b(x_1, x_2)) \times \overline{x'_2})))$$

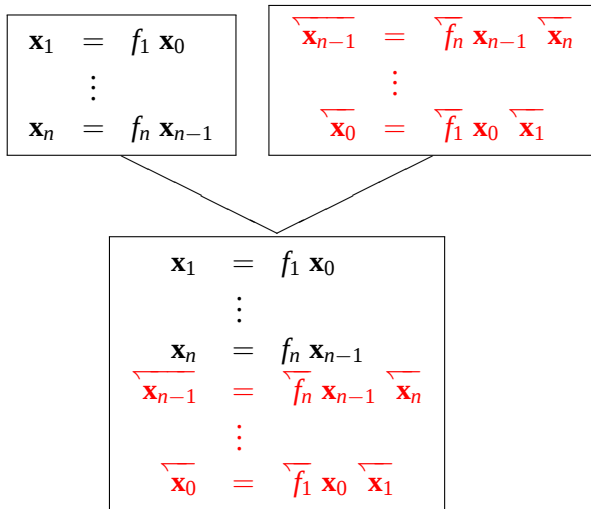
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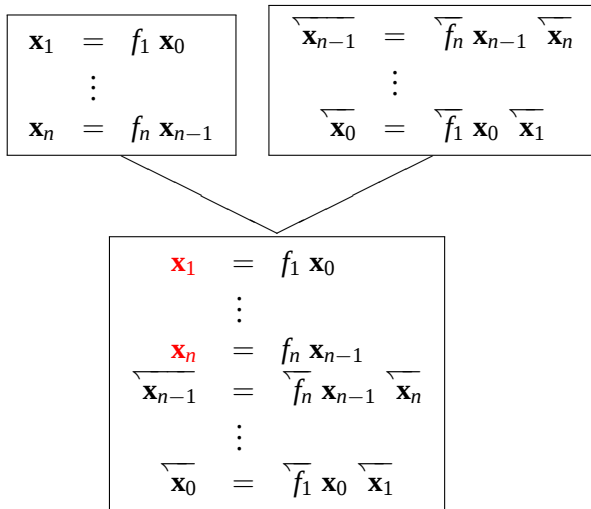
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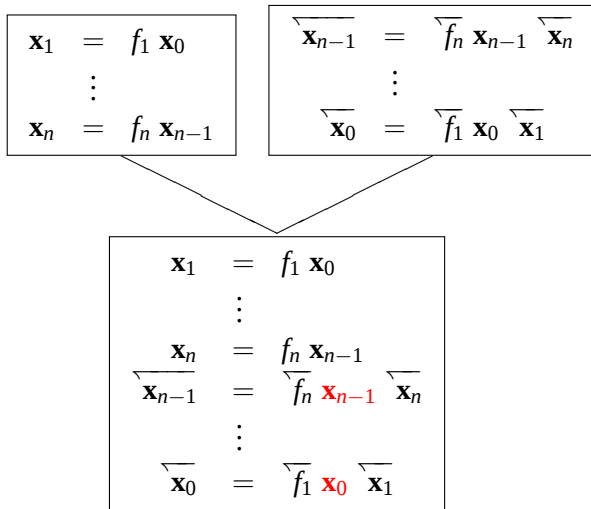
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$$\mathbf{x}_1 = f_1 \mathbf{x}_0$$

$$\overline{\mathbf{x}}_1 = \lambda \overline{\mathbf{x}} \quad \overline{\mathbf{x}}_0 \left(\overline{f}_1 \mathbf{x}_0 \quad \overline{\mathbf{x}} \right)$$

$$\vdots$$

$$\mathbf{x}_n = f_n \mathbf{x}_{n-1}$$

$$\overline{\mathbf{x}}_n = \lambda \overline{\mathbf{x}} \quad \overline{\mathbf{x}}_{n-1} \left(\overline{f}_1 \mathbf{x}_0 \quad \overline{\mathbf{x}} \right)$$

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$$\begin{aligned}
 \mathbf{x}_1 &= f_1 \mathbf{x}_0 \\
 \overline{\mathbf{x}}_1 &= \lambda \overline{\mathbf{x}} \quad \overline{\mathbf{x}}_0 \quad (\overline{f_1} \mathbf{x}_0 \quad \overline{\mathbf{x}}) \\
 &\vdots \\
 \mathbf{x}_n &= f_n \mathbf{x}_{n-1} \\
 \overline{\mathbf{x}}_n &= \lambda \overline{\mathbf{x}} \quad \overline{\mathbf{x}}_{n-1} \quad (\overline{f_1} \mathbf{x}_0 \quad \overline{\mathbf{x}})
 \end{aligned}$$

$$\begin{aligned}
 (\mathbf{x}_1, \overline{\mathbf{x}}_1) &= ((f_1 \mathbf{x}_0), (\lambda \overline{\mathbf{x}} \quad \overline{\mathbf{x}}_0 \quad (\overline{f_1} \mathbf{x}_0 \quad \overline{\mathbf{x}}))) \\
 &\vdots \\
 (\mathbf{x}_n, \overline{\mathbf{x}}_n) &= ((f_n \mathbf{x}_{n-1}), (\lambda \overline{\mathbf{x}} \quad \overline{\mathbf{x}}_{n-1} \quad (\overline{f_1} \mathbf{x}_0 \quad \overline{\mathbf{x}})))
 \end{aligned}$$

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$$\left. \begin{array}{l} \mathbf{x}_1 = f_1 \mathbf{x}_0 \\ \vdots \\ \mathbf{x}_n = f_n \mathbf{x}_{n-1} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{l} \overleftarrow{\mathbf{x}}_1 = \overleftarrow{f}_1 \overleftarrow{\mathbf{x}}_0 \\ \vdots \\ \overleftarrow{\mathbf{x}}_n = \overleftarrow{f}_n \overleftarrow{\mathbf{x}}_{n-1} \end{array} \right.$$

$$\begin{aligned} \overleftarrow{\mathbf{x}} &\equiv (\mathbf{x}, \overline{\mathbf{x}}) \\ \overleftarrow{f} \overleftarrow{\mathbf{x}} &\equiv ((f \mathbf{x}), (\lambda \overline{\mathbf{x}} \overline{\mathbf{x}} (\overleftarrow{f} \mathbf{x} \overline{\mathbf{x}}))) \end{aligned}$$

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$$\overleftarrow{f} \mathbf{x} \equiv \mathbf{begin} \ \bar{x} := \lambda \overleftarrow{\mathbf{x}} \ \bar{x} (\overleftarrow{f} \mathbf{x} \ \overleftarrow{\mathbf{x}});$$
$$(f \mathbf{x}) \mathbf{end}$$

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$$\begin{aligned}
 x_{L_i} &:= u_i \ x_{R_i} & \rightsquigarrow & \left\{ \begin{array}{l} \bar{x} := \lambda[] \textbf{begin} \ \overline{x_{R_i}} +:= (\mathcal{D} \ u_i \ \overline{x_{R_i}}) \times \overline{x_{L_i}}; \\ \overline{x_{L_i}} := 0; \\ \bar{x} [] \textbf{end} \\ \overline{x_{L_i}} := u_i \ \overline{x_{R_i}} \end{array} \right. \\
 x_{L_i} &:= b_i (x_{R_i}, x_{S_i}) & \rightsquigarrow & \left\{ \begin{array}{l} \bar{x} := \lambda[] \textbf{begin} \ \overline{x_{R_i}} +:= (\mathcal{D}_1 \ b_i (\overline{x_{R_i}}, \overline{x_{S_i}})) \times \overline{x_{L_i}}; \\ \overline{x_{S_i}} +:= (\mathcal{D}_2 \ b_i (\overline{x_{R_i}}, \overline{x_{S_i}})) \times \overline{x_{L_i}}; \\ \overline{x_{L_i}} := 0; \\ \bar{x} [] \textbf{end} \\ \overline{x_{L_i}} := b_i (\overline{x_{R_i}}, \overline{x_{S_i}}) \end{array} \right.
 \end{aligned}$$

$$\overline{x} \equiv x$$

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$$\begin{aligned}
 x_{L_i} &:= u_i \ x_{R_i} & \rightsquigarrow & \left\{ \begin{array}{l} \overleftarrow{x_{L_i}} := u_i \ \overleftarrow{x_{R_i}} \\ \vdots \\ \overleftarrow{x_{R_i}} + := (\mathcal{D} \ u_i \ \overleftarrow{x_{R_i}}) \times \overleftarrow{x_{L_i}}; \\ \overleftarrow{x_{L_i}} := 0 \end{array} \right. \\
 x_{L_i} &:= b_i (x_{R_i}, x_{S_i}) & \rightsquigarrow & \left\{ \begin{array}{l} \overleftarrow{x_{L_i}} := b_i (\overleftarrow{x_{R_i}}, \overleftarrow{x_{S_i}}) \\ \vdots \\ \overleftarrow{x_{R_i}} + := (\mathcal{D}_1 \ b_i (\overleftarrow{x_{R_i}}, \overleftarrow{x_{S_i}})) \times \overleftarrow{x_{L_i}}; \\ \overleftarrow{x_{S_i}} + := (\mathcal{D}_2 \ b_i (\overleftarrow{x_{R_i}}, \overleftarrow{x_{S_i}})) \times \overleftarrow{x_{L_i}}; \\ \overleftarrow{x_{L_i}} := 0 \end{array} \right.
 \end{aligned}$$

$$\overleftarrow{x} \equiv x$$

The Functional Reverse-Mode Transformation

$$x_{L_1} := u_1 \ x_{S_1}$$

$$\vdots$$

$$x_{L_n} := u_n \ x_{S_n}$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_{L_1} & := & u_1 x_{S_1} \\ \vdots & & \\ x_{L_n} & := & u_n x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} x_{L_1} & := & u_1 x_{S_1} \\ \vdots & & \\ x_{L_n} & := & u_n x_{S_n} \\ \overline{x_1} & := & 0 \\ \vdots & & \\ \overline{x_m} & := & 0 \\ \overline{x_{S_n}} & +: = & (\mathcal{D} u_n x_{S_n}) \times \overline{x_{L_n}} \\ \overline{x_{L_n}} & := & 0 \\ \vdots & & \\ \overline{x_{S_1}} & +: = & (\mathcal{D} u_1 x_{S_1}) \times \overline{x_{L_1}} \\ \overline{x_{L_1}} & := & 0 \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_{L_1} & \textcolor{red}{:=} & u_1 \ x_{S_1} \\ \vdots & & \\ x_{L_n} & \textcolor{red}{:=} & u_n \ x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} x_{L_1} & := & u_1 \ x_{S_1} \\ \vdots & & \\ x_{L_n} & := & u_n \ x_{S_n} \\ \overline{x_1} & := & 0 \\ \vdots & & \\ \overline{x_m} & := & 0 \\ \overline{x_{S_n}} & +:= & (\mathcal{D} \ u_n \ x_{S_n}) \times \overline{x_{L_n}} \\ \overline{x_{L_n}} & := & 0 \\ \vdots & & \\ \overline{x_{S_1}} & +:= & (\mathcal{D} \ u_1 \ x_{S_1}) \times \overline{x_{L_1}} \\ \overline{x_{L_1}} & := & 0 \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_{L_1} & := & u_1 \ x_{S_1} \\ \vdots & & \\ x_{L_n} & := & u_n \ x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} x_{L_1} & \textcolor{red}{:=} & u_1 \ x_{S_1} \\ \vdots & & \\ x_{L_n} & \textcolor{red}{:=} & u_n \ x_{S_n} \\ \overline{x_1} & := & 0 \\ \vdots & & \\ \overline{x_m} & := & 0 \\ \overline{x_{S_n}} & +:= & (\mathcal{D} \ u_n \ x_{S_n}) \times \overline{x_{L_n}} \\ \overline{x_{L_n}} & := & 0 \\ \vdots & & \\ \overline{x_{S_1}} & +:= & (\mathcal{D} \ u_1 \ x_{S_1}) \times \overline{x_{L_1}} \\ \overline{x_{L_1}} & := & 0 \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_{L_1} & := & u_1 \ x_{S_1} \\ \vdots & & \\ x_{L_n} & := & u_n \ x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} x_{L_1} & := & u_1 \ x_{S_1} \\ \vdots & & \\ x_{L_n} & := & u_n \ x_{S_n} \\ \overline{x_1} & := & 0 \\ \vdots & & \\ \overline{x_m} & := & 0 \\ \overline{x_{S_n}} & +: = & (\mathcal{D} \ u_n \ x_{S_n}) \times \overline{x_{L_n}} \\ \overline{x_{L_n}} & := & 0 \\ \vdots & & \\ \overline{x_{S_1}} & +: = & (\mathcal{D} \ u_1 \ x_{S_1}) \times \overline{x_{L_1}} \\ \overline{x_{L_1}} & := & 0 \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_1 & = & u_1 x_{S_1} \\ \vdots & & \\ x_n & = & u_n x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} x_1 & = & u_1 x_{S_1} \\ \vdots & & \\ x_n & = & u_n x_{S_n} \\ \overline{x_0} & := & 0 \\ \vdots & & \\ \overline{x_{n-1}} & := & 0 \\ \overline{x_{S_n}} & +: = & (\mathcal{D} u_n x_{S_n}) \times \overline{x_n} \\ \vdots & & \\ \overline{x_{S_1}} & +: = & (\mathcal{D} u_1 x_{S_1}) \times \overline{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_1 & = & u_1 x_{S_1} \\ \vdots & & \\ x_n & = & u_n x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} x_1 & = & u_1 x_{S_1} \\ \vdots & & \\ x_n & = & u_n x_{S_n} \\ \overline{x_0} & := & 0 \\ \vdots & & \\ \overline{x_{n-1}} & := & 0 \\ \overline{x_{S_n}} & +:= & (\mathcal{D} u_n x_{S_n}) \times \overline{x_n} \\ \vdots & & \\ \overline{x_{S_1}} & +:= & (\mathcal{D} u_1 x_{S_1}) \times \overline{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\overline{u_i} \triangleq \lambda \overline{x} (\mathcal{D} u_i x_{S_i}) \times \overline{x}$$

$$\left. \begin{array}{l} x_1 = u_1 x_{S_1} \\ \vdots \\ x_n = u_n x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{ll} x_1 = & u_1 x_{S_1} \\ \vdots & \\ x_n = & u_n x_{S_n} \\ \overline{x_0} := & 0 \\ \vdots & \\ \overline{x_{n-1}} := & 0 \\ \overline{x_{S_n}} +:= & \overline{u_n} \overline{x_n} \\ \vdots & \\ \overline{x_{S_1}} +:= & \overline{u_1} \overline{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\overleftarrow{u} \ x \triangleq ((u \ x), (\lambda \overleftarrow{x} (\mathcal{D} \ u \ x) \times \overleftarrow{x})))$$

$$\left. \begin{array}{l} x_1 = u_1 \ x_{S_1} \\ \vdots \\ x_n = u_n \ x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{ll} (x_1, \overline{x_1}) & = \overleftarrow{u_1} \ x_{S_1} \\ \vdots & \\ (x_n, \overline{x_n}) & = \overleftarrow{u_n} \ x_{S_n} \\ \overline{x_0} & := 0 \\ \vdots & \\ \overline{x_{n-1}} & := 0 \\ \overline{x_{S_n}} & +:= \overline{x_n} \ \overleftarrow{x_n} \\ \vdots & \\ \overline{x_{S_1}} & +:= \overline{x_1} \ \overleftarrow{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\overleftarrow{u} \ x \triangleq ((u \ x), (\lambda \overleftarrow{x} (\mathcal{D} \ u \ x) \times \overleftarrow{x}))$$

$$\left. \begin{array}{l} x_1 = \textcolor{red}{u}_1 x_{S_1} \\ \vdots \\ x_n = \textcolor{red}{u}_n x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{ll} (x_1, \overleftarrow{x}_1) & = \overleftarrow{u}_1 x_{S_1} \\ \vdots & \\ (x_n, \overleftarrow{x}_n) & = \overleftarrow{u}_n x_{S_n} \\ \overleftarrow{x}_0 & := 0 \\ \vdots & \\ \overleftarrow{x}_{n-1} & := 0 \\ \overleftarrow{x}_{S_n} & +:= \overleftarrow{x}_n \overleftarrow{x}_n \\ \vdots & \\ \overleftarrow{x}_{S_1} & +:= \overleftarrow{x}_1 \overleftarrow{x}_1 \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_1 & = & x_{R_1} x_{S_1} \\ \vdots & & \\ x_n & = & x_{R_n} x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} (\overleftarrow{x_1}, \overleftarrow{x_1}) & = & \overleftarrow{x_{R_1}} \overleftarrow{x_{S_1}} \\ \vdots & & \vdots \\ (\overleftarrow{x_n}, \overleftarrow{x_n}) & = & \overleftarrow{x_{R_n}} \overleftarrow{x_{S_n}} \\ \overleftarrow{x_0} & := & 0 \\ \vdots & & \vdots \\ \overleftarrow{x_{n-1}} & := & 0 \\ \overleftarrow{x_{S_n}} & +: = & \overleftarrow{x_n} \overleftarrow{x_n} \\ \vdots & & \vdots \\ \overleftarrow{x_{S_1}} & +: = & \overleftarrow{x_1} \overleftarrow{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_1 & = & x_{R_1} x_{S_1} \\ \vdots & & \\ x_n & = & x_{R_n} x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} (\overleftarrow{x_1}, \overline{x_1}) & = & \overleftarrow{x_{R_1}} \overleftarrow{x_{S_1}} \\ \vdots & & \vdots \\ (\overleftarrow{x_n}, \overline{x_n}) & = & \overleftarrow{x_{R_n}} \overleftarrow{x_{S_n}} \\ \overline{x_0} & := & 0 \\ \vdots & & \vdots \\ \overline{x_{n-1}} & := & 0 \\ \overline{x_{S_n}} & +: = & \overline{x_n} \overline{x_n} \\ \vdots & & \vdots \\ \overline{x_{S_1}} & +: = & \overline{x_1} \overline{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_1 & = & x_{R_1} x_{S_1} \\ \vdots & & \\ x_n & = & x_{R_n} x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{ll} (\overleftarrow{x_1}, \overleftarrow{x_1}) & = \overleftarrow{x_{R_1}} \overleftarrow{x_{S_1}} \\ \vdots & \\ (\overleftarrow{x_n}, \overleftarrow{x_n}) & = \overleftarrow{x_{R_n}} \overleftarrow{x_{S_n}} \\ \overleftarrow{x_0} & := 0 \\ \vdots & \\ \overleftarrow{x_{n-1}} & := 0 \\ \overleftarrow{x_{S_n}} & +:= \overleftarrow{x_n} \overleftarrow{x_n} \\ \vdots & \\ \overleftarrow{x_{S_1}} & +:= \overleftarrow{x_1} \overleftarrow{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_1 & = & x_{R_1} x_{S_1} \\ \vdots & & \\ x_n & = & x_{R_n} x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{lcl} (\overleftarrow{x_1}, \overleftarrow{x_1}) & = & \overleftarrow{x_{R_1}} \overleftarrow{x_{S_1}} \\ \vdots & & \\ (\overleftarrow{x_n}, \overleftarrow{x_n}) & = & \overleftarrow{x_{R_n}} \overleftarrow{x_{S_n}} \\ \overleftarrow{x_0} & := & 0 \\ \vdots & & \\ \overleftarrow{x_{n-1}} & := & 0 \\ \overleftarrow{x_{S_n}} & \textcolor{red}{+} := & \overleftarrow{x_n} \overleftarrow{x_n} \\ \vdots & & \\ \overleftarrow{x_{S_1}} & \textcolor{red}{+} := & \overleftarrow{x_1} \overleftarrow{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{lcl} x_1 & = & x_{R_1} x_{S_1} \\ \vdots & & \\ x_n & = & x_{R_n} x_{S_n} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{ll} (\overleftarrow{x_1}, \overline{x_1}) & = \quad \overleftarrow{x_{R_1}} \overleftarrow{x_{S_1}} \\ \vdots & \\ (\overleftarrow{x_n}, \overline{x_n}) & = \quad \overleftarrow{x_{R_n}} \overleftarrow{x_{S_n}} \\ \overline{x_0} & := \quad \mathbf{0} \ (\overleftarrow{\mathcal{J}}^{-1} \overleftarrow{x_0}) \\ \vdots & \\ \overline{x_{n-1}} & := \quad \mathbf{0} \ (\overleftarrow{\mathcal{J}}^{-1} \overleftarrow{x_{n-1}}) \\ \overline{x_{S_n}} & \oplus := \quad \overline{x_n} \overline{x_n} \\ \vdots & \\ \overline{x_{S_1}} & \oplus := \quad \overline{x_1} \overline{x_1} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{l} \lambda x_0 \text{ let } x_1 \triangleq x_{R_1} x_{S_1}; \\ \quad \vdots \\ \quad x_n \triangleq x_{R_n} x_{S_n} \\ \text{in } x_n \text{ end} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{l} \lambda \overleftarrow{x_0} \text{ let } (\overleftarrow{x_1}, \overline{x_1}) \triangleq \overleftarrow{x_{R_1}} \overleftarrow{x_{S_1}}; \\ \quad \vdots \\ \quad (\overleftarrow{x_n}, \overline{x_n}) \triangleq \overleftarrow{x_{R_n}} \overleftarrow{x_{S_n}} \\ \text{in } (\overleftarrow{x_n}, (\lambda \overleftarrow{x_n} \text{ let } \overleftarrow{x_0} \triangleq \mathbf{0} (\overleftarrow{\mathcal{J}}^{-1} \overleftarrow{x_0}); \\ \quad \quad \vdots \\ \quad \quad \overleftarrow{x_{n-1}} \triangleq \mathbf{0} (\overleftarrow{\mathcal{J}}^{-1} \overleftarrow{x_{n-1}}); \\ \quad \quad \overleftarrow{x_{S_n}} \oplus \triangleq \overline{x_n} \overleftarrow{x_n}; \\ \quad \quad \vdots \\ \quad \quad \overleftarrow{x_{S_1}} \oplus \triangleq \overline{x_1} \overleftarrow{x_1} \\ \text{in } \overleftarrow{x_0} \text{ end})) \text{ end} \end{array} \right.$$

The Functional Reverse-Mode Transformation

$$\left. \begin{array}{l} \lambda x_0 \text{ let } x_1 \triangleq x_{R_1} x_{S_1}; \\ \quad \vdots \\ \quad x_n \triangleq x_{R_n} x_{S_n} \\ \text{in } x_n \text{ end} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{l} \lambda \overleftarrow{x_0} \text{ let } (\overleftarrow{x_1}, \overline{x_1}) \triangleq \overleftarrow{x_{R_1}} \overleftarrow{x_{S_1}}; \\ \quad \vdots \\ \quad (\overleftarrow{x_n}, \overline{x_n}) \triangleq \overleftarrow{x_{R_n}} \overleftarrow{x_{S_n}} \\ \text{in } (\overleftarrow{x_n}, (\lambda \overline{x_n} \text{ let } \overline{x_0} \triangleq \mathbf{0} (\overleftarrow{\mathcal{J}}^{-1} \overleftarrow{x_0}); \\ \quad \quad \vdots \\ \quad \quad \overline{x_{n-1}} \triangleq \mathbf{0} (\overleftarrow{\mathcal{J}}^{-1} \overleftarrow{x_{n-1}}); \\ \quad \quad \overline{x_{S_n}} \oplus \triangleq \overline{x_n} \overline{x_n}; \\ \quad \quad \vdots \\ \quad \quad \overline{x_{S_1}} \oplus \triangleq \overline{x_1} \overline{x_1} \\ \text{in } \overline{x_0} \text{ end})) \text{ end} \end{array} \right.$$

The above is a white lie. The truth is a lot more complicated.

Modularity

$$\nabla f \mathbf{x}$$

$$\triangleq \frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$$

Modularity

$$\begin{aligned}\nabla f \mathbf{x} &\triangleq \frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n} \\ \text{GRADIENTDESCENT } f \mathbf{x}_0 &\triangleq \dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots\end{aligned}$$

Modularity

$$\nabla f \mathbf{x} \triangleq \frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$$

$$\text{GRADIENTDESCENT } f \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$$

$$\operatorname{argmin} f \triangleq \dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$$

Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$\frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$
<code>GRADIENTDESCENT</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
<code>argmin</code> f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
<code>NEUTRONFLUX</code> r	$\triangleq$	<i>classified</i>

Modularity

$$\nabla f \mathbf{x} \triangleq \frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$$

$$\text{GRADIENTDESCENT } f \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$$

$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\textit{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$\frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$
<code>GRADIENTDESCENT</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
<code>argmin</code> f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
<code>NEUTRONFLUX</code> r	$\triangleq$	<i>classified</i>
<code>DEVIATION</code> r	$\triangleq$	$((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$
r^*	$\triangleq$	<code>argmin</code> <code>DEVIATION</code>

Fermi, E. (1946). *The Development of the first chain reaction pile*.
 Proceedings of the American Philosophy Society, **90**:20–4.

Breaking Modularity

$$\nabla f \mathbf{x} \triangleq (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_1'), \dots, (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } f \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$$

$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$r^* \triangleq \text{argmin DEVIATION}$$

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 Proceedings of the American Philosophy Society, **90**:20–4.

Breaking Modularity

$$\nabla \vec{f} \mathbf{x} \triangleq (\vec{f} \mathbf{x} \triangleright \vec{e}_1), \dots, (\vec{f} \mathbf{x} \triangleright \vec{e}_n)$$

$$\text{GRADIENTDESCENT } f \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$$

$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$r^* \triangleq \text{argmin DEVIATION}$$

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Breaking Modularity

$$\nabla \overrightarrow{f} \mathbf{x} \triangleq (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_1'), \dots, (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } f \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \overrightarrow{f} \mathbf{x}_i \dots$$

$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$r^* \triangleq \text{argmin DEVIATION}$$

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Breaking Modularity

$$\nabla \vec{f} \mathbf{x} \triangleq (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_1'), \dots, (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } \vec{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \vec{f} \mathbf{x}_i \dots$$

$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$r^* \triangleq \text{argmin DEVIATION}$$

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Breaking Modularity

$$\nabla \vec{f} \mathbf{x} \triangleq (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_1'), \dots, (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } \vec{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \vec{f} \mathbf{x}_i \dots$$

$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } \vec{f} \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$r^* \triangleq \text{argmin DEVIATION}$$

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Breaking Modularity

$$\nabla \overrightarrow{f} \mathbf{x} \triangleq (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_1'), \dots, (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \overrightarrow{f} \mathbf{x}_i \dots$$

$$\text{argmin } \overrightarrow{f} \triangleq \dots \text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

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Breaking Modularity

$$\nabla \overrightarrow{f} \mathbf{x} \triangleq (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_1'), \dots, (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \overrightarrow{f} \mathbf{x}_i \dots$$

$$\text{argmin } \overrightarrow{f} \triangleq \dots \text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$r^* \triangleq \text{argmin } \overrightarrow{\text{DEVIATION}}$$

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Breaking Modularity

$$\nabla \overrightarrow{f} \mathbf{x} \triangleq (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_1'), \dots, (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \overrightarrow{f} \mathbf{x}_i \dots$$

$$\text{argmin } \overrightarrow{f} \triangleq \dots \text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$\text{DEVIATION} \xrightarrow{\text{ADIFOR}} \overrightarrow{\text{DEVIATION}}$$

$$r^* \triangleq \text{argmin } \overrightarrow{\text{DEVIATION}}$$

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 Proceedings of the American Philosophy Society, **90**:20–4.

Breaking Modularity

$$\nabla \overrightarrow{f} \mathbf{x} \triangleq (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_1'), \dots, (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \overrightarrow{f} \mathbf{x}_i \dots$$

$$\text{argmin } \overrightarrow{f} \triangleq \dots \text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{NEUTRONFLUX} \xrightarrow[\rightsquigarrow]{\text{ADIFOR}} \overrightarrow{\text{NEUTRONFLUX}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$\text{DEVIATION} \xrightarrow[\rightsquigarrow]{\text{ADIFOR}} \overrightarrow{\text{DEVIATION}}$$

$$r^* \triangleq \text{argmin } \overrightarrow{\text{DEVIATION}}$$

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Breaking Modularity

$$\nabla \overrightarrow{f} \mathbf{x} \triangleq (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_1'), \dots, (\overrightarrow{f} \mathbf{x} \triangleright \overrightarrow{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \overrightarrow{f} \mathbf{x}_i \dots$$

$$\text{argmin } \overrightarrow{f} \triangleq \dots \text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } \mathbf{r} \triangleq \boxed{\text{classified}}$$

$$\text{NEUTRONFLUX} \xrightarrow[\rightsquigarrow]{\text{ADIFOR}} \overrightarrow{\text{NEUTRONFLUX}}$$

$$\text{DEVIATION } \mathbf{r} \triangleq ((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

$$\text{DEVIATION} \xrightarrow[\rightsquigarrow]{\text{ADIFOR}} \overrightarrow{\text{DEVIATION}}$$

$$\mathbf{r}^* \triangleq \text{argmin } \overrightarrow{\text{DEVIATION}}$$

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Breaking Modularity

$$\nabla \overrightarrow{f} \mathbf{x} \triangleq \dots \overleftarrow{f} \mathbf{x} \dots$$

$$\text{GRADIENTDESCENT } \overrightarrow{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \overrightarrow{f} \mathbf{x}_i \dots$$

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$$\text{NEUTRONFLUX } \mathbf{r} \triangleq \boxed{\text{classified}}$$

$$\text{NEUTRONFLUX} \xrightarrow[\rightsquigarrow]{\text{ADIFOR}} \overrightarrow{\text{NEUTRONFLUX}}$$

$$\text{DEVIATION } \mathbf{r} \triangleq ((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

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Breaking Modularity

$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
NEWTONSMETHOD $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
argmin $\overleftarrow{f}$	$\triangleq$	$\dots \text{GRADIENTDESCENT } \overleftarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\overset{\text{TAPENADE}}{\rightsquigarrow}$	$\overleftarrow{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\overset{\text{TAPENADE}}{\rightsquigarrow}$	$\overleftarrow{\text{DEVIATION}}$
$\mathbf{r}^*$	$\triangleq$	argmin $\overleftarrow{\text{DEVIATION}}$

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$$\text{argmin} \overleftarrow{f} \triangleq \dots \text{NEWTONSMETHOD} \overleftarrow{f} \mathbf{x}_0 \dots$$

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Breaking Modularity

$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
NEWTONSMETHOD $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
$\operatorname{argmin} \overleftarrow{f}$	$\triangleq$	$\dots \operatorname{NEWTONSMETHOD} \overleftarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overline{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overline{\text{DEVIATION}}$
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$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
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$\operatorname{argmin} \overleftarrow{f}$	$\triangleq$	$\dots \operatorname{NEWTONSMETHOD} \overleftarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overline{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overline{\text{DEVIATION}}$
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$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
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GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
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argmin $\overleftarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
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$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} \overrightarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
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argmin $\overleftarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overleftarrow{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overleftarrow{\text{DEVIATION}}$
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$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} \overrightarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
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argmin $\overleftarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
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GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
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argmin $\overleftarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \overrightarrow{f} \mathbf{x}_0 \dots$
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DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
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$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
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argmin $\overleftarrow{f} \overrightarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \overrightarrow{f} \mathbf{x}_0 \dots$
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$\mathcal{H} \overrightarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
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argmin $\overleftarrow{f} \overrightarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \overrightarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overline{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overline{\text{DEVIATION}}$
$\mathbf{r}^*$	$\triangleq$	argmin $\overleftarrow{\text{DEVIATION}} \overrightarrow{\text{DEVIATION}}$

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$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} \overrightarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
NEWTONSMETHOD $\overleftarrow{f} \overrightarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots \mathcal{H} \overrightarrow{f} \mathbf{x}_i \dots$
argmin $\overleftarrow{f} \overrightarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \overrightarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\xrightarrow{\text{TAPENADE}}$	$\overleftarrow{\text{NEUTRONFLUX}}$
$\overleftarrow{\text{NEUTRONFLUX}}$	$\xrightarrow{\text{TAPENADE}}$	$\overrightarrow{\overleftarrow{\text{NEUTRONFLUX}}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\xrightarrow{\text{TAPENADE}}$	$\overleftarrow{\text{DEVIATION}}$
$\overleftarrow{\text{DEVIATION}}$	$\xrightarrow{\text{TAPENADE}}$	$\overrightarrow{\overleftarrow{\text{DEVIATION}}}$
$\mathbf{r}^*$	$\triangleq$	argmin $\overleftarrow{\text{DEVIATION}} \overrightarrow{\overleftarrow{\text{DEVIATION}}}$

Fermi, E. (1946). *The Development of the first chain reaction pile*.
 Proceedings of the American Philosophy Society, **90**:20–4.

Restoring Modularity

$\nabla f \mathbf{x}$	$\triangleq$	
$\mathcal{H} f \mathbf{x}$	$\triangleq$	
GRADIENTDESCENT $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
NEWTONSMETHOD $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
argmin f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	argmin DEVIATION

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Restoring Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$((\vec{\mathcal{J}} f) \mathbf{x} \triangleright \vec{\mathbf{e}}_1'), \dots, ((\vec{\mathcal{J}} f) \mathbf{x} \triangleright \vec{\mathbf{e}}_n')$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	
GRADIENTDESCENT $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
NEWTONSMETHOD $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
argmin f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	argmin DEVIATION

Fermi, E. (1946). *The Development of the first chain reaction pile*.
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Restoring Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$\dots (\overleftarrow{\mathcal{J}} f) \mathbf{x} \dots$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	
<code>GRADIENTDESCENT</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
<code>NEWTONSMETHOD</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
<code>argmin</code> f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
<code>NEUTRONFLUX</code> $\mathbf{r}$	$\triangleq$	classified
<code>DEVIATION</code> $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	<code>argmin</code> <code>DEVIATION</code>

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Restoring Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$\dots (\overleftarrow{\mathcal{J}} f) \mathbf{x} \dots$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	$\dots (\overrightarrow{\mathcal{J}} (\overleftarrow{\mathcal{J}} f)) \dots \mathbf{x} \dots$
<code>GRADIENTDESCENT</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
<code>NEWTONSMETHOD</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
<code>argmin</code> f	$\triangleq$	$\dots \text{NEWTONSMETHOD } f \mathbf{x}_0 \dots$
<code>NEUTRONFLUX</code> $\mathbf{r}$	$\triangleq$	classified
<code>DEVIATION</code> $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	<code>argmin</code> <code>DEVIATION</code>

Fermi, E. (1946). *The Development of the first chain reaction pile*.
 Proceedings of the American Philosophy Society, **90**:20–4.

Having your cake and eating it too

- Convenient

- Fast

Having your cake and eating it too

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 - $\mathcal{D}$ formulated as a higher-order function in the language
- Fast

Having your cake and eating it too

- Convenient
 - $\mathcal{D}$ formulated as a higher-order function in the language
 - no arbitrary restrictions
 - applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
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- $\mathcal{D}$ formulated as a higher-order function in the language
- no arbitrary restrictions
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 - $(\mathcal{D} (\mathcal{D} f))$

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- higher-order derivatives
 $(\mathcal{D} (\mathcal{D} f))$
- nesting
 $(\mathcal{D} (\text{lambda } (...) \dots (\mathcal{D} (\text{lambda } (...) \dots)) \dots))$

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- Fast

- $\mathcal{D}$ implemented by reflective transformation of environments and code associated with closures

Having your cake and eating it too

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- no arbitrary restrictions
applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
- higher-order derivatives
 $(\mathcal{D} (\mathcal{D} f))$
- nesting
 $(\mathcal{D} (\text{lambda } (...) \dots (\mathcal{D} (\text{lambda } (...) \dots)) \dots))$

- Fast

- $\mathcal{D}$ implemented by reflective transformation of environments and code associated with closures
- compile away reflection with partial evaluation implemented by flow analysis

Monovariant Flow Analysis: 0-CFA

```
(define ( $\mathcal{D}$  f)  
...)
```

Monovariant Flow Analysis: 0-CFA

```
(define ( $\mathcal{D}$  f)  
  ...)
```

```
( $\mathcal{D}$  (lambda (x)  $2x^3$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:  $\lambda x. 2x^3$ )  
  ...)
```

```
(D (lambda (x)  $2x^3$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ ))  
  ...:( $\lambda x\ 6x^2$ )  
  
(D (lambda (x)  $2x^3$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ ))  
  ...:( $\lambda x\ 6x^2$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ ))  
  ...:( $\lambda x\ 6x^2$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

```
(D (lambda (x)  $3x^4$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ )  $\cup$  ( $\lambda x\ 3x^4$ ))  
  ...:( $\lambda x\ 6x^2$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

```
(D (lambda (x)  $3x^4$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ )  $\cup$  ( $\lambda x\ 3x^4$ ))  
  ...:( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

```
(D (lambda (x)  $3x^4$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ )  $\cup$  ( $\lambda x\ 3x^4$ ))  
  ...:( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

```
(D (lambda (x)  $3x^4$ )):( $\lambda x\ 12x^3$ )
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ )  $\cup$  ( $\lambda x\ 3x^4$ ))  
  ...:( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ )
```

```
(D (lambda (x)  $3x^4$ )):( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ )
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:  $(\lambda x \ 2x^3) \cup (\lambda x \ 3x^4)$ )  
  ...:  $(\lambda x \ 6x^2) \cup (\lambda x \ 12x^3)$ )
```

```
(D (lambda (x)  $2x^3$ )):  $(\lambda x \ 6x^2) \cup (\lambda x \ 12x^3)$ 
```

```
(D (lambda (x)  $3x^4$ )):  $(\lambda x \ 6x^2) \cup (\lambda x \ 12x^3)$ 
```

Monovariant Flow Analysis: 0-CFA

```
(define ( $\mathcal{D}$  f)  
  ...)
```

Monovariant Flow Analysis: 0-CFA

```
(define ( $\mathcal{D}$  f)  
  ...)
```

```
( $\mathcal{D}$  ( $\mathcal{D}$  (lambda (x)  $e^{2x}$ )))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ ))  
  ...)  
  
(D (D (lambda (x)  $e^{2x}$ )))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ )  
  
(D (D (lambda (x)  $e^{2x}$ )))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ )
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))  
...:( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ )  $\cup$  ...)
...:( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ )  $\cup$  ...)
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ )  $\cup$  ...))
```

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}$ f) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}$  f) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define (Dg f:(λx 2x3)) ...:(λx 6x2))
```

```
(define (g ...) ... (D (lambda (x) 2x3)):(λx 6x2) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define (Dg f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define (g ...) ... (D (lambda (x)  $2x^3$ )):( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... (D (lambda (x)  $3x^4$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f : ( $\lambda x$   $2x^3$ )) ... : ( $\lambda x$   $6x^2$ ))
```

```
(define ( $\mathcal{D}_h$  f) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) : ( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... ( $\mathcal{D}$  (lambda (x)  $3x^4$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define ( $\mathcal{D}_h$  f:( $\lambda x$   $3x^4$ )) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )):( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... ( $\mathcal{D}$  (lambda (x)  $3x^4$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define ( $\mathcal{D}_h$  f:( $\lambda x$   $3x^4$ )) ...:( $\lambda x$   $12x^3$ ))
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )):( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... ( $\mathcal{D}$  (lambda (x)  $3x^4$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define ( $\mathcal{D}_h$  f:( $\lambda x$   $3x^4$ )) ...:( $\lambda x$   $12x^3$ ))
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )):( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... ( $\mathcal{D}$  (lambda (x)  $3x^4$ )):( $\lambda x$   $12x^3$ ) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
  (if (zero? n) x ((compose (- n 1) f) (f x))))
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
  (if (zero? n) x ((compose (- n 1) f) (f x))))

((compose k  $\mathcal{D}$ ) g)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
  (if (zero? n) x ((compose (- n 1) f) (f x))))
```

```
((compose k  $\mathcal{D}$ ) g)
```

```
(define ( $\mathcal{D}_{\text{compose}}$  f:g) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
  (if (zero? n) x ((compose (- n 1) f) (f x))))
```

```
((compose k  $\mathcal{D}$ ) g)
```

```
(define ( $\mathcal{D}_{\text{compose}}$  f:g) ...)
```

```
(define ( $\mathcal{D}_{\text{compose:compose}}$  f:g') ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
  (if (zero? n) x ((compose (- n 1) f) (f x))))
```

```
((compose k  $\mathcal{D}$ ) g)
```

```
(define ( $\mathcal{D}_{\text{compose}}$   $f:g$ ) ...)
```

```
(define ( $\mathcal{D}_{\text{compose:compose}}$   $f:g'$ ) ...)
```

```
(define ( $\mathcal{D}_{\text{compose:compose:compose}}$   $f:g''$ ) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
  (if (zero? n) x ((compose (- n 1) f) (f x))))
```

```
((compose k  $\mathcal{D}$ ) g)
```

```
(define ( $\mathcal{D}_{\text{compose}}$   $f:g$ ) ...)
```

```
(define ( $\mathcal{D}_{\text{compose:compose}}$   $f:g'$ ) ...)
```

```
(define ( $\mathcal{D}_{\text{compose:compose:compose}}$   $f:g''$ ) ...)
```

⋮

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \dots\}$$

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \dots\}$$

$$\bar{v} ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (\bar{v}_1, \bar{v}_2) \mid \langle \bar{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\bar{\sigma} ::= \{x_1 \mapsto \bar{v}_1, \dots\}$$

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \dots\}$$

$$\bar{\mathcal{E}} : e \times \bar{\sigma} \rightarrow \bar{v}$$

$$\bar{v} ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (\bar{v}_1, \bar{v}_2) \mid \langle \bar{\sigma}, e \rangle \mid \bar{\mathbb{R}}$$

$$\bar{\sigma} ::= \{x_1 \mapsto \bar{v}_1, \dots\}$$

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \dots\}$$

$$\bar{\mathcal{E}} : e \times \bar{\sigma} \rightarrow \bar{v}$$

$$\bar{v} ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (\bar{v}_1, \bar{v}_2) \mid \langle \bar{\sigma}, e \rangle \mid \bar{\mathbb{R}}$$

$$\bar{\sigma} ::= \{x_1 \mapsto \bar{v}_1, \dots\}$$

Memoize $\bar{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\bar{\sigma}$.

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \dots\}$$

$$\bar{\mathcal{E}} : e \times \bar{\sigma} \rightarrow \bar{v}$$

$$\bar{v} ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (\bar{v}_1, \bar{v}_2) \mid \langle \bar{\sigma}, e \rangle \mid \bar{\mathbb{R}}$$

$$\bar{\sigma} ::= \{x_1 \mapsto \bar{v}_1, \dots\}$$

Memoize $\bar{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\bar{\sigma}$.

Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \dots\}$$

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Necessary for migrating reflective source-code transformation to compile time.

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Side benefit: union-free

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No tags, tag checking, tag dispatching, indirect calls

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No tags, tag checking, tag dispatching, indirect calls

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Necessary for migrating reflective source-code transformation to compile time.

Side benefits: union-free, no cyclic abstract values

No tags, tag checking, tag dispatching, indirect calls

Allows complete unboxing: no allocation, reclamation, indirection

Game Theory

		B			
		b_1	$\dots$	b_j	$\dots b_n$
A	a_1				
	$\vdots$		$\ddots$	$\vdots$	
	a_i	$\dots$		PAYOFF(a_i, b_j)	$\dots$
	$\vdots$			$\vdots$	$\ddots$
	a_m				

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Game Theory

		B			
		b_1	$\dots$	b_j	$\dots$ b_n
A	a_1				
	$\vdots$		$\ddots$	$\vdots$	
	a_i	$\dots$		$\text{PAYOFF}(a_i, b_j)$	$\dots$
	$\vdots$			$\vdots$	$\ddots$
	a_m				

$$\max_{a \in A} \min_{b \in B} \text{PAYOFF}(a, b)$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

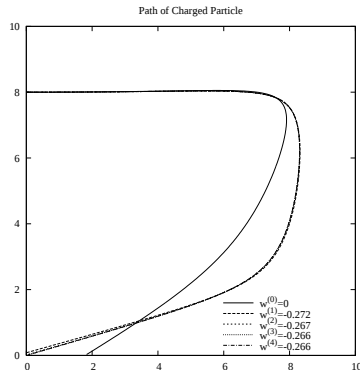
Game Theory

		$\mathbb{R}^n$		
		...	b	...
$\mathbb{R}^m$	a	...	PAYOFF(a , b)	...
		...		...
		...		...

$$\max_{\mathbf{a} \in \mathbb{R}^m} \min_{\mathbf{b} \in \mathbb{R}^n} \text{PAYOFF}(\mathbf{a}, \mathbf{b})$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Cathode Ray Tubes



$$\text{potential: } p(\mathbf{x}; w) = \|\mathbf{x} - (10, 10 - w)\|^{-1} + \|\mathbf{x} - (10, 0)\|^{-1}$$

$$\ddot{\mathbf{x}}(t) = -\nabla_{\mathbf{x}} p(\mathbf{x})|_{\mathbf{x}=\mathbf{x}(t)}$$

$$\dot{\mathbf{x}}(t + \Delta t) = \dot{\mathbf{x}}(t) + \Delta t \ddot{\mathbf{x}}(t)$$

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t \dot{\mathbf{x}}(t)$$

$$\text{When: } x_1(t + \Delta t) \leq 0$$

$$\text{let: } \Delta t_f = -x_1(t) / \dot{x}_1(t)$$

$$t_f = t + \Delta t_f$$

$$\mathbf{x}(t_f) = \mathbf{x}(t) + \Delta t_f \dot{\mathbf{x}}(t)$$

$$\text{Error: } E(w) = x_0(t_f)^2$$

$$\text{Find: } \underset{w}{\operatorname{argmin}} E(w)$$

Sprague, C. S. and George, R. H. (1939). *Cathode Ray Deflecting Electrode*. US Patent 2,161,437.

George, R. H. (1940). *Cathode Ray Tube*. US Patent 2,222,942.

Performance Comparison

		particle				saddle			
		FF	FR	RF	RR	FF	FR	RF	RR
VLAD	STALIN ∇	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
FORTRAN	ADIFOR	1.52	■	■	■	2.07	■	■	■
	TAPENADE	3.40	■	■	■	2.56	■	■	■
C++	FADBAD++	65.69	■	■	■	22.44	■	■	■
ML	MLTON	53.89	88.88	16.08	28.06	40.39	51.21	1.86	2.67
	OCAML	160.50	340.35	147.91	263.66	107.71	156.33	6.75	13.51
	SML/NJ	106.21	182.45	105.04	185.15	84.38	106.01	3.55	6.31
HASKELL	GHC	165.22	■	■	■	121.18	■	■	■
SCHEME	BIGLOO	505.90	761.40	104.81	228.56	423.69	440.25	15.77	24.59
	CHICKEN	1120.37	2026.31	425.60	1872.85	889.58	1144.65	35.73	68.94
	GAMBIT	444.13	752.63	138.34	256.30	362.65	420.48	14.08	23.87
	IKARUS	192.07	312.28	61.79	114.87	158.88	205.97	6.75	11.40
	LARCENY	726.59	1108.18	144.55	270.14	571.81	613.65	19.14	29.77
	MIT SCHEME	1472.26	2500.00	309.66	591.36	1243.26	1428.57	51.36	79.10
	MzC	2073.26	3434.64	340.30	655.83	2436.26	1996.40	72.45	150.02
	MzSCHEME	2344.70	4076.16	409.95	843.68	2000.89	2332.43	80.78	134.00
	SCHEME->C	391.42	605.26	109.77	198.43	324.95	328.84	12.74	18.28
	SCMUTILS	3321.20	■	■	■	2800.71	■	■	■
	STALIN	208.10	366.08	51.84	91.86	166.96	212.93	7.68	11.40

- not implemented but could implement
- not implemented in existing tool
- can't implement

Gradient-Based Optimization

```
(define (e i n)
  (if (zero? n)
      '()
      (cons (if (zero? i) 1.0 0.0)
              (e (- i 1) (- n 1)))))
```

Gradient-Based Optimization

```
(define (e i n)
  (if (zero? n)
      '()
      (cons (if (zero? i) 1.0 0.0)
              (e (- i 1) (- n 1)))))

(define ((gradient f) x)
  (let ((n (length x)))
    (map (lambda (i) (tangent ((j* f) (bundle x (e i n)))))
          (iota n))))
```

Gradient-Based Optimization

```
(define (e i n)
  (if (zero? n)
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      (cons (if (zero? i) 1.0 0.0)
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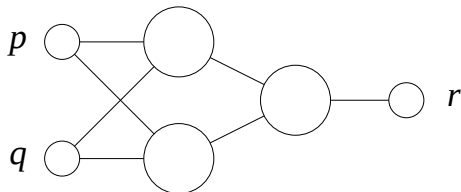
(define (gradient-ascent f x0 n eta)
  (if (zero? n)
      (list x0 (f x0) ((gradient f) x0))
      (gradient-ascent f
                        (zip (lambda (xi gi) (+ xi (* eta gi)))
                            x0
                            ((gradient f) x0))
                        (- n 1)
                        eta)))
```

Gradient-Based Optimization

```
(define ((gradient f) x) (cdr ((cdr ((*j f) (*j x))) 1.0)))
```

```
(define (gradient-ascent f x0 n eta)
  (if (zero? n)
      (list x0 (f x0) ((gradient f) x0))
      (gradient-ascent f
                        (zip (lambda (xi gi) (+ xi (* eta gi)))
                            x0
                            ((gradient f) x0))
                        (- n 1)
                        eta)))
```

Neural Networks



p	q	r
0	0	0
0	1	1
1	0	1
1	1	0

Rumelhart, D. E., Hinton, G. E., and Williams, R. J. (1986). *Learning representations by back-propagating errors*. Nature, **323**:533–6.

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
  ((fold + bias) (zip * ws activities)))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
  ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
  (map (sum-activities activities) ws-layer))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
  ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
  (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
  ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
  (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))

(define ((forward-pass ws-layers) in)
  (if (null? ws-layers)
      in
      ((forward-pass (cdr ws-layers))
       (map sigmoid (sum-layer in (car ws-layers))))))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
  ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
  (map (sum-activities activities) ws-layer))

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(define ((forward-pass ws-layers) in)
  (if (null? ws-layers)
      in
      ((forward-pass (cdr ws-layers))
       (map sigmoid (sum-layer in (car ws-layers))))))

(define ((error-on-dataset dataset) ws-layers)
  ((fold + 0)
   (map (lambda ((list in target))
          (* 0.5 (magnitude-squared (v- ((forward-pass ws-layers) in) target))))
        dataset)))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
  ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
  (map (sum-activities activities) ws-layer))

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(define ((forward-pass ws-layers) in)
  (if (null? ws-layers)
      in
      ((forward-pass (cdr ws-layers))
       (map sigmoid (sum-layer in (car ws-layers))))))

(define ((error-on-dataset dataset) ws-layers)
  ((fold + 0)
   (map (lambda ((list in target))
        (* 0.5 (magnitude-squared (v- ((forward-pass ws-layers) in) target))))
        dataset)))

(gradient-descent (error-on-dataset '(((0 0) (0))
                                       ((0 1) (1))
                                       ((1 0) (1))
                                       ((1 1) (0))))
                  '(((0 -0.284227 1.16054) (0 0.617194 1.30467))
                    ((0 -0.084395 0.648461)))
                  1000.0
                  0.3)
```

Performance Comparison

		backprop		
		Fs	Fv	R
VLAD	STALIN ∇	1.00	■	1.00
FORTRAN	ADIFOR	11.84	2.68	■
	TAPENADE	11.35	4.33	6.24
C	ADIC	16.33	3.93	■
C++	ADOL-C	12.34	3.89	35.53
	CppAD	42.15	■	23.69
	FADBAD++	98.96	33.15	53.03
ML	MLTON	73.94	■	37.94
	OCAML	157.75	■	149.14
	SML/NJ	142.71	■	94.97
HASKELL	GHC	■	■	■
SCHEME	BIGLOO	577.45	■	306.60
	CHICKEN	1391.75	■	971.91
	GAMBIT	545.20	■	341.73
	IKARUS	216.42	■	147.49
	LARCENY	955.98	■	486.64
	MIT SCHEME	1900.04	■	1141.22
	MzC	2439.93	■	1571.52
	MzSCHEME	3477.86	■	1866.28
	SCHEME->C	484.24	■	233.75
	SCMUTILS	4544.48	■	■
	STALIN	832.68	■	367.84

- not implemented but could implement
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$P = \text{if } x_0 \text{ then } 0 \text{ else if } x_1 \text{ then } 1 \text{ else } 2$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

$P = \text{if } x_0 \text{ then } 0 \text{ else if } x_1 \text{ then } 1 \text{ else } 2$

$$\Pr(x_0 \mapsto \mathbf{true}) = p_0$$

$$\Pr(x_0 \mapsto \mathbf{false}) = 1 - p_0$$

$$\Pr(x_1 \mapsto \mathbf{true}) = p_1$$

$$\Pr(x_1 \mapsto \mathbf{false}) = 1 - p_1$$

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$$\Pr(\mathcal{E}(P) = 0 | p_0, p_1) = p_0$$

$$\Pr(\mathcal{E}(P) = 1 | p_0, p_1) = (1 - p_0)p_1$$

$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

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$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

$$\prod_{v \in \{0,1,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

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$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

$$\prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

$$\operatorname{argmax}_{p_0, p_1} \prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = \left\langle \frac{1}{4}, \frac{1}{3} \right\rangle$$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Prolog

$p(\theta).$

$p(X) :- q(X).$

$q(1).$

$q(2).$

Probabilistic Prolog

$$\Pr(p(\theta) \text{ .}) = p_0$$

$$\Pr(p(X) : \neg q(X) \text{ .}) = 1 - p_0$$

$$\Pr(q(1) \text{ .}) = p_1$$

$$\Pr(q(2) \text{ .}) = 1 - p_1$$

Probabilistic Prolog

$$\Pr(p(\Theta) \text{ .}) = p_0$$

$$\Pr(p(X) : \neg q(X) \text{ .}) = 1 - p_0$$

$$\Pr(q(1) \text{ .}) = p_1$$

$$\Pr(q(2) \text{ .}) = 1 - p_1$$

$$\Pr(? \text{ - } p(\Theta) \text{ .}) = p_0$$

$$\Pr(? \text{ - } p(1) \text{ .}) = (1 - p_0)p_1$$

$$\Pr(? \text{ - } p(2) \text{ .}) = (1 - p_0)(1 - p_1)$$

Probabilistic Prolog

$$\Pr(p(\theta) \text{ .}) = p_0$$

$$\Pr(p(X) : \neg q(X) \text{ .}) = 1 - p_0$$

$$\Pr(q(1) \text{ .}) = p_1$$

$$\Pr(q(2) \text{ .}) = 1 - p_1$$

$$\Pr(? - p(\theta) \text{ .}) = p_0$$

$$\Pr(? - p(1) \text{ .}) = (1 - p_0)p_1$$

$$\Pr(? - p(2) \text{ .}) = (1 - p_0)(1 - p_1)$$

$$\prod_{q \in \{p(\theta), p(1), p(2), p(2)\}} \Pr(? - q \text{ .}) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

Probabilistic Prolog

$$\Pr(p(\theta) \text{ .}) = p_0$$

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$$\prod_{q \in \{p(\theta), p(1), p(2), p(2)\}} \Pr(? - q \text{ .}) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

$$\operatorname{argmax}_{p_0, p_1} \prod_{q \in \{p(\theta), p(1), p(2), p(2)\}} \Pr(? - q \text{ .}) = \left\langle \frac{1}{4}, \frac{1}{3} \right\rangle$$

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
  (cond
    ((constant-expression? expression)
     (singleton-tagged-distribution
      (constant-expression-value expression)))
    ((variable-access-expression? expression)
     (lookup-value
      (variable-access-expression-variable expression) environment))
    ((lambda-expression? expression)
     (singleton-tagged-distribution
      (lambda (tagged-distribution)
        (evaluate
         (lambda-expression-body expression)
         (cons (make-binding (lambda-expression-variable expression)
                             tagged-distribution)
               environment))))))
    (else (let ((tagged-distribution
                  (evaluate (application-argument expression)
                           environment)))
                (map-tagged-distribution
                 (lambda (value) (value tagged-distribution))
                 (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
  (cond
    ((constant-expression? expression)
     (singleton-tagged-distribution
      (constant-expression-value expression)))
    ((variable-access-expression? expression)
     (lookup-value
      (variable-access-expression-variable expression) environment))
    ((lambda-expression? expression)
     (singleton-tagged-distribution
      (lambda (tagged-distribution)
        (evaluate
         (lambda-expression-body expression)
         (cons (make-binding (lambda-expression-variable expression)
                             tagged-distribution)
               environment))))))
    (else (let ((tagged-distribution
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Probabilistic Lambda Calculus

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Probabilistic Lambda Calculus

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  (lambda (value)
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Probabilistic Lambda Calculus

(gradient-ascent

(lambda (p)

(let ((tagged-distribution

(evaluate **if** x_0 **then** 0 **else if** x_1 **then** 1 **else** 2

(list $\Pr(x_0 \mapsto \mathbf{true}) = p_0$ $\Pr(x_0 \mapsto \mathbf{false}) = 1 - p_0$

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Probabilistic Prolog

```
(define (proof-distribution term clauses)
  (let ((offset ...))
    (map-reduce
      append
      '()
      (lambda (clause)
        (let ((clause (alpha-rename clause offset)))
          (let loop ((p (clause-p clause))
                     (substitution (unify term (clause-term clause)))
                     (terms (clause-terms clause)))
            (if (boolean? substitution)
                '()
                (if (null? terms)
                    (list (make-double p substitution))
                    (map-reduce
                      append
                      '()
                      (lambda (double)
                        (loop (* p (double-p double))
                              (append substitution (double-substitution double))
                              (rest terms)))
                      (proof-distribution
                       (apply-substitution substitution (first terms)) clauses)))))))
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Probabilistic Prolog

```
(gradient-ascent
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  (let ((clauses (list Pr(p(0).) = p0
                        Pr(p(X):-q(X).) = 1 - p0
                        Pr(q(1).) = p1
                        Pr(q(2).) = 1 - p1))))
    (map-reduce
     *
     1.0
     (lambda (query)
      (likelihood (proof-distribution query clauses)))
      '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
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     (lambda (query)
      (likelihood (proof-distribution query clauses)))
     '(p(0) p(1) p(2) p(2)))))
'(0.5 0.5)
1000.0
0.1)
```

Generated Code

```
static void f2679(double a_f2679_0,double a_f2679_1,double a_f2679_2,double a_f2679_3){
    int t272381=((a_f2679_2==0.)?0:1);
    double t272406;
    double t272405;
    double t272404;
    double t272403;
    double t272402;
    if((t272381==0)){
        double t272480=(1.-a_f2679_0);
        double t272572=(1.-a_f2679_1);
        double t273043=(a_f2679_0+0.);
        double t274185=(t272480*a_f2679_1);
        double t274426=(t274185+0.);
        double t275653=(t272480*t272572);
        double t275894=(t275653+0.);
        double t277121=(t272480*t272572);
        double t277362=(t277121+0.);
        double t277431=(t277362*1.);
        double t277436=(t275894*t277431);
        double t277441=(t274426*t277436);
        double t277446=(t273043*t277441);
        ...
        double t1777107=(t1774696+t1715394);
        double t1777194=(0.-t1745420);
        double t1778533=(t1777194+t1419700);
        t272406=a_f2679_0;
        t272405=a_f2679_1;
        t272404=t277446;
        t272403=t1778533;
        t272402=t1777107;}
    else {...}
    r_f2679_0=t272406;
    r_f2679_1=t272405;
    r_f2679_2=t272404;
    r_f2679_3=t272403;
    r_f2679_4=t272402;}
```

Performance Comparison

		probabilistic-lambda-calculus		probabilistic-prolog	
		F	R	F	R
VLAD	STALIN ∇	1.00	1.00	1.00	1.00
ML	MLTON	106.45	124.95	789.41	483.47
	OCAML	215.73	538.68	1207.13	1534.61
	SML/NJ	197.75	272.45	2448.02	1471.94
HASKELL	GHC	■	■	■	■
SCHEME	BIGLOO	832.92	1048.11	14422.16	8286.06
	CHICKEN	2305.98	3283.00	66948.70	37792.84
	GAMBIT	879.88	1153.86	24316.03	13649.81
	IKARUS	437.46	531.10	8242.92	4845.86
	LARCENY	1651.01	1673.22	25589.62	14833.53
	MIT SCHEME	3491.10	4130.19	85819.57	48335.38
	MzC	5289.17	5929.14	154206.95	83480.27
	MzSCHEME	6235.78	7134.71	166129.12	91630.70
	SCHEME->C	682.15	794.31	10530.66	5980.27
	SCMUTILS	6456.99	■	80100.23	■
	STALIN	1240.73	1137.41	22511.79	10986.43

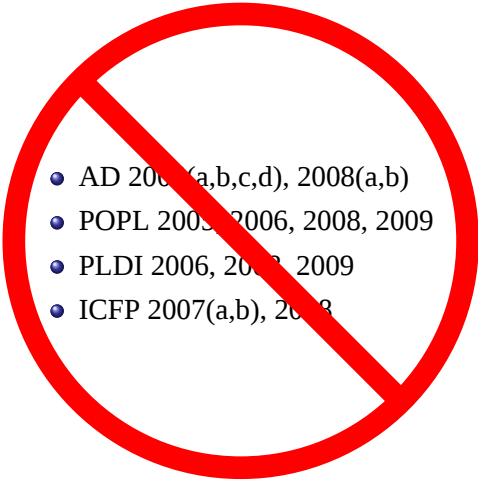
- not implemented but could implement, including FORTRAN, C, and C++
- not implemented in existing tool
- can't implement

Where You Can Read About This Work

Where You Can Read About This Work

- AD 2004(a,b,c,d), 2008(a,b)
- POPL 2005, 2006, 2008, 2009
- PLDI 2006, 2008, 2009
- ICFP 2007(a,b), 2008

Where You Can Read About This Work

- 
- AD 2005(a,b,c,d), 2008(a,b)
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 - PLDI 2006, 2008, 2009
 - ICFP 2007(a,b), 2008

Evaluation: C

It is not clear to me how this approach is better than automatic differentiation frameworks using C++ templates and the C++ type inference algorithm.

The paper seems correct, and I agree that it is hard to implement automatic differentiation both conveniently and efficiently without the proper programming language support (i.e., efficiently implemented generics with a minimal amount of type inference).

It's not clear to me that an efficient C++ templates implementation doesn't give the benefits listed here: it is "callee derives" (mostly) and efficiently implemented (using inlining). (The extent that it's not "callee derives" is that you need to put "template<class X>" in front of every function and declare all your variables as being of type "X" instead of "double".)

I was somewhat shocked that the only C++ implementation you compared to was apparently one that uses virtual function dispatch (FADBAD++). I Googled "automatic differentiation C++ template" and found two papers that do what I expected you to compare to:

Dan Piponi: Automatic Differentiation, C++ Templates and Photogrammetry, "to be published in the Journal of Graphics Tools", Sep. 2004.

and

M.E. Jerrell: Function Minimization and Automatic Differentiation Using C++, OOPSLA, 1989.

Evaluation: C. Weak paper, but it will not be an embarrassment to have it in POPL.
Confidence: Z. I am an informed outsider and tried my best to understand the paper.

===== Summary =====

Shows how to optimize a functional language with a built-in automatic differentiation operator.

===== Detailed Comments =====

The results look useful, but I wonder whether POPL is the right place to present them. Yes, the development involves functional programming. But it also involves a lot of concepts from scientific computing that may be unfamiliar to many and that are explained only minimally or not at all.

It is, of course, not excluded that the range of arguments or range of values of a function should consist wholly or partly of functions. The derivative, as this notion appears in the elementary differential calculus, is a familiar mathematical example of a function for which both ranges consist of functions.

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(¶4)

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Gottfried Leibniz
|
Jacob Bernoulli
|
Johann Bernoulli
|
Leonhard Euler
|
Joseph Louis Lagrange
|
Simeon Poisson
|
Michel Chasles
|
Hubert Anson Newton
|
Eliakim Hastings Moore
|
Oswald Veblen
|
Alonzo Church