Binomial Checkpointing for Arbitrary Programs with No User Annotation

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Joint work with Barak Avrum Pearlmutter

Automatic Differentiation (AD)

$$f = f_t \circ f_{t-1} \circ \cdots \circ f_2 \circ f_1$$

Automatic Differentiation (AD)

$$
\begin{aligned}
f &= f_t \circ f_{t-1} \circ \cdots \circ f_2 \circ f_1 \\
f'(x_1) &= f'_t(x_t) \times f'_{t-1}(x_{t-1}) \times \cdots \times f'_2(x_2) \times f'_1(x_1)
\end{aligned}
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f'(x_1) &= f_t'(x_t) \times f_{t-1}'(x_{t-1}) \times \cdots \times f_2'(x_2) \times f_1'(x_1) \\
f'(x_1)^\top &= f_1'(x_1)^\top \times f_2(x_2)^\top \times \cdots \times f_{t-1}'(x_{t-1})^\top \times f_t'(x_t)^\top
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\begin{aligned}
f &= f_t \circ f_{t-1} \circ \cdots \circ f_2 \circ f_1 \\
f'(x_1) \times \acute{x}_1 &= f_t'(x_t) \times f_{t-1}'(x_{t-1}) \times \cdots \times f_2'(x_2) \times f_1'(x_1) \times \acute{x}_1 \\
f'(x_1)^\top \times \grave{y} &= f_1'(x_1)^\top \times f_2(x_2)^\top \times \cdots \times f_{t-1}'(x_{t-1})^\top \times f_t'(x_t)^\top \times \grave{y}
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$$x_2 = f_1(x_1)$$

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$$x_t = \quad f_{t-1}(\ldots f_2(f_1(x_1))\ldots)$$

Automatic Differentiation (AD)

$$f = f_t \circ f_{t-1} \circ \cdots \circ f_2 \circ f_1$$

$$y = f_t(f_{t-1}(\ldots f_2(f_1(x_1))\ldots))$$

Automatic Differentiation (AD)

$=$ $f_1'(x_1) \times \acute{x}_1$

x_1

Automatic Differentiation (AD)

$$= \quad f_2'(x_2) \times f_1'(x_1) \times \acute{x}_1$$

$$x_2 = f_1(x_1)$$

Automatic Differentiation (AD)

$$
\begin{aligned}
&= \quad f'_{t-1}(x_{t-1}) \times \cdots \times f'_2(x_2) \times f'_1(x_1) \times \acute{x}_1 \\
x_{t-1} &= \quad f_{t-2}(\ldots f_2(f_1(x_1))\ldots)
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$$f'(x_1) \times \acute{x}_1 = f_t'(x_t) \times f_{t-1}'(x_{t-1}) \times \cdots \times f_2'(x_2) \times f_1'(x_1) \times \acute{x}_1$$

$$x_t = \quad f_{t-1}(\ldots f_2(f_1(x_1)) \ldots)$$

Automatic Differentiation (AD)

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$$x_2 = f_1(x_1)$$

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$$x_{t-1} = f_{t-2}(\ldots f_2(f_1(x_1))\ldots)$$

$$x_{t-1}, \ldots, x_2, x_1$$

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$$x_t = \quad f_{t-1}(\ldots f_2(f_1(x_1))\ldots)$$

$$x_t, x_{t-1}, \ldots, x_2, x_1$$

Automatic Differentiation (AD)

$$f_t'(\color{red}{x_t})^\top \times \grave{y}$$

$$\color{red}{x_t}\color{black}{, x_{t-1}, \ldots, x_2, x_1}$$

Automatic Differentiation (AD)

$$f'_{t-1}(x_{t-1})^\top \times f'_t(x_t)^\top \times \grave{y}$$

$$x_{t-1}, \ldots, x_2, x_1$$

Automatic Differentiation (AD)

$$f_2(x_2)^\top \times \cdots \times f'_{t-1}(x_{t-1})^\top \times f'_t(x_t)^\top \times \grave{y}$$

x_2, x_1

Automatic Differentiation (AD)

$$f'(x_1)^\top \times \grave{y} = f_1'(x_1)^\top \times f_2(x_2)^\top \times \cdots \times f_{t-1}'(x_{t-1})^\top \times f_t'(x_t)^\top \times \grave{y}$$

x_1

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- $f : \mathbb{R}^n \to \mathbb{R}$

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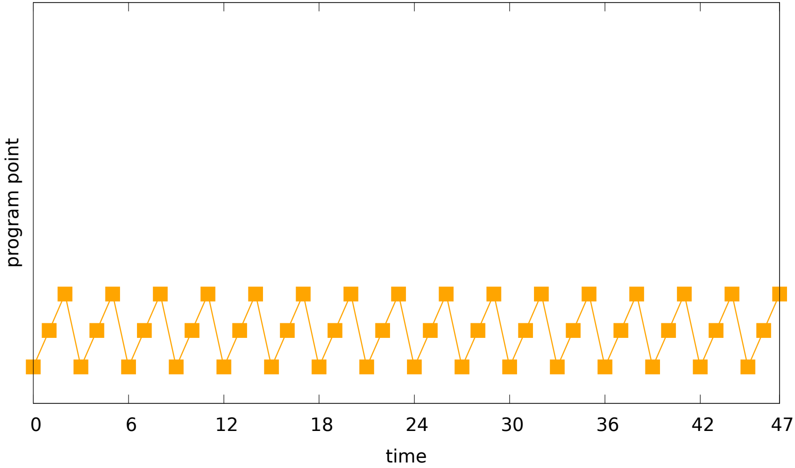
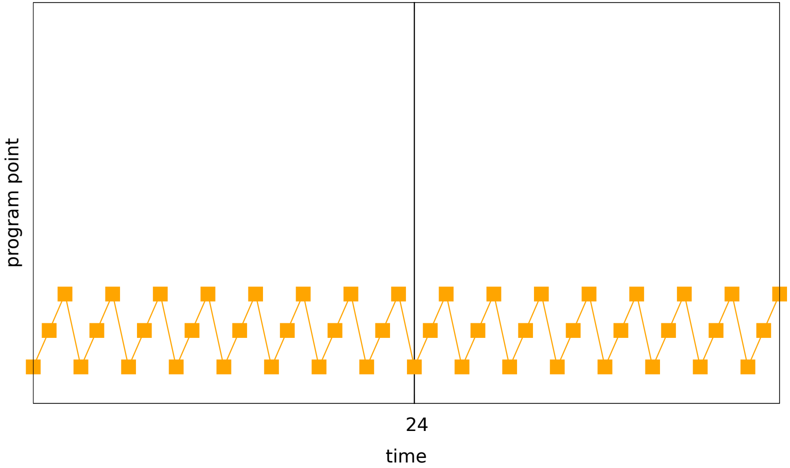
- both n and t are large
- today: computing ∇f with checkpointing reverse mode entails an increase of $O(\log n)$ time and $O(\log t)$ space over computing f

Execution Trace of Loop



Execution Trace of Loop

Easy to make regular and uniform checkpoints



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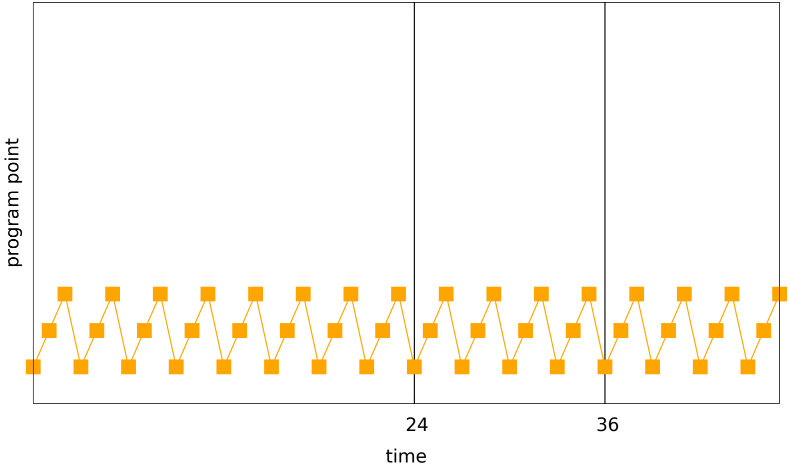

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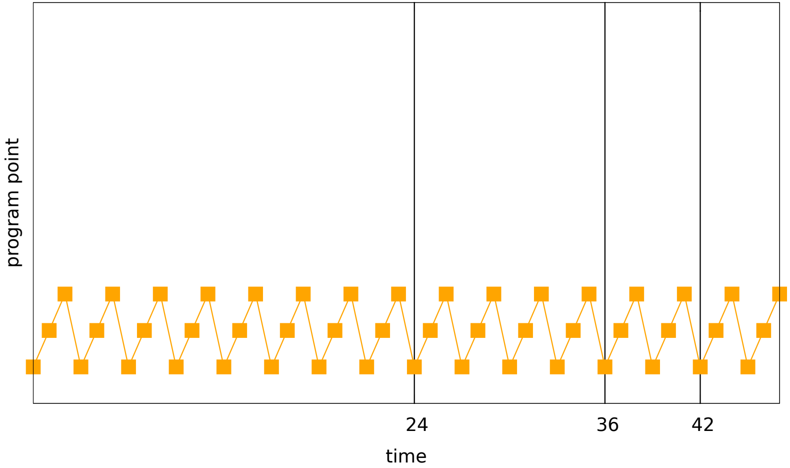

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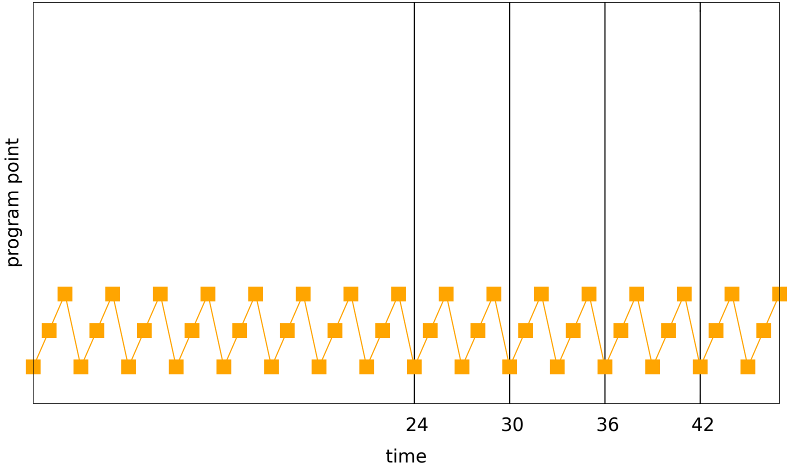

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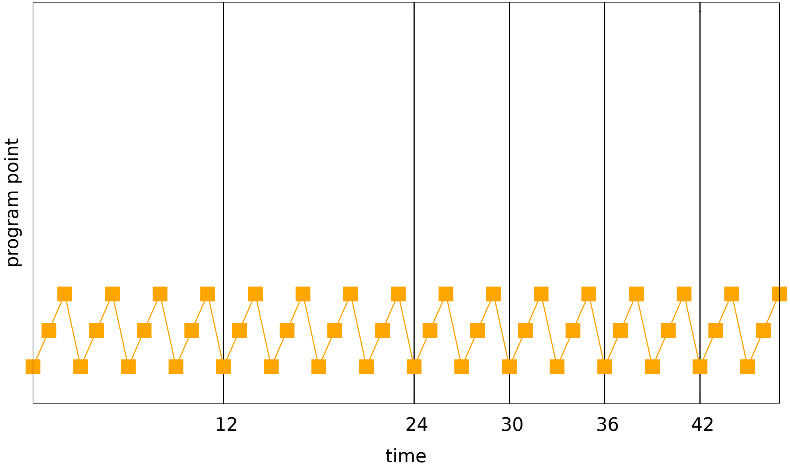

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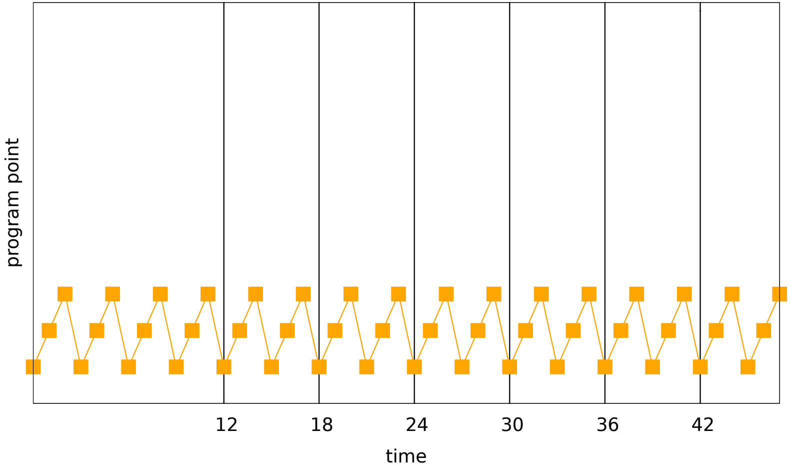

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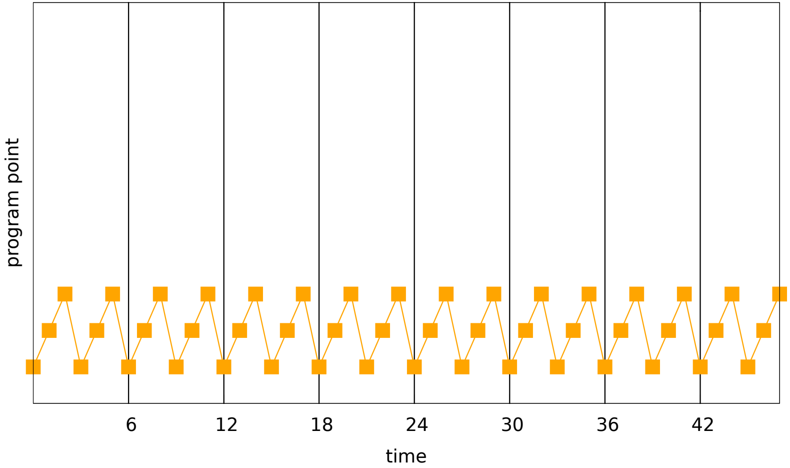
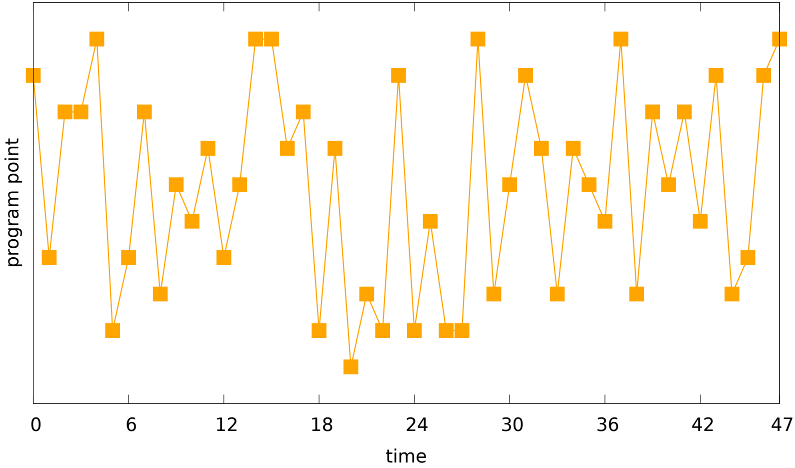


Execution Trace of Arbitrary Code



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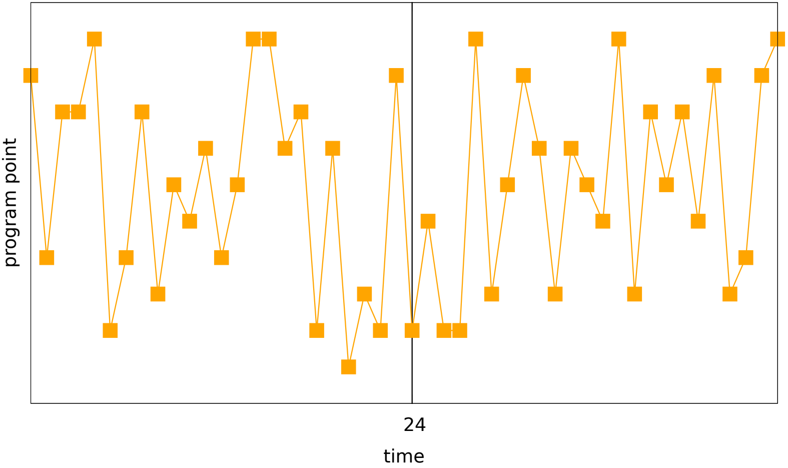

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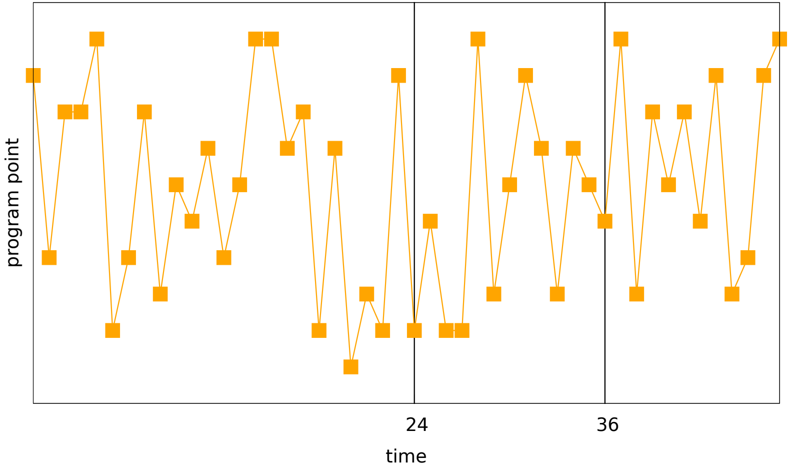

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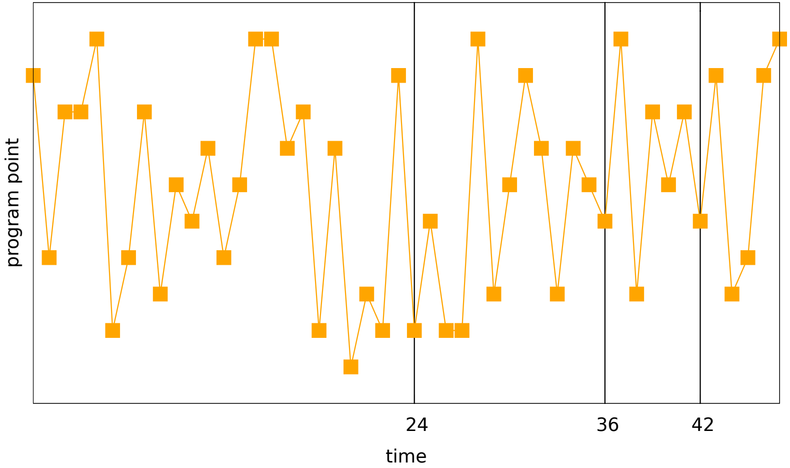

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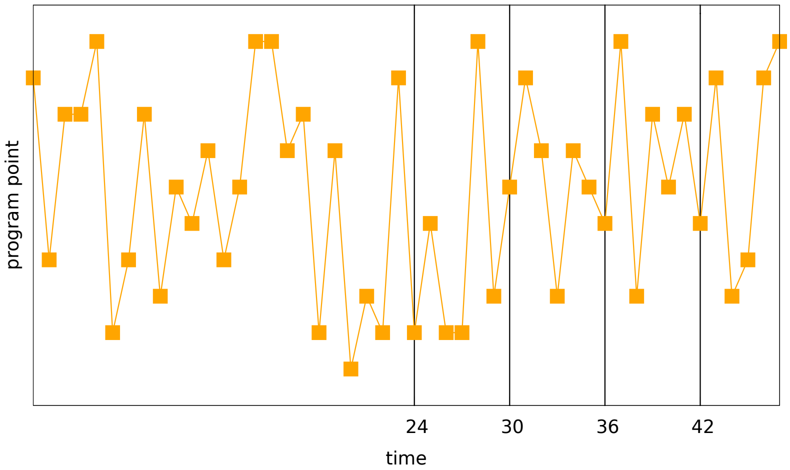

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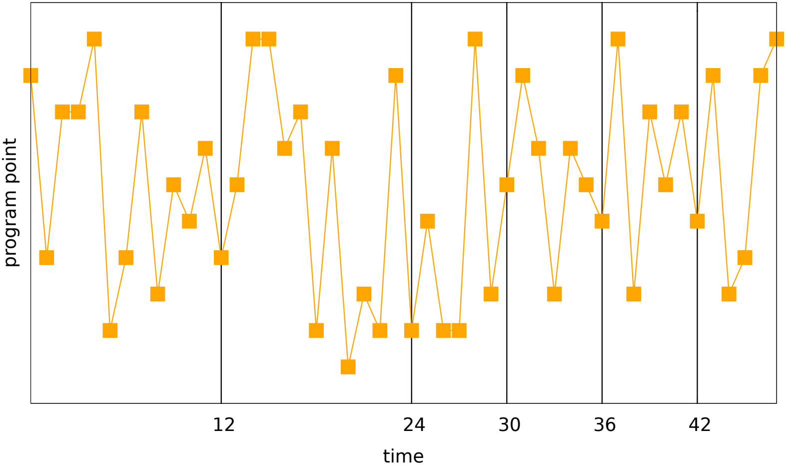

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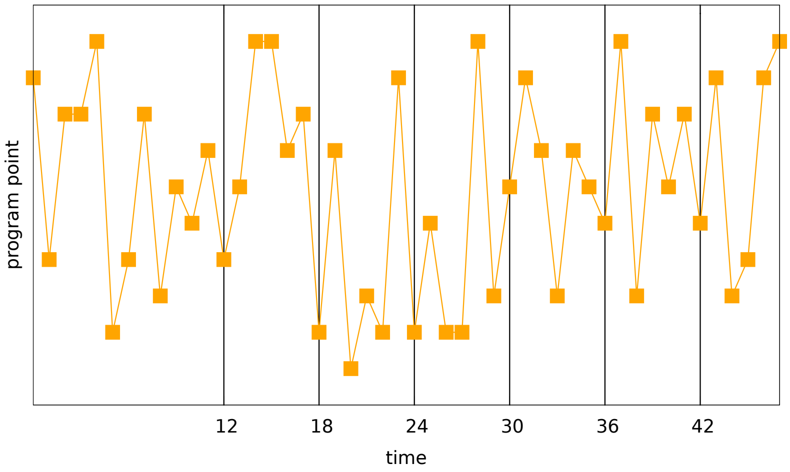

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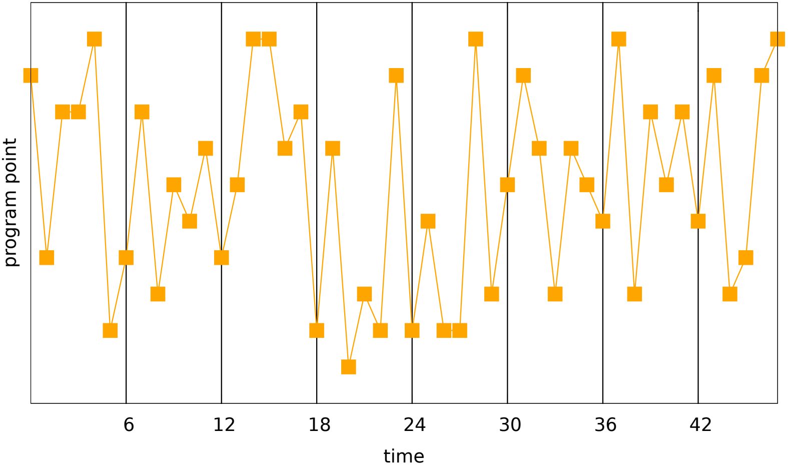

Key Idea

```
function main(w)
    local x = f(w)
    local y = h(g(x))
    local z = p(y)
    return z
end
```

$\leadsto$

```
function main(w)
    for i = 1, 5
    if i==1 then
        local x = f(w)
    elseif i==2 then
        local t = g(x)
    elseif i==3 then
        local y = h(t)
    elseif i==4 then
        local z = p(y)
    elseif i==5 then
        return z
    end
end
```

Algorithm for Bisection Checkpointing

To compute $(y, \grave{x}) = \overset{\surd}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y}$ (0)

inductive case:

$$h \circ g = f \quad (1)$$
$$u = g\ x \quad (2)$$
$$(y, \grave{u}) = \overset{\surd}{\mathcal{J}}\ h\ u\ \grave{y} \quad (3)$$
$$(u, \grave{x}) = \overset{\surd}{\mathcal{J}}\ g\ x\ \grave{u} \quad (4)$$

Algorithm for Bisection Checkpointing

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

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Algorithm for Bisection Checkpointing

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ {\color{red}f}\ x\ \grave{y}$:

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$$u = g\ x \tag{2}$$

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inductive case:

$$h \circ \color{red}{g} = f \tag{1}$$

$$u = \color{red}{g}\ x \tag{2}$$

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Algorithm for Bisection Checkpointing

To compute $(y, \grave{x}) = \overset{\surd}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y}$ (0)

inductive case:

$$h \circ g = f \quad (1)$$

$$u = g\ x \quad (2)$$

$$(y, \grave{u}) = \overset{\surd}{\mathcal{J}}\ h\ u\ \grave{y} \quad (3)$$

$$(u, \grave{x}) = \overset{\surd}{\mathcal{J}}\ g\ x\ \grave{u} \quad (4)$$

A General Checkpointing API

$\text{PRIMOPS}\ f\ x \mapsto (y, n)$	Return $y = f(x)$ along with the number n of steps needed to compute y.
$\text{CHECKPOINT}\ f\ x\ n \mapsto u$	Run the first n steps of the computation of $f(x)$ and return a checkpoint u.
$\text{RESUME}\ u \mapsto y$	If $u = (\text{CHECKPOINT}\ f\ x\ n)$, return $y = f(x)$.

A General Checkpointing API

PRIMOPS $f\ x \mapsto (y, n)$	Return $y = f(x)$ along with the number n of steps needed to compute y.
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RESUME $u \mapsto y$	If $u = (\text{CHECKPOINT } f\ x\ n)$, return $y = f(x)$.

A General Checkpointing API

$\textsc{primops}\ f\ x \mapsto (y, n)$	Return $y = f(x)$ along with the number n of steps needed to compute y.
$\textsc{checkpoint}\ f\ x\ n \mapsto u$	Run the first n steps of the computation of $f(x)$ and return a checkpoint u.
$\textsc{resume}\ u \mapsto y$	If $u = (\textsc{checkpoint}\ f\ x\ n)$, return $y = f(x)$.

A General Checkpointing API

PRIMOPS $f\ x \mapsto (y, n)$	Return $y = f(x)$ along with the number n of steps needed to compute y.
CHECKPOINT $f\ x\ n \mapsto u$	Run the first n steps of the computation of $f(x)$ and return a checkpoint u.
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$\textsc{primops}\ f\ x \mapsto (y, n)$	Return $y = f(x)$ along with the number n of steps needed to compute y.
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A General Checkpointing API

$\textsc{primops}\ f\ x \mapsto (y, n)$	Return $y = f(x)$ along with the number n of steps needed to compute y.
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A General Checkpointing API

$\textsc{primops}\ f\ x \mapsto (y, n)$	Return $y = f(x)$ along with the number n of steps needed to compute y.
$\textsc{checkpoint}\ f\ x\ n \mapsto u$	Run the first n steps of the computation of $f(x)$ and return a checkpoint u.
$\textsc{resume}\ u \mapsto y$	If $u = (\textsc{checkpoint}\ f\ x\ n)$, return $y = f(x)$.

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y} \tag{0}$$

inductive case:

$$h \circ g = f \tag{1}$$

$$u = g\ x \tag{2}$$

$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y} \tag{3}$$

$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y} \tag{0}$$

inductive case:
$$h \circ g = f \tag{1}$$
$$u = g\ x \tag{2}$$
$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y} \tag{3}$$
$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y} \tag{0}$$

inductive case:
$$h \circ g = f \tag{1}$$
$$u = g\ x \tag{2}$$
$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y} \tag{3}$$
$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

$$\begin{aligned}
&\textbf{base case } (f\ x \text{ fast}): && (y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y} && (0)\\
&\textbf{inductive case}: && (y, 2n) = \text{PRIMOPS}\ f\ x && (1)\\
& && u = g\ x && (2)\\
& && (y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y} && (3)\\
& && (u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u} && (4)
\end{aligned}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y}$ (0)

inductive case:

$$(y, 2n) = \text{PRIMOPS}\ f\ x \tag{1}$$

$$u = g\ x \tag{2}$$

$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y} \tag{3}$$

$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\; f\; x\; \grave{y}$:

base case ($f\; x$ fast): $$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\; f\; x\; \grave{y} \tag{0}$$

inductive case:
$$(y, 2n) = \text{PRIMOPS}\; f\; x \tag{1}$$
$$u = \color{red}{g}\; x \tag{2}$$
$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\; h\; u\; \grave{y} \tag{3}$$
$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\; g\; x\; \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\, {\color{red}f}\, x\, \grave{y}$:

base case (f x fast): $$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\, f\, x\, \grave{y} \tag{0}$$

inductive case:

$$(y, {\color{red}2n}) = \text{PRIMOPS}\, f\, x \tag{1}$$

$$u = {\color{red}\text{CHECKPOINT}\, f\, x\, n} \tag{2}$$

$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\, h\, u\, \grave{y} \tag{3}$$

$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\, g\, x\, \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

$$\begin{aligned}
&\textbf{base case } (f\ x \text{ fast}): && (y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y} && (0)\\
&\textbf{inductive case}: && (y, 2n) = \text{PRIMOPS}\ f\ x && (1)\\
& && u = \text{CHECKPOINT}\ f\ x\ n && (2)\\
& && {\color{red}(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y}} && (3)\\
& && (u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u} && (4)
\end{aligned}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y}$ (0)

inductive case:

$$(y, 2n) = \text{PRIMOPS}\ f\ x \tag{1}$$

$$u = \text{CHECKPOINT}\ f\ x\ n \tag{2}$$

$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y} \tag{3}$$

$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\, f\ x\ \grave{y}$:

base case ($f\ x$ fast): $$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\, f\ x\ \grave{y} \tag{0}$$

inductive case:

$$(y, 2n) = \text{PRIMOPS}\ f\ x \tag{1}$$

$$u = \text{CHECKPOINT}\ f\ x\ n \tag{2}$$

$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ (\lambda u.\text{RESUME}\ u)\ u\ \grave{y} \tag{3}$$

$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\, f\; x\; \grave{y}$:

base case ($f\; x$ fast): $$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\, f\; x\; \grave{y} \tag{0}$$

inductive case:
$$(y, 2n) = \text{PRIMOPS}\; f\; x \tag{1}$$
$$u = \text{CHECKPOINT}\; f\; x\; n \tag{2}$$
$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\, (\lambda u.\text{RESUME}\; u)\; u\; \grave{y} \tag{3}$$
$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\, g\; x\; \grave{u} \tag{4}$$

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\; f\; x\; \grave{y}$:

base case ($f\; x$ fast): $$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\; f\; x\; \grave{y} \tag{0}$$

inductive case:
$$(y, 2n) = \text{PRIMOPS}\; f\; x \tag{1}$$
$$u = \text{CHECKPOINT}\; f\; x\; n \tag{2}$$
$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\; (\lambda u.\text{RESUME}\; u)\; u\; \grave{y} \tag{3}$$
$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\; g\; x\; \grave{u} \tag{4}$$

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

$$
\begin{array}{lll}
\textbf{base case } (f\ x \text{ fast}): & (y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y} & (0) \\
\textbf{inductive case}: & (y, 2n) = \text{PRIMOPS}\ f\ x & (1) \\
& u = \text{CHECKPOINT}\ f\ x\ n & (2) \\
& (y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ (\lambda u.\text{RESUME}\ u)\ u\ \grave{y} & (3) \\
& (u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ (\lambda x.\text{CHECKPOINT}\ f\ x\ n)\ x\ \grave{u} & (4)
\end{array}
$$

CPS Conversion

```
function f(x)
   return q(p(g(x), h(x)))
end
```

$\leadsto$

```
function f(c, x)
   return g(function(t1)
               return h(function(t2)
                           return p(function(t3)
                                       return q(c, t3)
                                    end, t1, t2)
                        end, x)
            end, x)
end
```

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\,x \\
[\lambda x.e]_c & \rightsquigarrow & c\,\lambda c\,x.[e]_c \\
[e_1\,e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\,c\,x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\; x \\
[\lambda x.e]_c & \rightsquigarrow & c\; \lambda c\; x.[e]_c \\
[e_1\; e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\; c\; x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_{\color{red}c} & \rightsquigarrow & c\;x \\
[\lambda x.e]_c & \rightsquigarrow & c\;\lambda c\;x.[e]_c \\
[e_1\;e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\;c\;x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_{\color{red}c} & \rightsquigarrow & {\color{red}c}\ x \\
[\lambda x.e]_c & \rightsquigarrow & c\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \leadsto & c\ x \\
[\lambda x.e]_c & \leadsto & c\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_c & \leadsto & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \leadsto & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\;x \\
[\lambda x.e]_c & \rightsquigarrow & c\;\lambda c\;x.[e]_c \\
[e_1\;e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\;c\;x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{aligned}
[x]_c &\leadsto c\,x\\
[\lambda x.e]_{\color{red}c} &\leadsto c\,\lambda c\,x.[e]_c\\
[e_1\,e_2]_c &\leadsto [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\,c\,x_2}}\\
e_0 &\leadsto [e_0]_{\lambda x.x}
\end{aligned}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\,x \\
[\lambda x.e]_c & \rightsquigarrow & c\,\lambda {\color{red}c}\,x.[e]_c \\
[e_1\,e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\,c\,x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

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[x]_c & \rightsquigarrow & c\; x \\
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CPS Conversion is a Program Transformation

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[\lambda x.e]_c & \rightsquigarrow & c\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
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$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \leadsto & c\;x \\
[\lambda x.e]_c & \leadsto & c\;\lambda c\;x.[e]_c \\
[e_1\;e_2]_c & \leadsto & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\;c\;x_2}} \\
e_0 & \leadsto & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\ x \\
[\lambda x.e]_{\color{red}c} & \rightsquigarrow & {\color{red}c}\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \leadsto & c\,x \\
[\lambda x.e]_c & \leadsto & c\,\lambda c\,x.[e]_c \\
[{\color{red}e_1\ e_2}]_c & \leadsto & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\,c\,x_2}} \\
e_0 & \leadsto & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\ x \\
[\lambda x.e]_c & \rightsquigarrow & c\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_{\color{red}{c}} & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$\begin{array}{lcl}
[x]_c & \leadsto & c\;x \\
[\lambda x.e]_c & \leadsto & c\;\lambda c\;x.[e]_c \\
[e_1\;e_2]_c & \leadsto & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\;c\;x_2}} \\
e_0 & \leadsto & [e_0]_{\lambda x.x}
\end{array}$$

CPS Conversion is a Program Transformation

$$
\begin{aligned}
[x]_c &\rightsquigarrow c\,x\\
[\lambda x.e]_c &\rightsquigarrow c\,\lambda c\,x.[e]_c\\
[e_1\,e_2]_c &\rightsquigarrow [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\,c\,x_2}}\\
e_0 &\rightsquigarrow [e_0]_{\lambda x.x}
\end{aligned}
$$

CPS Conversion is a Program Transformation

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[x]_c & \rightsquigarrow & c\ x \\
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[e_1\ e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \leadsto & c\ x \\
[\lambda x.e]_c & \leadsto & c\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_c & \leadsto & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \leadsto & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\;x \\
[\lambda x.e]_c & \rightsquigarrow & c\;\lambda c\;x.[e]_c \\
[e_1\;e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\;c\;x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

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\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\;x \\
[\lambda x.e]_c & \rightsquigarrow & c\;\lambda c\;x.[e]_c \\
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\end{array}
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CPS Conversion is a Program Transformation

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\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\ x \\
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[e_1\ e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\ x \\
[\lambda x.e]_c & \rightsquigarrow & c\ \lambda c\ x.[e]_c \\
[e_1\ {\color{red}e_2}]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda {\color{red}x_2}.x_1\ c\ {\color{red}x_2}}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\; x \\
[\lambda x.e]_c & \rightsquigarrow & c\; \lambda c\; x.[e]_c \\
[e_1\; e_2]_{\color{red}{c}} & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\; {\color{red}{c}}\; x_2}} \\
e_0 & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\,x \\
[\lambda x.e]_c & \rightsquigarrow & c\,\lambda c\,x.[e]_c \\
[e_1\;e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\,c\,x_2}} \\
{\color{red}e_0} & \rightsquigarrow & [e_0]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\ x \\
[\lambda x.e]_c & \rightsquigarrow & c\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 & \rightsquigarrow & [{\color{red}e_0}]_{\lambda x.x}
\end{array}
$$

CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \leadsto & c\;x \\
[\lambda x.e]_c & \leadsto & c\;\lambda c\;x.[e]_c \\
[e_1\;e_2]_c & \leadsto & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\;c\;x_2}} \\
e_0 & \leadsto & [e_0]_{\color{red}{\lambda x.x}}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lll}
[x]_c & \rightsquigarrow & c \qquad x \\
[\lambda x.e]_c & \rightsquigarrow & c \qquad \lambda c \quad x.[e]_c \\
[e_1\ e_2]_c & \rightsquigarrow & [e_1]_{(\lambda \quad x_1.[e_2]_{(\lambda \quad x_2.x_1\ c \quad x_2)} \quad)}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n} & \rightsquigarrow & c \quad n \qquad x \\
[\lambda x.e]_{c,n} & \rightsquigarrow & c \quad n \qquad \lambda c\ n \quad x.[e]_{c,n} \\
[e_1\ e_2]_{c,n} & \rightsquigarrow & [e_1]_{(\lambda n_1 \quad x_1.[e_2]_{(\lambda n_2 \quad x_2.x_1\ c\ n_2 \quad x_2),n_1} \quad),\ n}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n} & \rightsquigarrow & c\ (n+1)\ \ x \\
[\lambda x.e]_{c,n} & \rightsquigarrow & c\ (n+1)\ \ \lambda c\ n\ \ x.[e]_{c,n} \\
[e_1\ e_2]_{c,n} & \rightsquigarrow & [e_1]_{(\lambda n_1\ \ x_1.[e_2]_{(\lambda n_2\ \ x_2.x_1\ c\ n_2\ \ x_2),n_1}\ ),(n+1)}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \leadsto & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \leadsto & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\langle e\rangle_{c,n,l} & \leadsto & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \leadsto & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \leadsto & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\langle e\rangle_{c,n,l} & \leadsto & \textbf{if } n = l \textbf{ then } [\![c, n, \lambda c\ n\ l.e]\!] \textbf{ else } e
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\langle e\rangle_{{\color{red}c},n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![{\color{red}c},n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \leadsto & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \leadsto & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\langle e\rangle_{c,{\color{red}n},l} & \leadsto & \textbf{if}\ n = l\ \textbf{then}\ [\![c, {\color{red}n}, \lambda c\ n\ l.e]\!]\ \textbf{else}\ e
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \leadsto & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \leadsto & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\langle e\rangle_{c,n,l} & \leadsto & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\color{red}{\lambda c\ n\ l.e}]\!]\ \textbf{else}\ e
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ {\color{red}e}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l}\\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l}\\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l}\\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e\\
\text{PRIMOPS}\ e & \rightsquigarrow & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e \\
\text{PRIMOPS}\ e & \rightsquigarrow & [e]_{(\lambda n\ l\ x.(x,n)),{\color{red}0},\infty}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ x\rangle_{c,n,l}\\
[\lambda x.e]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l}\\
[e_1\ e_2]_{c,n,l} & \leadsto & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l}\\
\langle e\rangle_{c,n,l} & \leadsto & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e\\
\text{PRIMOPS}\ e & \leadsto & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if } n=l \textbf{ then } [\![c,n,\lambda c\ n\ l.e]\!] \textbf{ else } e \\
\text{PRIMOPS } e & \rightsquigarrow & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e \\
\text{PRIMOPS}\ e & \rightsquigarrow & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty} \\
\color{red}{\text{CHECKPOINT}}\ e\ n & \rightsquigarrow & [e]_{\perp,0,n}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l}\\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l}\\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l}\\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e\\
\text{PRIMOPS}\ e & \rightsquigarrow & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty}\\
\text{CHECKPOINT}\ e\ n & \rightsquigarrow & [e]_{\perp,{\color{red}0},n}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \leadsto & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \leadsto & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e \\
\text{PRIMOPS}\ e & \leadsto & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty} \\
\text{CHECKPOINT}\ e\ {\color{red}n} & \leadsto & [e]_{\perp,0,{\color{red}n}}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e \\
\text{PRIMOPS}\ e & \rightsquigarrow & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty} \\
\text{CHECKPOINT}\ e\ n & \rightsquigarrow & [e]_{\perp,0,n}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \leadsto & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \leadsto & \textbf{if } n=l \textbf{ then } [\![c,n,\lambda c\ n\ l.e]\!] \textbf{ else } e \\
\text{PRIMOPS } e & \leadsto & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty} \\
\text{CHECKPOINT } e\ n & \leadsto & [e]_{\perp,0,n} \\
\text{RESUME } [\![c,n,v]\!] & = & v\ c\ n\ \infty
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if}\ n=l\ \textbf{then}\ [\![c,n,\lambda c\ n\ l.e]\!]\ \textbf{else}\ e \\
\text{PRIMOPS}\ e & \rightsquigarrow & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty} \\
\text{CHECKPOINT}\ e\ n & \rightsquigarrow & [e]_{\bot,0,n} \\
\text{RESUME}\ [\![c,n,v]\!] & = & v\ c\ n\ \infty
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ x\rangle_{c,n,l} \\
[\lambda x.e]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \leadsto & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l} \\
\langle e\rangle_{c,n,l} & \leadsto & \textbf{if}\ n=l\ \textbf{then}\ [\![c, n, \lambda c\ n\ l.e]\!]\ \textbf{else}\ e \\
\text{PRIMOPS}\ e & \leadsto & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty} \\
\text{CHECKPOINT}\ e\ n & \leadsto & [e]_{\perp,0,n} \\
\text{RESUME}\ [\![c, n, v]\!] & = & v\ c\ n\ \infty
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ x\rangle_{c,n,l}\\
[\lambda x.e]_{c,n,l} & \leadsto & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l}\\
[e_1\ e_2]_{c,n,l} & \leadsto & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l}\\
\langle e\rangle_{c,n,l} & \leadsto & \textbf{if } n=l \textbf{ then } [\![c,n,\lambda c\ n\ l.e]\!] \textbf{ else } e\\
\text{PRIMOPS } e & \leadsto & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty}\\
\text{CHECKPOINT } e\ n & \leadsto & [e]_{\bot,0,n}\\
\text{RESUME } [\![c,n,v]\!] & = & v\ c\ n\ \infty
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ x\rangle_{c,n,l}\\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & \langle c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l}\rangle_{c,n,l}\\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & \langle [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}\rangle_{c,n,l}\\
\langle e\rangle_{c,n,l} & \rightsquigarrow & \textbf{if } n=l \textbf{ then } [\![c,n,\lambda c\ n\ l.e]\!] \textbf{ else } e\\
\text{PRIMOPS } e & \rightsquigarrow & [e]_{(\lambda n\ l\ x.(x,n)),0,\infty}\\
\text{CHECKPOINT } e\ n & \rightsquigarrow & [e]_{\bot,0,n}\\
\text{RESUME } [\![c,n,v]\!] & = & v\ c\ n\ \infty
\end{array}
$$

Space and Time Complexity

computation length	t
maximal live storage	w
space for checkpoints	$O(w \log t)$
space for tape	$O(w)$
time to (re)compute primal	$O(t \log t)$
time for reverse sweep	$O(t)$

FORTRAN Example

```
      subroutine f(x, y)
      n = 100003
      y = x
c$ad binomial-ckp n+1 30 1
      do i = 1, n
         m = l(x, i)
         do j = 1, m
            y = y*y
            y = sqrt(y)
         end do
      end do
      end
```

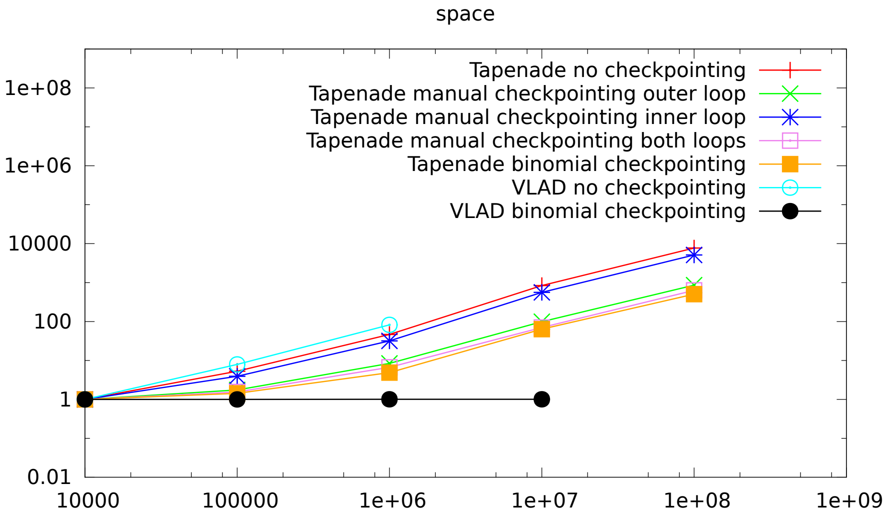
VLAD Example

```
(define (f x)
 (let ((n 100003))
  (let outer ((i 1) (y x))
   (if (> i n)
       y
       (outer (+ i 1)
              (let ((m (l x i)))
               (let inner ((j 1) (y y))
                (if (> j m)
                    y
                    (inner (+ j 1)
                           (sqrt (* y y))))))))))))
```

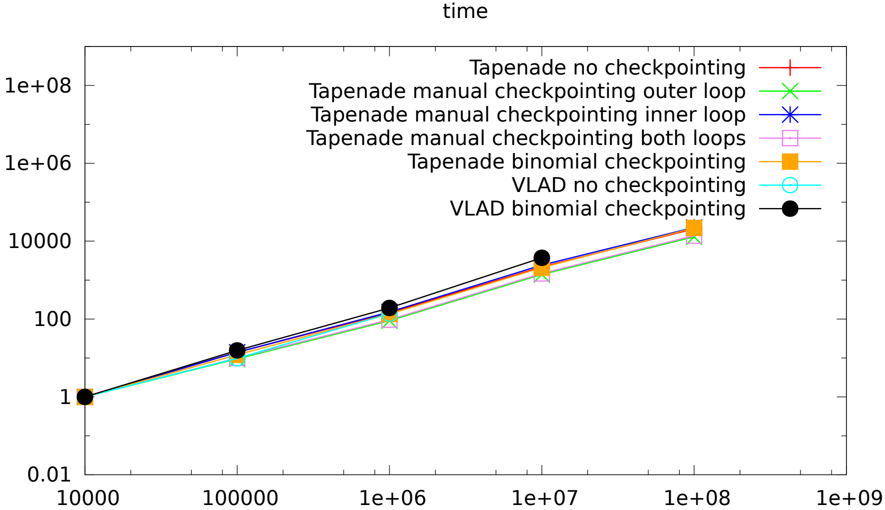
Space and Time Complexity of our Example

computation length	t	$= O(n)$
maximal live storage	w	$= O(1)$
space for checkpoints	$O(w \log t)$	$= O(\log n)$
space for tape	$O(w)$	$= O(1)$
time to (re)compute primal	$O(t \log t)$	$= O(n \log n)$
time for reverse sweep	$O(t)$	$= O(n)$

Space Usage of our Example



Time Usage of our Example



Take-Home Message

Checkpointing

- is traditionally formulated around loop iterations
- but can be extended to arbitrary code
- that doesn't have same-size iterations of a single loop
- using CPS to make arbitrary code look like it does

metaphor: a CPU is an instruction-execution loop