Automatic Differentiation for Probabilistic Programming

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Part of this talk covers joint work with Barak A. Pearlmutter.

The Essence

`(define (`f x`)` $2x^3$`)`

The Essence

$$\texttt{(define (}f\ x\texttt{)}\ 2x^3\texttt{)} \quad \rightsquigarrow \quad \texttt{(define (}f'\ x\texttt{)}\ 6x^2\texttt{)}$$

The Essence

`(define (`g x`)` $\sin f(x)$`)`

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$$\texttt{(define (}g\ \ x\texttt{)}\ \sin f(x)\texttt{)} \rightsquigarrow \texttt{(define (}g'\ \ x\texttt{)}\ f'(x)\cos f(x)\texttt{)}$$

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$$\texttt{(map-closure}\ f\ \langle\{x_1 \mapsto v_1, \ldots\}, e\rangle\texttt{)} \Longrightarrow \langle\{x_1 \mapsto f(v_1), \ldots\}, e\rangle$$

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need reflective transformation of closure bodies

The Essence

$$\texttt{(define (}f\ x\texttt{)}\ 2x^3\texttt{)} \rightsquigarrow \texttt{(define (}f'\ x\texttt{)}\ 6x^2\texttt{)}$$

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need reflective transformation of closure bodies
want transformation done at compile time

The Essence

$$\texttt{(define (}f\ x\texttt{)}\ 2x^3\texttt{)} \leadsto \texttt{(define (}f'\ x\texttt{)}\ 6x^2\texttt{)}$$

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need reflective transformation of closure bodies
want transformation done at compile time
need flow analysis

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need polyvariant flow analysis

Nesting

```
(sqrt (sqrt x))
```

Nesting

```
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```

$(\mathcal{D}\ (\mathcal{D}\ \texttt{f}))$

Nesting

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```
(map (lambda (x) ... (map (lambda (y) ...) ...) ...) ...)
```

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$$\max_x \min_y f(x, y)$$

The Essence of Forward-Mode AD

$$f(c+\varepsilon) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}\varepsilon + \frac{f''(c)}{2!}\varepsilon^2 + \cdots + \frac{f^{(i)}(c)}{i!}\varepsilon^i + \cdots$$

Taylor, B. (1715). *Methodus Incrementorum Directa et Inversa*. London.

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To compute $\mathcal{D}\, f\ c$:

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To compute $\mathcal{D}\; f\; c$:

- evaluate f

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To compute $\mathcal{D}\ f\ c$:

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- multiply by 1!

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Preserves control flow: Augments original values with derivatives.

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$(\mathcal{D}\ f)$ is $\mathcal{O}(1)$ relative to f (both space and time).

Arithmetic on Complex Numbers

$a + b\mathrm{i}$

Hamilton, W. R. (1837). *Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time*. Transactions of the Royal Irish Academy, **17**(1):293–422.

Arithmetic on Complex Numbers

$a + b\mathrm{i}$
$\mathrm{i}^2 = -1$

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$$(x_1 + x_1'\varepsilon) \times (x_2 + x_2'\varepsilon) = (x_1 \times x_2) + (x_1 \times x_2' + x_2 \times x_1')\varepsilon + (x_1' + x_2')\varepsilon^2$$

Clifford, W. K. (1873). *Preliminary Sketch of Bi-quaternions*. Proceedings of the London Mathematical Society, **4**:381–95.

Arithmetic on Dual Numbers

$x + x'\varepsilon$
$\varepsilon^2 = 0$, but $\varepsilon \neq 0$

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$\langle x, x' \rangle$

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Arithmetic on Dual Numbers

$x + x'\varepsilon$
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$\langle x, x' \rangle$

$$\langle x_1, x_1' \rangle + \langle x_2, x_2' \rangle = \langle (x_1 + x_2), (x_1' + x_2') \rangle$$
$$\langle x_1, x_1' \rangle \times \langle x_2, x_2' \rangle = \langle (x_1 \times x_2), (x_1 \times x_2' + x_2 \times x_1') \rangle$$

Clifford, W. K. (1873). *Preliminary Sketch of Bi-quaternions*. Proceedings of the London Mathematical Society, **4**:381–95.

Dynamic Overloading: SCMUTILS

```
(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define +
 (let ((+ +))
  (lambda (x1 x2)
   (make-bundle (+ (primal x1) (primal x2))
                (+ (tangent x1) (tangent x2))))))

(define *
 (let ((+ +) (* *))
  (lambda (x1 x2)
   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
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(define ((D f) x) (tangent (f (make-bundle x 1))))
```

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(define ((D f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))
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($\mathcal{D}$ f)

Dynamic Overloading: SCMUTILS

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(𝒟 f)
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(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient

Dynamic Overloading: SCMUTILS

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(𝒟 f)
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(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

Dynamic Overloading: SCMUTILS

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(𝒟 f)
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   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

Dynamic Overloading: SCMUTILS

```
(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define ((D f) x)
 (fluid-let ((+ (lambda (x1 x2)
                 (make-bundle (+ (primal x1) (primal x2))
                              (+ (tangent x1) (tangent x2)))))
             (* (lambda (x1 x2)
                 (make-bundle (* (primal x1) (primal x2))
                              (+ (* (primal x1) (tangent x2))
                                 (* (tangent x1) (primal x2)))))))
  (tangent (f (make-bundle x 1)))))
```

```
(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

Dynamic Overloading: SCMUTILS

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(define (primal p) (if (bundle? p) (bundle-primal p) p))
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(define ((D f) x)
 (fluid-let ((+ (lambda (x1 x2)
                 (make-bundle (+ (primal x1) (primal x2))
                              (+ (tangent x1) (tangent x2)))))
             (* (lambda (x1 x2)
                 (make-bundle (* (primal x1) (primal x2))
                              (+ (* (primal x1) (tangent x2))
                                 (* (tangent x1) (primal x2)))))))
  (tangent (f (make-bundle x 1)))))
```

```
(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = f
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = gf
AD_IVARS = x, gx
AD_DVARS = gf, gresult
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)                          AD_TOP = f
double precision x, f                  AD_IVARS = x
f = 2.0d0*x*x*x                        AD_DVARS = f
end

function gf(x, gx, gresult)            AD_TOP = gf
double precision x, gx, gf, gresult    AD_IVARS = x, gx
gf = 2.0d0*x*x*x                       AD_DVARS = gf, gresult
gresult = 6.0d0*x*x*gx
end

function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
gresult = 6.0d0*x*x*gx
ggresult = 6.0d0*x*x*ggx+12.0d0*x*gx*gx
end
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = gf
AD_IVARS = x, gx
AD_DVARS = gf, gresult
```

```
function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
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end
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)                        AD_TOP = f
double precision x, f                AD_IVARS = x
f = 2.0d0*x*x*x                      AD_DVARS = f
end

function gf(x, gx, gresult)          AD_TOP = gf
double precision x, gx, gf, gresult  AD_IVARS = x, gx
gf = 2.0d0*x*x*x                     AD_DVARS = gf, gresult
gresult = 6.0d0*x*x*gx
end

function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
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Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)                          AD_TOP = f
double precision x, f                  AD_IVARS = x
f = 2.0d0*x*x*x                        AD_DVARS = f
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function gf(x, gx, gresult)            AD_TOP = gf
double precision x, gx, gf, gresult    AD_IVARS = x, gx
gf = 2.0d0*x*x*x                       AD_DVARS = gf, gresult
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double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
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end
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Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = gf
AD_IVARS = x, gx
AD_DVARS = gf, gresult
AD_PREFIX = h
```

```
function hgf(x, hx, gx, hgx, gresult, hgresult, hresult)
double precision x, hx, gx, hgx, hgf, hresult, gresult, hgresult
hgf = 2.0d0*x*x*x
hresult = 6.0d0*x*x*hx
gresult = 6.0d0*x*x*gx
hgresult = 6.0d0*x*x*hgx+12.0d0*x*gx*hx
end
```

Fast but inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...
```

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
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F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...
```

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
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F<double> f(F<double> x) {return 2*x*x*x;}
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x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
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Slow

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x.diff(0, 1).diff(0,1);
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... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
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Slow

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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
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x.diff(0, 1);
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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
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Slow and inconvenient

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Slow and inconvenient

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Slow and inconvenient

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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
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Slow and inconvenient

Static Overloading: FADBAD++

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Slow and inconvenient

Static Overloading: FADBAD++

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F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
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x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...

template <typename T>
T f(T x) {return 2*x*x*x;}
T x;
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
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template <typename T>
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T x;
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Slow and inconvenient

Static Overloading: FADBAD++

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double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
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x.diff(0, 1);
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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
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... f(x).d(0).d(0) ...

template <typename T>
T f(T x) {return 2*x*x*x;}
T x;
```

Slow and inconvenient

Our API for Functional Forward AD

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

Our API for Functional Forward AD

$$\texttt{bundle} \quad : \quad \mathbb{R}^n \times \overrightarrow{\mathbb{R}^n} \rightarrow (\mathbb{R}^n \triangleright \overrightarrow{\mathbb{R}^n})$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

Our API for Functional Forward AD

$$\begin{aligned}
\texttt{bundle} &: \mathbb{R}^n \times \overline{\mathbb{R}^n} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^n}) \\
\texttt{primal} &: ({\color{red}\mathbb{R}^n} \triangleright \overline{\mathbb{R}^n}) \rightarrow {\color{red}\mathbb{R}^n} \\
\texttt{tangent} &: (\mathbb{R}^n \triangleright {\color{red}\overline{\mathbb{R}^n}}) \rightarrow {\color{red}\overline{\mathbb{R}^n}}
\end{aligned}$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \mathbb{R}^n \times \overline{\mathbb{R}^n} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^n})\\
\texttt{primal} &: (\mathbb{R}^n \triangleright \overline{\mathbb{R}^n}) \rightarrow \mathbb{R}^n\\
\texttt{tangent} &: (\mathbb{R}^n \triangleright \overline{\mathbb{R}^n}) \rightarrow \overline{\mathbb{R}^n}\\
\texttt{j*} &: \color{red}{(\mathbb{R}^n \rightarrow \mathbb{R}^m)} \rightarrow ((\mathbb{R}^n \triangleright \overline{\mathbb{R}^n}) \rightarrow (\mathbb{R}^m \triangleright \overline{\mathbb{R}^m}))
\end{aligned}
$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$
`j*` maps a function to its *push forward*

Our API for Functional Forward AD

$$\begin{aligned}
\texttt{bundle} &: \mathbb{R}^n \times \overline{\mathbb{R}^{\acute{n}}} \rightarrow (\mathbb{R}^n \rhd \overline{\mathbb{R}^{\acute{n}}})\\
\texttt{primal} &: (\mathbb{R}^n \rhd \overline{\mathbb{R}^{\acute{n}}}) \rightarrow \mathbb{R}^n\\
\texttt{tangent} &: (\mathbb{R}^n \rhd \overline{\mathbb{R}^{\acute{n}}}) \rightarrow \overline{\mathbb{R}^{\acute{n}}}\\
\texttt{j*} &: (\mathbb{R}^n \rightarrow \mathbb{R}^m) \rightarrow ((\mathbb{R}^n \rhd \overline{\mathbb{R}^{\acute{n}}}) \rightarrow (\mathbb{R}^m \rhd \overline{\mathbb{R}^{\acute{m}}}))
\end{aligned}$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^{\acute{n}}}$
`j*` maps a function to its *push forward*

Our API for Functional Forward AD

$$\begin{array}{rcl} \texttt{bundle} & : & \mathbb{R}^n \times \overrightarrow{\mathbb{R}^n} \rightarrow (\mathbb{R}^n \triangleright \overrightarrow{\mathbb{R}^n}) \\ \texttt{primal} & : & (\mathbb{R}^n \triangleright \overrightarrow{\mathbb{R}^n}) \rightarrow \mathbb{R}^n \\ \texttt{tangent} & : & (\mathbb{R}^n \triangleright \overrightarrow{\mathbb{R}^n}) \rightarrow \overrightarrow{\mathbb{R}^n} \\ \texttt{j*} & : & (\mathbb{R}^n \rightarrow \mathbb{R}^m) \rightarrow ((\mathbb{R}^n \triangleright \overrightarrow{\mathbb{R}^n}) \rightarrow (\mathbb{R}^m \triangleright \overrightarrow{\mathbb{R}^m})) \end{array}$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$
`j*` maps a function to its *push forward*

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \overline{\tau} \rightarrow (\tau \triangleright \overline{\tau}) \\
\texttt{primal} &: (\tau \triangleright \overline{\tau}) \rightarrow \tau \\
\texttt{tangent} &: (\tau \triangleright \overline{\tau}) \rightarrow \overline{\tau} \\
\texttt{j*} &: (\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \triangleright \overline{\tau_1}) \rightarrow (\tau_2 \triangleright \overline{\tau_2}))
\end{aligned}
$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \overline{\tau'} \rightarrow (\tau \rhd \overline{\tau'}) \\
\texttt{primal} &: (\tau \rhd \overline{\tau'}) \rightarrow \tau \\
\texttt{tangent} &: (\tau \rhd \overline{\tau'}) \rightarrow \overline{\tau'} \\
\texttt{j*} &: (\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \rhd \overline{\tau_1'}) \rightarrow (\tau_2 \rhd \overline{\tau_2'}))
\end{aligned}
$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^{n'}}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Our API for Functional Forward AD

$$
\begin{array}{rcl}
\texttt{bundle} & : & \tau \times \overline{\tau} \rightarrow (\tau \triangleright \overline{\tau}) \\
\texttt{primal} & : & (\tau \triangleright \overline{\tau}) \rightarrow \tau \\
\texttt{tangent} & : & (\tau \triangleright \overline{\tau}) \rightarrow \overline{\tau} \\
\texttt{j*} & : & (\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \triangleright \overline{\tau_1}) \rightarrow (\tau_2 \triangleright \overline{\tau_2}))
\end{array}
$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overline{\tau}$ as $\overrightarrow{\tau}$

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \overline{\tau} \rightarrow \overrightarrow{\tau} \\
\texttt{primal} &: \overrightarrow{\tau} \rightarrow \tau \\
\texttt{tangent} &: \overrightarrow{\tau} \rightarrow \overline{\tau} \\
\texttt{j*} &: (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})
\end{aligned}
$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overline{\tau}$ as $\overrightarrow{\tau}$

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \overline{\tau} \to \overrightharpoonup{\tau} \\
\texttt{primal} &: \overrightharpoonup{\tau} \to \tau \\
\texttt{tangent} &: \overrightharpoonup{\tau} \to \overline{\tau} \\
\overrightarrow{\mathcal{J}} &: (\tau_1 \to \tau_2) \to (\overrightharpoonup{\tau_1} \to \overrightharpoonup{\tau_2})
\end{aligned}
$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$
`j*` maps a function to its *push forward*
Generalize to arbitrary types
What is the tangent of a discrete value or a function?
Can abbreviate $\tau \triangleright \overline{\tau}$ as $\overrightharpoonup{\tau}$
Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \acute{\overline{\tau}} \rightarrow \overrightharpoonup{\tau}\\
\texttt{primal} &: \overrightharpoonup{\tau} \rightarrow \tau\\
\texttt{tangent} &: \overrightharpoonup{\tau} \rightarrow \acute{\overline{\tau}}\\
\texttt{j*} &: (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightharpoonup{\tau_1} \rightarrow \overrightharpoonup{\tau_2})
\end{aligned}
$$

```
(define ((𝒟 f) x) (tangent ((j* f) (bundle x 1))))
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\acute{\overline{\mathbb{R}^n}}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \acute{\overline{\tau}}$ as $\overrightharpoonup{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

Convenient

Our API for Functional Forward AD

$$\begin{array}{rcl}
\texttt{bundle} & : & \tau \times \overline{\tau}' \rightarrow \overrightharpoon{\tau} \\
\texttt{primal} & : & \overrightharpoon{\tau} \rightarrow \tau \\
\texttt{tangent} & : & \overrightharpoon{\tau} \rightarrow \overline{\tau}' \\
\texttt{j*} & : & (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightharpoon{\tau_1} \rightarrow \overrightharpoon{\tau_2})
\end{array}$$

```
(define ((𝒟 f) x) (tangent ((j* f) (bundle x 1))))
(𝒟 f)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}'$
`j*` maps a function to its *push forward*
Generalize to arbitrary types
What is the tangent of a discrete value or a function?
Can abbreviate $\tau \triangleright \overline{\tau}'$ as $\overrightharpoon{\tau}$
Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

Convenient

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \overline{\tau} \rightarrow \overrightarrow{\tau}\\
\texttt{primal} &: \overrightarrow{\tau} \rightarrow \tau\\
\texttt{tangent} &: \overrightarrow{\tau} \rightarrow \overline{\tau}\\
\texttt{j*} &: (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})
\end{aligned}
$$

```
(define ((𝒟 f) x) (tangent ((j* f) (bundle x 1))))
(𝒟 f)
(𝒟 (𝒟 f))
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$
`j*` maps a function to its *push forward*
Generalize to arbitrary types
What is the tangent of a discrete value or a function?
Can abbreviate $\tau \triangleright \overline{\tau}$ as $\overrightarrow{\tau}$
Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

Convenient

Our API for Functional Forward AD

$$
\begin{array}{rcl}
\texttt{bundle} & : & \tau \times \grave{\tau} \rightarrow \overrightharpoon{\tau} \\
\texttt{primal} & : & \overrightharpoon{\tau} \rightarrow \tau \\
\texttt{tangent} & : & \overrightharpoon{\tau} \rightarrow \grave{\tau} \\
\texttt{j*} & : & (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightharpoon{\tau_1} \rightarrow \overrightharpoon{\tau_2})
\end{array}
$$

```
(define ((𝒟 f) x) (tangent ((j* f) (bundle x 1))))
(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\grave{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \grave{\tau}$ as $\overrightharpoon{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

Convenient

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \overline{\tau}' \rightarrow \overrightarrow{\tau} \\
\texttt{primal} &: \overrightarrow{\tau} \rightarrow \tau \\
\texttt{tangent} &: \overrightarrow{\tau} \rightarrow \overline{\tau}' \\
\texttt{j*} &: (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})
\end{aligned}
$$

```
(define ((𝒟 f) x) (tangent ((j* f) (bundle x 1))))
(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}'$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overline{\tau}'$ as $\overrightarrow{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

What is `(j* j*)`?

Convenient

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \overline{\tau} \rightarrow \overrightarrow{\tau} \\
\texttt{primal} &: \overrightarrow{\tau} \rightarrow \tau \\
\texttt{tangent} &: \overrightarrow{\tau} \rightarrow \overline{\tau} \\
\texttt{j*} &: (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})
\end{aligned}
$$

```
(define ((𝒟 f) x) (tangent ((j* f) (bundle x 1))))
(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overline{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

What is `(j* j*)`?

Convenient

Our API for Functional Forward AD

$$
\begin{aligned}
\texttt{bundle} &: \tau \times \acute{\tau} \to \overrightharpoon{\tau} \\
\texttt{primal} &: \overrightharpoon{\tau} \to \tau \\
\texttt{tangent} &: \overrightharpoon{\tau} \to \acute{\tau} \\
\texttt{j*} &: (\tau_1 \to \tau_2) \to (\overrightharpoon{\tau_1} \to \overrightharpoon{\tau_2})
\end{aligned}
$$

```
(define ((𝒟 f) x) (tangent ((j* f) (bundle x 1))))
(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\acute{\mathbb{R}^n}$
`j*` maps a function to its *push forward*
Generalize to arbitrary types
What is the tangent of a discrete value or a function?
Can abbreviate $\tau \triangleright \acute{\tau}$ as $\overrightharpoon{\tau}$
Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$
What is `(j* j*)`?
Convenient and fast

A property

A property

$x : \mathbb{R}^n$

A property

$x : \mathbb{R}^n$ $\overleftarrow{x'} : \mathbb{R}^n$

A property

$x : \mathbb{R}^n$ $\overline{x'} : \mathbb{R}^n$ $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

A property

$x : \mathbb{R}^n$ $\qquad \overline{x'} : \mathbb{R}^n$ $\qquad f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$

A property

$x : \mathbb{R}^n$ $\qquad \overleftarrow{x'} : \mathbb{R}^n$ $\qquad f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\qquad ((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$

A property

$x : \mathbb{R}^n$ $\quad$ $\overleftarrow{x'} : \mathbb{R}^n$ $\quad$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\quad$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$ $\quad$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

A property

$x : \mathbb{R}^n$ $\qquad \overline{x'} : \mathbb{R}^n$ $\qquad f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\qquad ((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$ $\qquad ((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$\texttt{(primal ((j* } f\texttt{) (bundle } x\ \overline{x'}\texttt{)))} = \texttt{(}f\ x\texttt{)}$

A property

$x : \mathbb{R}^n$ $\quad \overrightharpoon{x'} : \mathbb{R}^n$ $\quad f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\quad ((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$ $\quad ((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

(primal ((j* f) (bundle x $\overrightharpoon{x'}$))) $=$ (f x)

(tangent ((j* f) (bundle x $\overrightharpoon{x'}$))) $= ((\mathcal{J}\ f)\ x) \times \overrightharpoon{x'}$

A property

$x : \mathbb{R}^n$ $\quad$ $\overrightarrow{x'} : \mathbb{R}^n$ $\quad$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\quad$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^{m\times n}$ $\quad$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$\texttt{(primal ((j* }f\texttt{) (bundle }x\ \overrightarrow{x'}\texttt{)))} = \texttt{(}f\ \ x\texttt{)}$

$\texttt{(tangent ((j* }f\texttt{) (bundle }x\ \overrightarrow{x'}\texttt{)))} = ((\mathcal{J}\ f)\ x) \times \overrightarrow{x'}$

$\texttt{(tangent ((j* }f\texttt{) (bundle }x\ \overrightarrow{x'}\texttt{)))} = \texttt{(}((\mathcal{J}\ f)\ x)\ \ \overrightarrow{x'}\texttt{)}$

A property

$x : \mathbb{R}^n$ $\quad \overline{x'} : \mathbb{R}^n$ $\quad f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\quad ((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$ $\quad ((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$\texttt{(primal ((j* } f\texttt{) (bundle } x\ \overline{x'}\texttt{)))} = \texttt{(}f\ x\texttt{)}$

$\texttt{(tangent ((j* } f\texttt{) (bundle } x\ \overline{x'}\texttt{)))} = ((\mathcal{J}\ f)\ x) \times \overline{x'}$

$\texttt{(tangent ((j* } f\texttt{) (bundle } x\ \overline{x'}\texttt{)))} = \texttt{(}((\mathcal{J}\ f)\ x)\ \ \overline{x'}\texttt{)}$

$\texttt{((j* } f\texttt{) } x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal } x\texttt{)) (}((\mathcal{J}\ f)\texttt{ (primal } x\texttt{))}\texttt{ (tangent } x\texttt{)))}$

A property

$x : \mathbb{R}^n$ $\overrightharpoon{x'} : \mathbb{R}^n$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^{m\times n}$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

`(primal ((j*` f`) (bundle` x $\overrightharpoon{x'}$`)))` $=$ `(`f x`)`

`(tangent ((j*` f`) (bundle` x $\overrightharpoon{x'}$`)))` $= ((\mathcal{J}\ f)\ x) \times \overrightharpoon{x'}$

`(tangent ((j*` f`) (bundle` x $\overrightharpoon{x'}$`)))` $=$ `(`$((\mathcal{J}\ f)\ x)$ $\overrightharpoon{x'}$`)`

`((j*` f`)` x`)` $=$ `(bundle (`f `(primal` x`))` `(`$((\mathcal{J}\ f)$ `(primal` x`))` `(tangent` x`)))`

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

A property

$x : \mathbb{R}^n$ $\qquad$ $\overrightharpoon{x} : \mathbb{R}^n$ $\qquad$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\qquad$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^{m\times n}$ $\qquad$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$\texttt{(primal ((j* } f\texttt{) (bundle } x\ \overrightharpoon{x}\texttt{)))} = \texttt{(}f\ x\texttt{)}$

$\texttt{(tangent ((j* } f\texttt{) (bundle } x\ \overrightharpoon{x}\texttt{)))} = ((\mathcal{J}\ f)\ x) \times \overrightharpoon{x}$

$\texttt{(tangent ((j* } f\texttt{) (bundle } x\ \overrightharpoon{x}\texttt{)))} = \texttt{(}((\mathcal{J}\ f)\ x)\ \ \overrightharpoon{x}\texttt{)}$

$\texttt{((j* } f\texttt{) } x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal } x\texttt{)) (}((\mathcal{J}\ f)\texttt{ (primal } x\texttt{)}) \texttt{ (tangent } x\texttt{)))}$

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

A property

$x : \mathbb{R}^n$ $\overrightharpoon{x} : \mathbb{R}^n$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

(primal ((j* f) (bundle x $\overrightharpoon{x}$))) $=$ (f x)

(tangent ((j* f) (bundle x $\overrightharpoon{x}$))) $= ((\mathcal{J}\ f)\ x) \times \overrightharpoon{x}$

(tangent ((j* f) (bundle x $\overrightharpoon{x}$))) $=$ ($(((\mathcal{J}\ f)\ x)$ $\overrightharpoon{x}$)

((j* f) x) $=$ (bundle (f (primal x)) ($((\mathcal{J}\ f)$ (primal x)$)$ (tangent x)))

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$0/1$ matrix, every row has exactly one 1

A property

$x : \mathbb{R}^n$ $\qquad \overrightharpoon{x} : \mathbb{R}^n$ $\qquad f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\qquad ((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$ $\qquad ((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

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$\texttt{((j* } f\texttt{) } x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal }x\texttt{)) (}((\mathcal{J}\ f)\texttt{ (primal }x\texttt{))}\texttt{ (tangent }x\texttt{)))}$

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$0/1$ matrix, every row has exactly one 1

$((\mathcal{J}\ f)\ x)$

A property

$x : \mathbb{R}^n$ $\qquad \overrightarrow{x'} : \mathbb{R}^n$ $\qquad f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\qquad ((\mathcal{J}\ f)\ x) : \mathbb{R}^{m\times n}$ $\qquad ((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

(primal ((j* f) (bundle x $\overrightarrow{x'}$))) $=$ (f x)

(tangent ((j* f) (bundle x $\overrightarrow{x'}$))) $= ((\mathcal{J}\ f)\ x) \times \overrightarrow{x'}$

(tangent ((j* f) (bundle x $\overrightarrow{x'}$))) $=$ ($((\mathcal{J}\ f)\ x)$ $\overrightarrow{x'}$)

((j* f) x) $=$ (bundle (f (primal x)) ($((\mathcal{J}\ f)$ (primal x)$)$ (tangent x)))

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$0/1$ matrix, every row has exactly one 1

$((\mathcal{J}\ f)\ x) = \frac{\partial f(x)[i]}{\partial x[j]}$

A property

$x : \mathbb{R}^n$ $\overrightarrow{x'} : \mathbb{R}^n$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^{m\times n}$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$(\texttt{primal}\ ((\texttt{j*}\ f)\ (\texttt{bundle}\ x\ \overrightarrow{x'}))) = (f\ x)$

$(\texttt{tangent}\ ((\texttt{j*}\ f)\ (\texttt{bundle}\ x\ \overrightarrow{x'}))) = ((\mathcal{J}\ f)\ x) \times \overrightarrow{x'}$

$(\texttt{tangent}\ ((\texttt{j*}\ f)\ (\texttt{bundle}\ x\ \overrightarrow{x'}))) = (((\mathcal{J}\ f)\ x)\ \ \overrightarrow{x'})$

$((\texttt{j*}\ f)\ x) = (\texttt{bundle}\ (f\ (\texttt{primal}\ x))\ (((\mathcal{J}\ f)\ (\texttt{primal}\ x))\ (\texttt{tangent}\ x)))$

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$0/1$ matrix, every row has exactly one 1

$$((\mathcal{J}\ f)\ x) = \frac{\partial f(x)[i]}{\partial x[j]} = \begin{cases} 1 & \text{when } (f\ x)[i] = x[j] \\ 0 & \text{otherwise} \end{cases}$$

A property

$x : \mathbb{R}^n$ $\quad$ $\overrightarrow{x'} : \mathbb{R}^n$ $\quad$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\quad$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$ $\quad$ $((\mathcal{J}\ f)\ x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$(\texttt{primal}\ ((\texttt{j*}\ f)\ (\texttt{bundle}\ x\ \overrightarrow{x'}))) = (f\ x)$

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$(\texttt{tangent}\ ((\texttt{j*}\ f)\ (\texttt{bundle}\ x\ \overrightarrow{x'}))) = (((\mathcal{J}\ f)\ x)\ \overrightarrow{x'})$

$((\texttt{j*}\ f)\ x) = (\texttt{bundle}\ (f\ (\texttt{primal}\ x))\ (((\mathcal{J}\ f)\ (\texttt{primal}\ x))\ (\texttt{tangent}\ x)))$

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A property

$x : \mathbb{R}^n$ $\qquad \overrightarrow{x'} : \mathbb{R}^n$ $\qquad f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

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`((j* `f`) `x`)` $=$ `(bundle (`f` (primal `x`)) (`$((\mathcal{J}\ f)$` (primal `x`)`$)$` (tangent `x`)))`

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

0/1 matrix, every row has exactly one 1

$$((\mathcal{J}\ f)\ x) = \frac{\partial f(x)[i]}{\partial x[j]} = \begin{cases} 1 & \text{when } (f\ x)[i] = x[j] \\ 0 & \text{otherwise} \end{cases} = f$$

when f is a rearrangement function

`((j* `f`) `x`)` $=$ `(bundle (`f` (primal `x`)) (`f` (tangent `x`)))`

A property

$x : \mathbb{R}^n$ $\quad\quad$ $\overrightharpoon{x'} : \mathbb{R}^n$ $\quad\quad$ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

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$\texttt{((j* } f\texttt{) } x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal } x\texttt{)) (}((\mathcal{J}\ f)\ \texttt{(primal } x\texttt{)})\texttt{ (tangent } x\texttt{)))}$

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

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when f is a rearrangement function

$\texttt{((j* } f\texttt{) } x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal } x\texttt{)) (}f\texttt{ (tangent } x\texttt{)))}$

A property

$x : \tau_1$ $\overrightarrow{x'} : \tau_1$ $f : \tau_1 \rightarrow \tau_2$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $((\mathcal{J}\ f)\ x) : \tau_1 \xrightarrow{L} \tau_2$

`(primal ((j*` f`) (bundle` x $\overrightarrow{x'}$`)))` $=$ `(`f x`)`

`(tangent ((j*` f`) (bundle` x $\overrightarrow{x'}$`)))` $= ((\mathcal{J}\ f)\ x) \times \overrightarrow{x'}$

`(tangent ((j*` f`) (bundle` x $\overrightarrow{x'}$`)))` $=$ `(`$(((\mathcal{J}\ f)\ x)$ $\overrightarrow{x'}$`)`

`((j*` f`)` x`)` $=$ `(bundle (`f `(primal` x`)) (`$((\mathcal{J}\ f)$ `(primal` x`))` $)$ `(tangent` x`)))`

rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

$f : \tau_1 \xrightarrow{L} \tau_2$

$0/1$ matrix, every row has exactly one 1

$$((\mathcal{J}\ f)\ x) = \frac{\partial f(x)[i]}{\partial x[j]} = \begin{cases} 1 & \text{when } (f\ x)[i] = x[j] \\ 0 & \text{otherwise} \end{cases} = f$$

when f is a rearrangement function
`((j*` f`)` x`)` $=$ `(bundle (`f `(primal` x`)) (`f `(tangent` x`)))`

A property

$x : \tau_1$ $\quad$ $\overrightarrow{x'} : \tau_1$ $\quad$ $f : \tau_1 \rightarrow \tau_2$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\quad$ $((\mathcal{J}\ f)\ x) : \tau_1 \xrightarrow{L} \tau_2$

`(primal ((j* `f`) (bundle `x $\overrightarrow{x'}$`))) = (`f x`)`

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`(tangent ((j* `f`) (bundle `x $\overrightarrow{x'}$`))) = (`$((\mathcal{J}\ f)\ x)$ $\overrightarrow{x'}$`)`

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rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

$f : \tau_1 \xrightarrow{L} \tau_2$

$0/1$ matrix, every row has exactly one 1

$$((\mathcal{J}\ f)\ x) = \frac{\partial f(x)[i]}{\partial x[j]} = \begin{cases} 1 & \text{when } (f\ x)[i] = x[j] \\ 0 & \text{otherwise} \end{cases} = f$$

when f is a rearrangement function
`((j* `f`) `x`) = (bundle (`f` (primal `x`)) (`f` (tangent `x`)))`

What is the tangent of `#t`?

What is the tangent of `#t`?

What if we take $\overrightharpoon{\texttt{\#t}} = \texttt{\#f}$?

What is the tangent of `#t`?

What if we take $\overrightarrow{\texttt{\#t}} = \texttt{\#f}$?

when f is a rearrangement function
$((\texttt{j*}\ f)\ x) = (\texttt{bundle}\ (f\ (\texttt{primal}\ x))\ (f\ (\texttt{tangent}\ x)))$

What is the tangent of #t?

What if we take $\overrightarrow{\texttt{\#t}} = \texttt{\#f}$?

when f is a rearrangement function

$(\texttt{(j*}\ f\texttt{)}\ x\texttt{)} = \texttt{(bundle (}f\ \texttt{(primal}\ x\texttt{)) (}f\ \texttt{(tangent}\ x\texttt{)))}$

$f : \texttt{(\#t}\ x\ y\texttt{)} \mapsto \texttt{(\#t}\ x\ y\texttt{)}$ but $f : \texttt{(\#f}\ x\ y\texttt{)} \mapsto \texttt{(\#f}\ y\ x\texttt{)}$

What is the tangent of `#t`?

What if we take $\overrightarrow{\texttt{\#t}} = \texttt{\#f}$?

when f is a rearrangement function

$$\texttt{((j* }f\texttt{) }x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal }x\texttt{)) (}f\texttt{ (tangent }x\texttt{)))}$$

$f : \texttt{(\#t }x\ y\texttt{)} \mapsto \texttt{(\#t }x\ y\texttt{)}$ but $f : \texttt{(\#f }x\ y\texttt{)} \mapsto \texttt{(\#f }y\ x\texttt{)}$

f is a rearrangement function

What is the tangent of `#t`?

What if we take $\overrightarrow{\texttt{\#t}} = \texttt{\#f}$?

when f is a rearrangement function

$(\texttt{(j* } f\texttt{)}\ x\texttt{)} = \texttt{(bundle (}f\ \texttt{(primal } x\texttt{)) (}f\ \texttt{(tangent } x\texttt{)))}$

$f : \texttt{(\#t } x\ y\texttt{)} \mapsto \texttt{(\#t } x\ y\texttt{)}$ but $f : \texttt{(\#f } x\ y\texttt{)} \mapsto \texttt{(\#f } y\ x\texttt{)}$

f is a rearrangement function

$\texttt{((j* } f\texttt{) (bundle (\#t } x\ y\texttt{) (\#f } \overrightarrow{x}\ \overrightarrow{y}\texttt{)))}$

What is the tangent of `#t`?

What if we take $\overrightarrow{\texttt{\#t}} = \texttt{\#f}$?

when f is a rearrangement function

$$\texttt{((j* }f\texttt{) }x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal }x\texttt{)) (}f\texttt{ (tangent }x\texttt{)))}$$

$$f : \texttt{(\#t }x\ y\texttt{)} \mapsto \texttt{(\#t }x\ y\texttt{)} \quad \text{but} \quad f : \texttt{(\#f }x\ y\texttt{)} \mapsto \texttt{(\#f }y\ x\texttt{)}$$

f is a rearrangement function

$$\texttt{((j* }f\texttt{) (bundle (\#t }x\ y\texttt{) (\#f }\overrightarrow{x}\ \overrightarrow{y}\texttt{)))}$$

$$= \texttt{(bundle (\#t }x\ y\texttt{) (\#f }\overrightarrow{y}\ \overrightarrow{x}\texttt{))}$$

What is the tangent of `#t`?

What if we take $\overrightarrow{\texttt{\#t}} = \texttt{\#f}$?

when f is a rearrangement function

$\texttt{((j* } f\texttt{) } x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal } x\texttt{)) (}f\texttt{ (tangent } x\texttt{)))}$

$f : \texttt{(\#t } x\ y\texttt{)} \mapsto \texttt{(\#t } x\ y\texttt{)}$ but $f : \texttt{(\#f } x\ y\texttt{)} \mapsto \texttt{(\#f } y\ x\texttt{)}$

f is a rearrangement function

$\texttt{((j* } f\texttt{) (bundle (\#t } x\ y\texttt{) (\#f } \overrightarrow{x}\ \overrightarrow{y}\texttt{)))}$

$= \texttt{(bundle (\#t } x\ y\texttt{) (\#f } \overrightarrow{y}\ \overrightarrow{x}\texttt{))}$

What is the tangent of #t?

What if we take $\overrightarrow{\texttt{\#t}} = \texttt{\#f}$?

when f is a rearrangement function

$$\texttt{((j* } f\texttt{) } x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal } x\texttt{)) (}f\texttt{ (tangent } x\texttt{)))}$$

$$f : \texttt{(\#t } x\ y\texttt{)} \mapsto \texttt{(\#t } x\ y\texttt{)} \qquad \text{but} \qquad f : \texttt{(\#f } x\ y\texttt{)} \mapsto \texttt{(\#f } y\ x\texttt{)}$$

f is a rearrangement function

$$\texttt{((j* } f\texttt{) (bundle (\#t } x\ y\texttt{) (\#f } \overrightarrow{x}\ \overrightarrow{y}\texttt{)))}$$

$$= \texttt{(bundle (\#t } x\ {\color{red}y}\texttt{) (\#f } \overrightarrow{y}\ {\color{red}\overrightarrow{x}}\texttt{))}$$

What is the tangent of `#t`?

What if we take $\acute{\overline{\texttt{\#t}}} = \texttt{\#f}$?

when f is a rearrangement function

$$\texttt{((j* } f\texttt{) } x\texttt{)} = \texttt{(bundle (}f\texttt{ (primal } x\texttt{)) (}f\texttt{ (tangent } x\texttt{)))}$$

$$f : \texttt{(\#t } x\ y\texttt{)} \mapsto \texttt{(\#t } x\ y\texttt{)} \qquad \text{but} \quad f : \texttt{(\#f } x\ y\texttt{)} \mapsto \texttt{(\#f } y\ x\texttt{)}$$

f is a rearrangement function

$$\texttt{((j* } f\texttt{) (bundle (\#t } x\ y\texttt{) (\#f } \acute{\overline{x}}\ \acute{\overline{y}}\texttt{)))}$$

$$= \texttt{(bundle (\#t } x\ y\texttt{) (\#f } \acute{\overline{y}}\ \acute{\overline{x}}\texttt{))}$$

Problem avoided if we take $\acute{\overline{\texttt{\#t}}} = \texttt{\#t}$

What is `(j* j*)`?

What is `(j* j*)`?

when f is a rearrangement function

`((j*` f`)` x`)` $=$ `(bundle (`f `(primal` x`)) (`f `(tangent` x`)))`

What is (j* j*)?

when f is a rearrangement function

((j* f) x) = (bundle (f (primal x)) (f (tangent x)))

bundle, primal, tangent, and j* are rearrangement functions

What is `(j* j*)`?

when f is a rearrangement function

`((j*` f`)` x`)` $=$ `(bundle (`f `(primal` x`)) (`f `(tangent` x`)))`

`bundle`, `primal`, `tangent`, and `j*` are rearrangement functions

```
  ((j* bundle) x)=(bundle (bundle (primal x)) (bundle (tangent x)))
  ((j* primal) x)=(bundle (primal (primal x)) (primal (tangent x)))
 ((j* tangent) x)=(bundle (tangent (primal x)) (tangent (tangent x)))
      ((j* j*) x)=(bundle (j* (primal x)) (j* (tangent x)))
```

A (Not So) Brief Tutorial on AD

$$z = g(f(x))$$

A (Not So) Brief Tutorial on AD

$$
\begin{aligned}
z &= g(f(x)) \\
&= (f \circ g)(x)
\end{aligned}
$$

A (Not So) Brief Tutorial on AD

$$
\begin{aligned}
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\end{aligned}
$$

$$
\begin{aligned}
y &= f(x) \\
z &= g(y)
\end{aligned}
$$

$$
\begin{aligned}
z &= g(f(x)) \\
&= (f \circ g)(x)
\end{aligned}
$$

$$
\begin{aligned}
y &= f(x) \\
z &= g(y)
\end{aligned}
$$

$$
\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}
$$

A (Not So) Brief Tutorial on AD

$$
\begin{aligned}
z &= g(f(x)) \\
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\end{aligned}
$$

$$
\begin{aligned}
y &= f(x) \\
z &= g(y)
\end{aligned}
$$

$$
\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}
$$

$$
\mathcal{D}\;(f \circ g)\;x = (\mathcal{D}\;g\;y) \times (\mathcal{D}\;f\;x)
$$

A (Not So) Brief Tutorial on AD

$$f = f_1 \circ \cdots \circ f_n$$

$$f = f_1 \circ \cdots \circ f_n$$

$$\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}$$

$$f = f_1 \circ \cdots \circ f_n$$

$$\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\ \vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}$$

$$\mathcal{J}\ f\ \mathbf{x}_0 = (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_1\ \mathbf{x}_0)$$

$$f = f_1 \circ \cdots \circ f_n$$

$$\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}\ f\ \mathbf{x}_0 &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_1\ \mathbf{x}_0)\\
(\mathcal{J}\ f\ \mathbf{x}_0)^\top &= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top
\end{aligned}$$

A (Not So) Brief Tutorial on AD

$$\overleftarrow{\mathbf{x}_n} = \mathcal{J}\ f\ \mathbf{x}_0$$

A (Not So) Brief Tutorial on AD

$$
\begin{aligned}
\overrightarrow{\mathbf{x}_n} &= \mathcal{J}\ f\ \mathbf{x}_0\\
&= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_2\ \mathbf{x}_1) \times (\mathcal{J}\ f_1\ \mathbf{x}_0)
\end{aligned}
$$

$$\begin{aligned}\overline{\mathbf{x}'_n} &= \mathcal{J}\ f\ \mathbf{x}_0\\ &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_2\ \mathbf{x}_1) \times (\mathcal{J}\ f_1\ \mathbf{x}_0)\end{aligned}$$

$$\overline{\mathbf{x}'_1} = (\mathcal{J}\ f_1\ \mathbf{x}_0)$$

A (Not So) Brief Tutorial on AD

$$\overleftarrow{\mathbf{X}_n} = \mathcal{J}\ f\ \mathbf{x}_0$$
$$= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_2\ \mathbf{x}_1) \times (\mathcal{J}\ f_1\ \mathbf{x}_0)$$

$$\overleftarrow{\mathbf{X}_1} = (\mathcal{J}\ f_1\ \mathbf{x}_0)$$
$$\overleftarrow{\mathbf{X}_2} = (\mathcal{J}\ f_2\ \mathbf{x}_1) \times \overleftarrow{\mathbf{X}_1}$$

$$
\begin{aligned}
\overrightarrow{\mathbf{X}}_n &= \mathcal{J}\ f\ \mathbf{x}_0 \\
&= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_2\ \mathbf{x}_1) \times (\mathcal{J}\ f_1\ \mathbf{x}_0)
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\mathbf{X}}_1 &= (\mathcal{J}\ f_1\ \mathbf{x}_0) \\
\overrightarrow{\mathbf{X}}_2 &= (\mathcal{J}\ f_2\ \mathbf{x}_1) \times \overrightarrow{\mathbf{X}}_1 \\
&\vdots \\
\overrightarrow{\mathbf{X}}_n &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \overrightarrow{\mathbf{X}_{n-1}}
\end{aligned}
$$

$$\overleftharpoon{\mathbf{X}_0} = (\mathcal{J}\ f\ \mathbf{x}_0)^\top$$

$$
\begin{aligned}
\grave{\mathbf{x}}_0 &= (\mathcal{J}\ f\ \mathbf{x}_0)^\top \\
&= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J}\ f_{n-1}\ \mathbf{x}_{n-2})^\top \times (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top
\end{aligned}
$$

$$
\begin{aligned}
\grave{\mathbf{x}}_0 &= (\mathcal{J}\ f\ \mathbf{x}_0)^\top \\
&= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J}\ f_{n-1}\ \mathbf{x}_{n-2})^\top \times (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top
\end{aligned}
$$

$$
\grave{\mathbf{x}}_{n-1} = (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top
$$

$$\grave{\overline{\mathbf{X}_0}} = (\mathcal{J}\ f\ \mathbf{x}_0)^\top$$
$$= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J}\ f_{n-1}\ \mathbf{x}_{n-2})^\top \times (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top$$

$$\grave{\overline{\mathbf{X}_{n-1}}} = (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top$$
$$\grave{\overline{\mathbf{X}_{n-2}}} = (\mathcal{J}\ f_{n-1}\ \mathbf{x}_{n-2})^\top \times \grave{\overline{\mathbf{X}_{n-1}}}$$

$$
\begin{aligned}
\grave{\mathbf{X}}_0 &= (\mathcal{J}\ f\ \mathbf{x}_0)^\top \\
&= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J}\ f_{n-1}\ \mathbf{x}_{n-2})^\top \times (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top
\end{aligned}
$$

$$
\begin{aligned}
\grave{\mathbf{X}}_{n-1} &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top \\
\grave{\mathbf{X}}_{n-2} &= (\mathcal{J}\ f_{n-1}\ \mathbf{x}_{n-2})^\top \times \grave{\mathbf{X}}_{n-1} \\
&\vdots \\
\grave{\mathbf{X}}_0 &= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \grave{\mathbf{X}}_1
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\mathbf{X}_1} &= (\mathcal{J}\; f_1\; \mathbf{x}_0) \\
\overrightarrow{\mathbf{X}_2} &= (\mathcal{J}\; f_2\; \mathbf{x}_1) \times \overrightarrow{\mathbf{X}_1} \\
&\vdots \\
\overrightarrow{\mathbf{X}_n} &= (\mathcal{J}\; f_n\; \mathbf{x}_{n-1}) \times \overrightarrow{\mathbf{X}_{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{X}_{n-1}} &= (\mathcal{J}\; f_n\; \mathbf{x}_{n-1})^\top \\
\overleftarrow{\mathbf{X}_{n-2}} &= (\mathcal{J}\; f_{n-1}\; \mathbf{x}_{n-2})^\top \times \overleftarrow{\mathbf{X}_{n-1}} \\
&\vdots \\
\overleftarrow{\mathbf{X}_0} &= (\mathcal{J}\; f_1\; \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{X}_1}
\end{aligned}
$$

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$$
\begin{aligned}
\overrightarrow{\mathbf{X}_1} &= (\mathcal{J}\ f_1\ \mathbf{x}_0) \\
\overrightarrow{\mathbf{X}_2} &= (\mathcal{J}\ f_2\ \mathbf{x}_1) \times \overrightarrow{\mathbf{X}_1} \\
&\ \vdots \\
\overrightarrow{\mathbf{X}_n} &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \overrightarrow{\mathbf{X}_{n-1}}
\end{aligned}
\qquad
\begin{aligned}
\overleftarrow{\mathbf{X}_{n-1}} &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top \\
\overleftarrow{\mathbf{X}_{n-2}} &= (\mathcal{J}\ f_{n-1}\ \mathbf{x}_{n-2})^\top \times \overleftarrow{\mathbf{X}_{n-1}} \\
&\ \vdots \\
\overleftarrow{\mathbf{X}_0} &= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{X}_1}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\mathbf{X}_1} &= (\mathcal{J}\ f_1\ \mathbf{x}_0) \\
\overrightarrow{\mathbf{X}_2} &= (\mathcal{J}\ f_2\ \mathbf{x}_1) \times \overrightarrow{\mathbf{X}_1} \\
&\vdots \\
\overrightarrow{\mathbf{X}_n} &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \overrightarrow{\mathbf{X}_{n-1}}
\end{aligned}
\qquad
\begin{aligned}
\overleftarrow{\mathbf{X}_{n-1}} &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top \\
\overleftarrow{\mathbf{X}_{n-2}} &= (\mathcal{J}\ f_{n-1}\ \mathbf{x}_{n-2})^\top \times \overleftarrow{\mathbf{X}_{n-1}} \\
&\vdots \\
\overleftarrow{\mathbf{X}_0} &= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{X}_1}
\end{aligned}
$$

$$\overline{\mathbf{x}'_n} = (\mathcal{J}\ f\ \mathbf{x}_0) \times \overline{\mathbf{x}'_0}$$

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$$\begin{aligned}\overrightarrow{\mathbf{x}_n} &= (\mathcal{J}\ f\ \mathbf{x}_0) \times \overrightarrow{\mathbf{x}_0}\\ &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_1\ \mathbf{x}_0) \times \overrightarrow{\mathbf{x}_0}\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{x}_n'} &= (\mathcal{J}\ f\ \mathbf{x}_0) \times \overline{\mathbf{x}_0'}\\ &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_1\ \mathbf{x}_0) \times \overline{\mathbf{x}_0'}\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{x}_1'} &= (\mathcal{J}\ f_1\ \mathbf{x}_0) \times \overline{\mathbf{x}_0'}\\ &\vdots\\ \overline{\mathbf{x}_n'} &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \overline{\mathbf{x}_{n-1}'}\end{aligned}$$

$$\overleftharpoon{\mathbf{x}_0} = (\mathcal{J}\ f\ \mathbf{x}_0)^\top \times \overleftharpoon{\mathbf{x}_n}$$

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$$
\begin{aligned}
\overleftarrow{\mathbf{x}_0} &= (\mathcal{J}\ f\ \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{x}_n}\\
&= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top \times \overleftarrow{\mathbf{x}_n}
\end{aligned}
$$

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$$
\begin{aligned}
\overleftarrow{\mathbf{x}_0} &= (\mathcal{J}\ f\ \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{x}_n}\\
&= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top \times \overleftarrow{\mathbf{x}_n}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{n-1}} &= (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top \times \overleftarrow{\mathbf{x}_n}\\
&\vdots\\
\overleftarrow{\mathbf{x}_0} &= (\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

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$$\mathbf{y} = \mathbf{A} \times \mathbf{x}$$

$$\begin{aligned}\mathbf{y} &= \mathbf{A} \times \mathbf{x}\\ &= f(\mathbf{x})\end{aligned}$$

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$$
\begin{aligned}
\mathbf{y} &= \mathbf{A} \times \mathbf{x} \\
&= f(\mathbf{x})
\end{aligned}
$$

$$
\overrightarrow{\mathbf{x}_n} = (\mathcal{J}\ f\ \mathbf{x}_0) \times \overrightarrow{\mathbf{x}_0}
$$

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$$\begin{aligned}\mathbf{y} &= \mathbf{A} \times \mathbf{x}\\ &= f(\mathbf{x})\end{aligned}$$

$$\begin{aligned}\overrightharpoon{\mathbf{x}_n} &= (\mathcal{J}\ f\ \mathbf{x}_0) \times \overrightharpoon{\mathbf{x}_0}\\ &= \overrightharpoon{f}\ \mathbf{x}_0\ \overrightharpoon{\mathbf{x}_0}\end{aligned}$$

A (Not So) Brief Tutorial on AD

$$\begin{aligned}\mathbf{y} &= \mathbf{A} \times \mathbf{x}\\ &= f(\mathbf{x})\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathbf{x}_n} &= (\mathcal{J}\ f\ \mathbf{x}_0) \times \overrightarrow{\mathbf{x}_0}\\ &= \overrightarrow{f}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0}\end{aligned}$$

$$\overleftarrow{\mathbf{x}_0} = (\mathcal{J}\ f\ \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{x}_n}$$

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$$
\begin{aligned}
\mathbf{y} &= \mathbf{A} \times \mathbf{x} \\
&= f(\mathbf{x})
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\mathbf{x}_n} &= (\mathcal{J}\ f\ \mathbf{x}_0) \times \overrightarrow{\mathbf{x}_0} \\
&= \overrightarrow{f}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_0} &= (\mathcal{J}\ f\ \mathbf{x}_0)^\top \times \overleftarrow{\mathbf{x}_n} \\
&= \overleftarrow{f}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_n}
\end{aligned}
$$

$$\overleftarrow{\mathbf{X}_n}[;j] = \overleftarrow{f}\ \mathbf{x}_0\ \overleftarrow{\mathbf{e}_j}$$

$$
\begin{aligned}
\overrightarrow{\mathbf{X}_n}[;j] &= \overrightarrow{f}\ \mathbf{x}_0\ \overrightarrow{\mathbf{e}_j}\\
&= (\mathcal{J}\ f\ \mathbf{x}_0)[;j]
\end{aligned}
$$

$$\begin{aligned}\overrightarrow{\mathbf{X}_n}[;j] &= \overrightarrow{f}\ \mathbf{x}_0\ \overrightarrow{\mathbf{e}_j}\\ &= (\mathcal{J}\ f\ \mathbf{x}_0)[;j]\end{aligned}$$

$$\overleftarrow{\mathbf{X}_0}[;i] = \overleftarrow{f}\ \mathbf{x}_0\ \overleftarrow{\mathbf{e}_i}$$

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$$
\begin{aligned}
\overrightarrow{\mathbf{X}_n}[;j] &= \overrightarrow{f}\ \mathbf{x}_0\ \overrightarrow{\mathbf{e}_j}\\
&= (\mathcal{J}\ f\ \mathbf{x}_0)[;j]
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{X}_0}[;i] &= \overleftarrow{f}\ \mathbf{x}_0\ \overleftarrow{\mathbf{e}_i}\\
&= (\mathcal{J}\ f\ \mathbf{x}_0)^\top[;i]
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\mathbf{X}_n}[;j] &= \overrightarrow{f}\ \mathbf{x}_0\ \overrightarrow{\mathbf{e}_j} \\
&= (\mathcal{J}\ f\ \mathbf{x}_0)[;j]
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{X}_0}[;i] &= \overleftarrow{f}\ \mathbf{x}_0\ \overleftarrow{\mathbf{e}_i} \\
&= (\mathcal{J}\ f\ \mathbf{x}_0)^\top[;i] \\
&= (\mathcal{J}\ f\ \mathbf{x}_0)[i;]
\end{aligned}
$$

A (Not So) Brief Tutorial on AD

$$\mathbf{y} = \mathbf{B} \times (\mathbf{A} \times \mathbf{x})$$

$$\begin{aligned}\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x})\\ &= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x}\end{aligned}$$

$$
\begin{aligned}
\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x}) \\
&= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x} \\
&= g(f(\mathbf{x}))
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x}) \\
&= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x} \\
&= g(f(\mathbf{x})) \\
&= (f \circ g)(\mathbf{x})
\end{aligned}
$$

$$\begin{aligned}\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x})\\ &= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x}\\ &= g(f(\mathbf{x}))\\ &= (f \circ g)(\mathbf{x})\end{aligned}$$

$$(\mathcal{J}\; f_n\; \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\; f_1\; \mathbf{x}_0) = (\overline{f_1}'\; \mathbf{x}_0) \circ \cdots \circ (\overline{f_n}'\; \mathbf{x}_{n-1})$$

$$
\begin{aligned}
\mathbf{y} &= \mathbf{B} \times (\mathbf{A} \times \mathbf{x}) \\
&= (\mathbf{B} \times \mathbf{A}) \times \mathbf{x} \\
&= g(f(\mathbf{x})) \\
&= (f \circ g)(\mathbf{x})
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{J}\ f_n\ \mathbf{x}_{n-1}) \times \cdots \times (\mathcal{J}\ f_1\ \mathbf{x}_0) &= (\overrightarrow{f_1}\ \mathbf{x}_0) \circ \cdots \circ (\overrightarrow{f_n}\ \mathbf{x}_{n-1}) \\
(\mathcal{J}\ f_1\ \mathbf{x}_0)^\top \times \cdots \times (\mathcal{J}\ f_n\ \mathbf{x}_{n-1})^\top &= (\overleftarrow{f_n}\ \mathbf{x}_{n-1}) \circ \cdots \circ (\overleftarrow{f_1}\ \mathbf{x}_0)
\end{aligned}
$$

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$$\mathbf{x}_{i-1} = \left(\boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{\mathbf{x}_{i-1}[L_i]} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]} \right)$$

$$\mathbf{x}_i = f_i\ \mathbf{x}_{i-1} = \left(\boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{u_i\ \mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]} \right)$$

$$\mathbf{x}[L_i] := u_i\ \mathbf{x}[R_i]$$

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$$\mathbf{x}_{i-1} = \left(\ \boxed{\mathbf{x}_{i-1}[1]} \cdots \ \boxed{\mathbf{x}_{i-1}[L_i]} \ \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]}\ \right)$$

$$\mathbf{x}_i = f_i\ \mathbf{x}_{i-1} = \left(\ \boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{{\color{red}u_i\ \mathbf{x}_{i-1}[R_i]}} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]}\ \right)$$

$${\color{red}\mathbf{x}[L_i]} := u_i\ \mathbf{x}[R_i]$$

A (Not So) Brief Tutorial on AD

$$\mathbf{x}_{i-1} = (\ \boxed{\mathbf{x}_{i-1}[1]}\cdots\ \boxed{\mathbf{x}_{i-1}[L_i]}\ \cdots\boxed{\mathbf{x}_{i-1}[R_i]}\cdots\boxed{\mathbf{x}_{i-1}[m]}\ )$$

$$\mathbf{x}_i = f_i\ \mathbf{x}_{i-1} = (\ \boxed{\mathbf{x}_{i-1}[1]}\cdots\boxed{u_i\ \mathbf{x}_{i-1}[R_i]}\cdots\boxed{\mathbf{x}_{i-1}[R_i]}\cdots\boxed{\mathbf{x}_{i-1}[m]}\ )$$

$$\mathbf{x}[L_i] := u_i\ \mathbf{x}[R_i]$$

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$$\mathbf{x}_{i-1} = (\ \boxed{\mathbf{x}_{i-1}[1]} \cdots \quad \boxed{\mathbf{x}_{i-1}[L_i]} \quad \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[S_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]}\ )$$

$$\mathbf{x}_i = f_i\ \mathbf{x}_{i-1} = (\ \boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[S_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]}\ )$$

$$\mathbf{x}[L_i] := b\ (\mathbf{x}[R_i], \mathbf{x}[S_i])$$

$$\mathbf{x}_{i-1} = (\ \boxed{\mathbf{x}_{i-1}[1]} \cdots \quad \boxed{\mathbf{x}_{i-1}[L_i]} \quad \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[S_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]}\ )$$

$$\mathbf{x}_i = f_i\ \mathbf{x}_{i-1} = (\ \boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{\color{red}{b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])}} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[S_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]}\ )$$

$$\color{red}{\mathbf{x}[L_i]} := b\ (\mathbf{x}[R_i], \mathbf{x}[S_i])$$

A (Not So) Brief Tutorial on AD

$$\mathbf{x}_{i-1} = (\ \boxed{\mathbf{x}_{i-1}[1]} \cdots \quad \boxed{\mathbf{x}_{i-1}[L_i]} \quad \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[S_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]}\ )$$

$$\mathbf{x}_i = f_i\ \mathbf{x}_{i-1} = (\ \boxed{\mathbf{x}_{i-1}[1]} \cdots \boxed{b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])} \cdots \boxed{\mathbf{x}_{i-1}[R_i]} \cdots \boxed{\mathbf{x}_{i-1}[S_i]} \cdots \boxed{\mathbf{x}_{i-1}[m]}\ )$$

$$\mathbf{x}[L_i] := b\ (\mathbf{x}[R_i], \mathbf{x}[S_i])$$

$$
\begin{array}{c|ccccccccc|}
 & & & & L_i & & & R_i & & \\
 & & & & \downarrow & & & \downarrow & & \\
\cline{2-10}
 & 1 & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & 1 & & & & & & \\
L_i \rightarrow & & & & 0 & & & u' & & \\
 & & & & & 1 & & & & \\
 & & & & & & \ddots & & & \\
R_i \rightarrow & & & & & & & 1 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & 1 \\
\cline{2-10}
\end{array}
$$

$$u' = \mathcal{D}\ u_i\ \mathbf{x}_{i-1}[R_i]$$

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$$
\begin{array}{c|ccccccccccc|}
 & & & & L_i & & & R_i & & S_i & & \\
 & & & & \downarrow & & & \downarrow & & \downarrow & & \\
\hline
 & 1 & & & & & & & & & & \\
 & & \ddots & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
L_i \longrightarrow & & & & 0 & & & b'_1 & & b'_2 & & \\
 & & & & & 1 & & & & & & \\
 & & & & & & \ddots & & & & & \\
R_i \longrightarrow & & & & & & & 1 & & & & \\
 & & & & & & & & \ddots & & & \\
S_i \longrightarrow & & & & & & & & & 1 & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & 1 \\
\hline
\end{array}
$$

$$
\begin{aligned}
b'_1 &= \mathcal{D}_1\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\
b'_2 &= \mathcal{D}_2\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])
\end{aligned}
$$

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$$\overline{\mathbf{x}_i'} = \overline{f_i'}\;\mathbf{x}_{i-1}\;\overline{\mathbf{x}_{i-1}'}$$

$$
\begin{aligned}
\overline{\mathbf{x}_i'} &= \overline{f_i'}\ \mathbf{x}_{i-1}\ \overline{\mathbf{x}_{i-1}'} \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}'}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_i} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_{i-1}} \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1}) \times \overleftarrow{\mathbf{x}_{i-1}}
\end{aligned}
$$

$$
\begin{pmatrix}
\overleftarrow{\mathbf{x}_{i-1}}[1] \\
\vdots \\
\overleftarrow{\mathbf{x}_{i-1}}[L_i - 1] \\
u' \times \overleftarrow{\mathbf{x}_{i-1}}[R_i] \\
\overleftarrow{\mathbf{x}_{i-1}}[L_i + 1] \\
\vdots \\
\overleftarrow{\mathbf{x}_{i-1}}[R_i] \\
\vdots \\
\overleftarrow{\mathbf{x}_{i-1}}[m]
\end{pmatrix}
=
\begin{pmatrix}
1 & & & & & & & & \\
& \ddots & & & & & & & \\
& & 1 & & & & & & \\
& & & 0 & & & u' & & \\
& & & & 1 & & & & \\
& & & & & \ddots & & & \\
& & & & & & 1 & & \\
& & & & & & & \ddots & \\
& & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
\overleftarrow{\mathbf{x}_{i-1}}[1] \\
\vdots \\
\overleftarrow{\mathbf{x}_{i-1}}[L_i - 1] \\
\overleftarrow{\mathbf{x}_{i-1}}[L_i] \\
\overleftarrow{\mathbf{x}_{i-1}}[L_i + 1] \\
\vdots \\
\overleftarrow{\mathbf{x}_{i-1}}[R_i] \\
\vdots \\
\overleftarrow{\mathbf{x}_{i-1}}[m]
\end{pmatrix}
$$

$$
u' = \mathcal{D}\ u_i\ \mathbf{x}_{i-1}[R_i]
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_i} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_{i-1}}\\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1}) \times \overleftarrow{\mathbf{x}_{i-1}}
\end{aligned}
$$

$$
\begin{pmatrix}
\overleftarrow{\mathbf{x}_{i-1}}[1]\\
\vdots\\
\overleftarrow{\mathbf{x}_{i-1}}[L_i-1]\\
\color{red}{u' \times \overleftarrow{\mathbf{x}_{i-1}}[R_i]}\\
\overleftarrow{\mathbf{x}_{i-1}}[L_i+1]\\
\vdots\\
\overleftarrow{\mathbf{x}_{i-1}}[R_i]\\
\vdots\\
\overleftarrow{\mathbf{x}_{i-1}}[m]
\end{pmatrix}
=
\begin{pmatrix}
1 &&&&&&&&\\
&\ddots&&&&&&&\\
&&1&&&&&&\\
&&&0&&&u'&&\\
&&&&1&&&&\\
&&&&&\ddots&&&\\
&&&&&&1&&\\
&&&&&&&\ddots&\\
&&&&&&&&1
\end{pmatrix}
\begin{pmatrix}
\overleftarrow{\mathbf{x}_{i-1}}[1]\\
\vdots\\
\overleftarrow{\mathbf{x}_{i-1}}[L_i-1]\\
\overleftarrow{\mathbf{x}_{i-1}}[L_i]\\
\overleftarrow{\mathbf{x}_{i-1}}[L_i+1]\\
\vdots\\
\overleftarrow{\mathbf{x}_{i-1}}[R_i]\\
\vdots\\
\overleftarrow{\mathbf{x}_{i-1}}[m]
\end{pmatrix}
$$

$$u' = \mathcal{D}\ u_i\ \mathbf{x}_{i-1}[R_i]$$

$$
\begin{aligned}
\overline{\mathbf{x}_i}' &= \overline{f_i}'\ \mathbf{x}_{i-1}\ \overline{\mathbf{x}_{i-1}}' \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}}'
\end{aligned}
$$

$$
\begin{pmatrix}
\overline{\mathbf{x}_{i-1}}'[1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[L_i-1] \\
\color{red}{u' \times \overline{\mathbf{x}_{i-1}}'[R_i]} \\
\overline{\mathbf{x}_{i-1}}'[L_i+1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[R_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[m]
\end{pmatrix}
=
\begin{pmatrix}
1 &&&&&&& \\
& \ddots &&&&&& \\
&& 1 &&&&& \\
&&& 0 &&& u' && \\
&&&& 1 &&& \\
&&&&& \ddots && \\
&&&&&& 1 & \\
&&&&&&& \ddots \\
&&&&&&&& 1
\end{pmatrix}
\begin{pmatrix}
\overline{\mathbf{x}_{i-1}}'[1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[L_i-1] \\
\overline{\mathbf{x}_{i-1}}'[L_i] \\
\overline{\mathbf{x}_{i-1}}'[L_i+1] \\
\vdots \\
\color{green}{\overline{\mathbf{x}_{i-1}}'[R_i]} \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[m]
\end{pmatrix}
$$

$$u' = \mathcal{D}\ u_i\ \mathbf{x}_{i-1}[R_i]$$

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$$\overline{\mathbf{x}_i'} = \overline{f_i'}\;\mathbf{x}_{i-1}\;\overline{\mathbf{x}_{i-1}'}$$

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$$
\begin{aligned}
\overleftarrow{\mathbf{x}_i} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_{i-1}} \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1}) \times \overleftarrow{\mathbf{x}_{i-1}}
\end{aligned}
$$

$$
\begin{aligned}
\overline{\mathbf{x}_i}' &= \overline{f_i}'\ \mathbf{x}_{i-1}\ \overline{\mathbf{x}_{i-1}}' \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}}'
\end{aligned}
$$

$$
\begin{pmatrix}
\overline{\mathbf{x}_{i-1}}'[1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[L_i-1] \\
b'_1 \times \overline{\mathbf{x}_{i-1}}'[R_i] + b'_2 \times \overline{\mathbf{x}_{i-1}}'[S_i] \\
\overline{\mathbf{x}_{i-1}}'[L_i+1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[R_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[S_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[m]
\end{pmatrix}
=
\begin{pmatrix}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 0 &&& b'_1 && b'_2 && \\
&&&& 1 &&&&&& \\
&&&&& \ddots &&&&& \\
&&&&&& 1 &&&& \\
&&&&&&& \ddots &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{pmatrix}
\begin{pmatrix}
\overline{\mathbf{x}_{i-1}}'[1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[L_i-1] \\
\overline{\mathbf{x}_{i-1}}'[L_i] \\
\overline{\mathbf{x}_{i-1}}'[L_i+1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[R_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[S_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}'[m]
\end{pmatrix}
$$

$$
\begin{aligned}
b'_1 &= \mathcal{D}_1\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\
b'_2 &= \mathcal{D}_2\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])
\end{aligned}
$$

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$$
\begin{aligned}
\overline{\mathbf{x}_i} &= \overline{f_i}\ \mathbf{x}_{i-1}\ \overline{\mathbf{x}_{i-1}} \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}}
\end{aligned}
$$

$$
\begin{pmatrix}
\overline{\mathbf{x}_{i-1}}[1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}[L_i-1] \\
\color{red}{b'_1 \times \overline{\mathbf{x}_{i-1}}[R_i] + b'_2 \times \overline{\mathbf{x}_{i-1}}[S_i]} \\
\overline{\mathbf{x}_{i-1}}[L_i+1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}[R_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}[S_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}[m]
\end{pmatrix}
=
\begin{pmatrix}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 0 &&& b'_1 && b'_2 && \\
&&&& 1 &&&&&& \\
&&&&& \ddots &&&&& \\
&&&&&& 1 &&&& \\
&&&&&&& \ddots &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{pmatrix}
\begin{pmatrix}
\overline{\mathbf{x}_{i-1}}[1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}[L_i-1] \\
\overline{\mathbf{x}_{i-1}}[L_i] \\
\overline{\mathbf{x}_{i-1}}[L_i+1] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}[R_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}[S_i] \\
\vdots \\
\overline{\mathbf{x}_{i-1}}[m]
\end{pmatrix}
$$

$$
\begin{aligned}
b'_1 &= \mathcal{D}_1\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\
b'_2 &= \mathcal{D}_2\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])
\end{aligned}
$$

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$$\begin{aligned}\overline{\mathbf{x}_i'} &= \overline{f_i'}\ \mathbf{x}_{i-1}\ \overline{\mathbf{x}_{i-1}'}\\ &= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1}) \times \overline{\mathbf{x}_{i-1}'}\end{aligned}$$

$$\begin{pmatrix} \overline{\mathbf{x}_{i-1}'}[1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}'}[L_i-1] \\ \color{red}{b_1' \times \overline{\mathbf{x}_{i-1}'}[R_i] + b_2' \times \overline{\mathbf{x}_{i-1}'}[S_i]} \\ \overline{\mathbf{x}_{i-1}'}[L_i+1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}'}[R_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}'}[S_i] \\ \vdots \\ \overline{\mathbf{x}_{i-1}'}[m] \end{pmatrix} = \begin{pmatrix} 1 &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& 0 &&& b_1' && b_2' &&& \\ &&&& 1 &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& 1 &&&& \\ &&&&&&& \ddots &&& \\ &&&&&&&& 1 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}_{i-1}'}[1] \\ \vdots \\ \overline{\mathbf{x}_{i-1}'}[L_i-1] \\ \overline{\mathbf{x}_{i-1}'}[L_i] \\ \overline{\mathbf{x}_{i-1}'}[L_i+1] \\ \vdots \\ \color{green}{\overline{\mathbf{x}_{i-1}'}[R_i]} \\ \vdots \\ \color{green}{\overline{\mathbf{x}_{i-1}'}[S_i]} \\ \vdots \\ \overline{\mathbf{x}_{i-1}'}[m] \end{pmatrix}$$

$$\begin{aligned} b_1' &= \mathcal{D}_1\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\ b_2' &= \mathcal{D}_2\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \end{aligned}$$

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$$\overleftarrow{\mathbf{x}_{i-1}} = \overleftarrow{f_i}\;\mathbf{x}_{i-1}\;\overleftarrow{\mathbf{x}_i}$$

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$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{i-1}} &= \overleftarrow{f_i}\; \mathbf{x}_{i-1}\; \overleftarrow{\mathbf{x}_i}\\
&= (\mathcal{J}\; f_i\; \mathbf{x}_{i-1})^{\top} \times \overleftarrow{\mathbf{x}_i}
\end{aligned}
$$

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$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{i-1}} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_i} \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1})^{\top} \times \overleftarrow{\mathbf{x}_i}
\end{aligned}
$$

$$
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[L_i - 1] \\
0 \\
\overleftarrow{\mathbf{x}_i}[L_i + 1] \\
\vdots \\
u' \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[R_i] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
=
\begin{pmatrix}
1 & & & & & & & \\
& \ddots & & & & & & \\
& & 1 & & & & & \\
& & & 0 & & & & \\
& & & & 1 & & & \\
& & & & & \ddots & & \\
& & & u' & & & 1 & \\
& & & & & & & \ddots & \\
& & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[L_i - 1] \\
\overleftarrow{\mathbf{x}_i}[L_i] \\
\overleftarrow{\mathbf{x}_i}[L_i + 1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[R_i] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
$$

$$u' = \mathcal{D}\ u_i\ \mathbf{x}_{i-1}[R_i]$$

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$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{i-1}} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_i} \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1})^\top \times \overleftarrow{\mathbf{x}_i}
\end{aligned}
$$

$$
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[L_i-1] \\
\color{red}{0} \\
\overleftarrow{\mathbf{x}_i}[L_i+1] \\
\vdots \\
\color{red}{u' \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[R_i]} \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
=
\begin{pmatrix}
1 & & & & & & & & \\
& \ddots & & & & & & & \\
& & 1 & & & & & & \\
& & & 0 & & & & & \\
& & & & 1 & & & & \\
& & & & & \ddots & & & \\
& & & u' & & & 1 & & \\
& & & & & & & \ddots & \\
& & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[L_i-1] \\
\overleftarrow{\mathbf{x}_i}[L_i] \\
\overleftarrow{\mathbf{x}_i}[L_i+1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[R_i] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
$$

$$u' = \mathcal{D}\ u_i\ \mathbf{x}_{i-1}[R_i]$$

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$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{i-1}} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_i} \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1})^\top \times \overleftarrow{\mathbf{x}_i}
\end{aligned}
$$

$$
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[L_i - 1] \\
0 \\
\overleftarrow{\mathbf{x}_i}[L_i + 1] \\
\vdots \\
u' \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[R_i] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
=
\begin{pmatrix}
1 & & & & & & & & \\
& \ddots & & & & & & & \\
& & 1 & & & & & & \\
& & & 0 & & & & & \\
& & & & 1 & & & & \\
& & & & & \ddots & & & \\
& & & u' & & & 1 & & \\
& & & & & & & \ddots & \\
& & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[L_i - 1] \\
\overleftarrow{\mathbf{x}_i}[L_i] \\
\overleftarrow{\mathbf{x}_i}[L_i + 1] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[R_i] \\
\vdots \\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
$$

$$u' = \mathcal{D}\ u_i\ \mathbf{x}_{i-1}[R_i]$$

$$\overleftharpoon{\mathbf{x}_{i-1}} = \overleftharpoon{f_i}\ \mathbf{x}_{i-1}\ \overleftharpoon{\mathbf{x}_i}$$

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$$
\begin{aligned}
\overleftharpoon{\mathbf{x}_{i-1}} &= \overleftharpoon{f_i}\ \mathbf{x}_{i-1}\ \overleftharpoon{\mathbf{x}_i} \\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1})^\top \times \overleftharpoon{\mathbf{x}_i}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{i-1}} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_i}\\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1})^\top \times \overleftarrow{\mathbf{x}_i}
\end{aligned}
$$

$$
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[L_i-1]\\
0\\
\overleftarrow{\mathbf{x}_i}[L_i+1]\\
\vdots\\
b_1' \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[R_i]\\
\vdots\\
b_2' \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[S_i]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
=
\begin{pmatrix}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & 0 & & & & & & & \\
 & & & & 1 & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & b_1' & & & 1 & & & & \\
 & & & & & & & \ddots & & & \\
 & & & b_2' & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[L_i-1]\\
\overleftarrow{\mathbf{x}_i}[L_i]\\
\overleftarrow{\mathbf{x}_i}[L_i+1]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[R_i]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[S_i]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
$$

$$
\begin{aligned}
b_1' &= \mathcal{D}_1\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])\\
b_2' &= \mathcal{D}_2\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{i-1}} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_i}\\
&= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1})^\top \times \overleftarrow{\mathbf{x}_i}
\end{aligned}
$$

$$
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[L_i-1]\\
\color{red}{0}\\
\overleftarrow{\mathbf{x}_i}[L_i+1]\\
\vdots\\
\color{red}{b'_1 \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[R_i]}\\
\vdots\\
\color{red}{b'_2 \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[S_i]}\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
=
\begin{pmatrix}
1 &&&&&&&&&&\\
&\ddots&&&&&&&&&\\
&&1&&&&&&&&\\
&&&0&&&&&&&\\
&&&&1&&&&&&\\
&&&&&\ddots&&&&&\\
&&&b'_1&&&1&&&&\\
&&&&&&&\ddots&&&\\
&&&b'_2&&&&&1&&\\
&&&&&&&&&\ddots&\\
&&&&&&&&&&1
\end{pmatrix}
\begin{pmatrix}
\overleftarrow{\mathbf{x}_i}[1]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[L_i-1]\\
\overleftarrow{\mathbf{x}_i}[L_i]\\
\overleftarrow{\mathbf{x}_i}[L_i+1]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[R_i]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[S_i]\\
\vdots\\
\overleftarrow{\mathbf{x}_i}[m]
\end{pmatrix}
$$

$$
\begin{aligned}
b'_1 &= \mathcal{D}_1\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])\\
b'_2 &= \mathcal{D}_2\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i])
\end{aligned}
$$

$$\begin{aligned}\overleftarrow{\mathbf{x}_{i-1}} &= \overleftarrow{f_i}\ \mathbf{x}_{i-1}\ \overleftarrow{\mathbf{x}_i}\\ &= (\mathcal{J}\ f_i\ \mathbf{x}_{i-1})^\top \times \overleftarrow{\mathbf{x}_i}\end{aligned}$$

$$\begin{pmatrix} \overleftarrow{\mathbf{x}_i}[1] \\ \vdots \\ \overleftarrow{\mathbf{x}_i}[L_i-1] \\ 0 \\ \overleftarrow{\mathbf{x}_i}[L_i+1] \\ \vdots \\ b'_1 \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[R_i] \\ \vdots \\ b'_2 \times \overleftarrow{\mathbf{x}_i}[L_i] + \overleftarrow{\mathbf{x}_i}[S_i] \\ \vdots \\ \overleftarrow{\mathbf{x}_i}[m] \end{pmatrix} = \begin{pmatrix} 1 &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& 0 &&&&&&& \\ &&&& 1 &&&&&& \\ &&&&& \ddots &&&&& \\ &&& b'_1 &&& 1 &&&& \\ &&&&&&& \ddots &&& \\ &&& b'_2 &&&&& 1 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{pmatrix} \begin{pmatrix} \overleftarrow{\mathbf{x}_i}[1] \\ \vdots \\ \overleftarrow{\mathbf{x}_i}[L_i-1] \\ \overleftarrow{\mathbf{x}_i}[L_i] \\ \overleftarrow{\mathbf{x}_i}[L_i+1] \\ \vdots \\ \overleftarrow{\mathbf{x}_i}[R_i] \\ \vdots \\ \overleftarrow{\mathbf{x}_i}[S_i] \\ \vdots \\ \overleftarrow{\mathbf{x}_i}[m] \end{pmatrix}$$

$$\begin{aligned} b'_1 &= \mathcal{D}_1\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \\ b'_2 &= \mathcal{D}_2\ b_i\ (\mathbf{x}_{i-1}[R_i], \mathbf{x}_{i-1}[S_i]) \end{aligned}$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\mathbf{x}_1} &= \overrightarrow{f_1}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0}\\
&\vdots\\
\overrightarrow{\mathbf{x}_n} &= \overrightarrow{f_n}\ \mathbf{x}_{n-1}\ \overrightarrow{\mathbf{x}_{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
\overrightarrow{\mathbf{x}_1} &= \overrightarrow{f_1}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0}\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}\\
\overrightarrow{\mathbf{x}_n} &= \overrightarrow{f_n}\ \mathbf{x}_{n-1}\ \overrightarrow{\mathbf{x}_{n-1}}
\end{aligned}
$$

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$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\ \vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\mathbf{x}_1} &= \overrightarrow{f_1}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0}\\
&\ \vdots\\
\overrightarrow{\mathbf{x}_n} &= \overrightarrow{f_n}\ \mathbf{x}_{n-1}\ \overrightarrow{\mathbf{x}_{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
\overrightarrow{\mathbf{x}_1} &= \overrightarrow{f_1}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0}\\
&\ \vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}\\
\overrightarrow{\mathbf{x}_n} &= \overrightarrow{f_n}\ \mathbf{x}_{n-1}\ \overrightarrow{\mathbf{x}_{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\color{red}{\mathbf{x}_n} &\color{red}{=} \color{red}{f_n\ \mathbf{x}_{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\mathbf{x}_1} &= \overrightarrow{f_1}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0}\\
&\vdots\\
\color{green}{\overrightarrow{\mathbf{x}_n}} &\color{green}{=} \color{green}{\overrightarrow{f_n}\ \mathbf{x}_{n-1}\ \overrightarrow{\mathbf{x}_{n-1}}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
\overrightarrow{\mathbf{x}_1} &= \overrightarrow{f_1}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0}\\
&\vdots\\
\color{red}{\mathbf{x}_n} &\color{red}{=} \color{red}{f_n\ \mathbf{x}_{n-1}}\\
\color{green}{\overrightarrow{\mathbf{x}_n}} &\color{green}{=} \color{green}{\overrightarrow{f_n}\ \mathbf{x}_{n-1}\ \overrightarrow{\mathbf{x}_{n-1}}}
\end{aligned}
$$

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$$
\boxed{
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0 \\
\overrightarrow{\mathbf{x}_1} &= \overrightarrow{f_1}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0} \\
&\ \vdots \\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1} \\
\overrightarrow{\mathbf{x}_n} &= \overrightarrow{f_n}\ \mathbf{x}_{n-1}\ \overrightarrow{\mathbf{x}_{n-1}}
\end{aligned}
}
$$

$$
\boxed{
\begin{aligned}
(\mathbf{x}_1, \overrightarrow{\mathbf{x}_1}) &= ((f_1\ \mathbf{x}_0), (\overrightarrow{f_1}\ \mathbf{x}_0\ \overrightarrow{\mathbf{x}_0})) \\
&\ \vdots \\
(\mathbf{x}_n, \overrightarrow{\mathbf{x}_n}) &= ((f_n\ \mathbf{x}_{n-1}), (\overrightarrow{f_n}\ \mathbf{x}_{n-1}\ \overrightarrow{\mathbf{x}_{n-1}}))
\end{aligned}
}
$$

$$\left.\begin{array}{l} \mathbf{x}_1 = f_1\ \mathbf{x}_0 \\ \quad\vdots \\ \mathbf{x}_n = f_n\ \mathbf{x}_{n-1} \end{array}\right\} \rightsquigarrow \left\{\begin{array}{l} \overrightarrow{\mathbf{x}_1} = \overrightarrow{f_1}\ \overrightarrow{\mathbf{x}_0} \\ \quad\vdots \\ \overrightarrow{\mathbf{x}_n} = \overrightarrow{f_n}\ \overrightarrow{\mathbf{x}_{n-1}} \end{array}\right.$$

$$\begin{aligned} \overrightarrow{\mathbf{x}} &\equiv (\mathbf{x}, \acute{\mathbf{x}}) \\ \overrightarrow{f}\ \overrightarrow{\mathbf{x}} &\equiv ((f\ \mathbf{x}), (\acute{f}\ \mathbf{x}\ \acute{\mathbf{x}})) \end{aligned}$$

A (Not So) Brief Tutorial on AD

$$
\begin{array}{rcl}
x_{L_i} := u_i\ x_{R_i} & \rightsquigarrow & \overrightarrow{x_{L_i}} := \overrightarrow{u_i}\ \overrightarrow{x_{R_i}} \\
x_{L_i} := b_i\ (x_{R_i}, x_{S_i}) & \rightsquigarrow & \overrightarrow{x_{L_i}} := \overrightarrow{b_i}\ (\overrightarrow{x_{R_i}}, \overrightarrow{x_{S_i}})
\end{array}
$$

$$
\overrightarrow{x} \equiv (x, \acute{x})
$$

$$
\begin{array}{rl}
\overrightarrow{u}\ \overrightarrow{x} & \equiv ((u\ x), ((\mathcal{D}\ u\ x) \times \acute{x})) \\
\overrightarrow{b}\ (\overrightarrow{x_1}, \overrightarrow{x_2}) & \equiv ((b\ (x_1, x_2)), ((\mathcal{D}_1\ b\ (x_1, x_2)) \times \acute{x_1}) + ((\mathcal{D}_2\ b\ (x_1, x_2)) \times \acute{x_2}))
\end{array}
$$

A (Not So) Brief Tutorial on AD

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0 \\
&\vdots \\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n} \\
&\vdots \\
\overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0 \\
&\vdots \\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1} \\
\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n} \\
&\vdots \\
\overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

A (Not So) Brief Tutorial on AD

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n}\\
&\vdots\\
\overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}\\
\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n}\\
&\vdots\\
\overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

$$\begin{aligned}\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\ &\vdots\\ \mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}\end{aligned}$$

$$\begin{aligned}\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n}\\ &\vdots\\ \overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}\end{aligned}$$

$$\begin{aligned}\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\ &\vdots\\ \mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}\\ \overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n}\\ &\vdots\\ \overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}\end{aligned}$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n}\\
&\vdots\\
\overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

$$
\begin{aligned}
\color{red}{\mathbf{x}_1} &= f_1\ \mathbf{x}_0\\
&\vdots\\
\color{red}{\mathbf{x}_n} &= f_n\ \mathbf{x}_{n-1}\\
\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n}\\
&\vdots\\
\overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n}\\
&\vdots\\
\overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
&\vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}\\
\overleftarrow{\mathbf{x}_{n-1}} &= \overleftarrow{f_n}\ \mathbf{x}_{n-1}\ \overleftarrow{\mathbf{x}_n}\\
&\vdots\\
\overleftarrow{\mathbf{x}_0} &= \overleftarrow{f_1}\ \mathbf{x}_0\ \overleftarrow{\mathbf{x}_1}
\end{aligned}
$$

A (Not So) Brief Tutorial on AD

$$
\begin{aligned}
\mathbf{x}_1 &= f_1\ \mathbf{x}_0\\
\overline{\mathbf{x}_1} &= \lambda\,\overleftarrow{\mathbf{x}}\quad \overline{\mathbf{x}_0}\ (\overleftarrow{f_1}\ \mathbf{x}_0\ \ \overleftarrow{\mathbf{x}})\\
&\ \vdots\\
\mathbf{x}_n &= f_n\ \mathbf{x}_{n-1}\\
\overline{\mathbf{x}_n} &= \lambda\,\overleftarrow{\mathbf{x}}\quad \overline{\mathbf{x}_{n-1}}\ (\overleftarrow{f_1}\ \mathbf{x}_0\ \ \overleftarrow{\mathbf{x}})
\end{aligned}
$$

$$
\boxed{
\begin{array}{rcl}
\mathbf{x}_1 & = & f_1\ \mathbf{x}_0 \\
\overline{\mathbf{x}_1} & = & \lambda\ \overleftarrow{\mathbf{x}}\quad \overline{\mathbf{x}_0}\ (\overleftarrow{f_1}\ \mathbf{x}_0\ \ \overleftarrow{\mathbf{x}}) \\
 & \vdots & \\
\mathbf{x}_n & = & f_n\ \mathbf{x}_{n-1} \\
\overline{\mathbf{x}_n} & = & \lambda\ \overleftarrow{\mathbf{x}}\quad \overline{\mathbf{x}_{n-1}}\ (\overleftarrow{f_1}\ \mathbf{x}_0\ \ \overleftarrow{\mathbf{x}})
\end{array}
}
$$

$$
\boxed{
\begin{array}{rcl}
(\mathbf{x}_1, \overline{\mathbf{x}_1}) & = & ((f_1\ \mathbf{x}_0), (\lambda\ \overleftarrow{\mathbf{x}}\quad \overline{\mathbf{x}_0}\ (\overleftarrow{f_1}\ \mathbf{x}_0\ \ \overleftarrow{\mathbf{x}}))) \\
 & \vdots & \\
(\mathbf{x}_n, \overline{\mathbf{x}_n}) & = & ((f_n\ \mathbf{x}_{n-1}), (\lambda\ \overleftarrow{\mathbf{x}}\quad \overline{\mathbf{x}_{n-1}}\ (\overleftarrow{f_1}\ \mathbf{x}_0\ \ \overleftarrow{\mathbf{x}})))
\end{array}
}
$$

$$\left.\begin{array}{l}\mathbf{x}_1 = f_1\ \mathbf{x}_0\\ \quad\vdots\\ \mathbf{x}_n = f_n\ \mathbf{x}_{n-1}\end{array}\right\} \rightsquigarrow \left\{\begin{array}{l}\overleftarrow{\mathbf{x}_1} = \overleftarrow{f_1}\ \overleftarrow{\mathbf{x}_0}\\ \quad\vdots\\ \overleftarrow{\mathbf{x}_n} = \overleftarrow{f_n}\ \overleftarrow{\mathbf{x}_{n-1}}\end{array}\right.$$

$$\begin{aligned}\overleftarrow{\mathbf{x}} &\equiv (\mathbf{x}, \bar{\mathbf{x}})\\ \overleftarrow{f}\ \overleftarrow{\mathbf{x}} &\equiv ((f\ \mathbf{x}), (\lambda\, \grave{\overline{\mathbf{x}}}\ \ \bar{\mathbf{x}}\ (\grave{\overline{f}}\ \mathbf{x}\ \grave{\overline{\mathbf{x}}})))\end{aligned}$$

$$\overleftharpoonup{f}\;\mathbf{x} \equiv \mathbf{begin}\; \bar{x} := \lambda\, \overleftharpoondown{\mathbf{x}}\quad \bar{x}\,(\overleftharpoondown{f}\;\mathbf{x}\;\overleftharpoondown{\mathbf{x}});$$
$$(f\;\mathbf{x})\;\mathbf{end}$$

$$
x_{L_i} := u_i\ x_{R_i} \quad \rightsquigarrow \quad
\begin{cases}
\bar{x} := \lambda[\,]\ \textbf{begin}\ \overleftharpoondown{x_{R_i}} \mathrel{+}:= (\mathcal{D}\ u_i\ \overleftharpoonup{x_{R_i}}) \times \overleftharpoondown{x_{L_i}}; \\
\qquad\qquad\qquad\quad \overleftharpoondown{x_{L_i}} := 0; \\
\qquad\qquad\qquad\quad \bar{x}\ [\,]\ \textbf{end} \\
\overleftharpoonup{x_{L_i}} := u_i\ \overleftharpoonup{x_{R_i}}
\end{cases}
$$

$$
x_{L_i} := b_i\ (x_{R_i}, x_{S_i}) \quad \rightsquigarrow \quad
\begin{cases}
\bar{x} := \lambda[\,]\ \textbf{begin}\ \overleftharpoondown{x_{R_i}} \mathrel{+}:= (\mathcal{D}_1\ b_i\ (\overleftharpoonup{x_{R_i}}, \overleftharpoonup{x_{S_i}})) \times \overleftharpoondown{x_{L_i}}; \\
\qquad\qquad\qquad\quad \overleftharpoondown{x_{S_i}} \mathrel{+}:= (\mathcal{D}_2\ b_i\ (\overleftharpoonup{x_{R_i}}, \overleftharpoonup{x_{S_i}})) \times \overleftharpoondown{x_{L_i}}; \\
\qquad\qquad\qquad\quad \overleftharpoondown{x_{L_i}} := 0; \\
\qquad\qquad\qquad\quad \bar{x}\ [\,]\ \textbf{end} \\
\overleftharpoonup{x_{L_i}} := b_i\ (\overleftharpoonup{x_{R_i}}, \overleftharpoonup{x_{S_i}})
\end{cases}
$$

$$
\overleftharpoonup{x} \equiv x
$$

$$x_{L_i} := u_i\ x_{R_i} \quad\rightsquigarrow\quad \left\{\begin{array}{l} \overset{\leftharpoonup}{x_{L_i}} := u_i\ \overset{\leftharpoonup}{x_{R_i}} \\ \quad\vdots \\ \overset{\leftharpoondown}{x_{R_i}} \mathrel{+}:= (\mathcal{D}\ u_i\ \overset{\leftharpoonup}{x_{R_i}}) \times \overset{\leftharpoondown}{x_{L_i}}; \\ \overset{\leftharpoondown}{x_{L_i}} := 0 \end{array}\right.$$

$$x_{L_i} := b_i\ (x_{R_i}, x_{S_i}) \quad\rightsquigarrow\quad \left\{\begin{array}{l} \overset{\leftharpoonup}{x_{L_i}} := b_i\ (\overset{\leftharpoonup}{x_{R_i}}, \overset{\leftharpoonup}{x_{S_i}}) \\ \quad\vdots \\ \overset{\leftharpoondown}{x_{R_i}} \mathrel{+}:= (\mathcal{D}_1\ b_i\ (\overset{\leftharpoonup}{x_{R_i}}, \overset{\leftharpoonup}{x_{S_i}})) \times \overset{\leftharpoondown}{x_{L_i}}; \\ \overset{\leftharpoondown}{x_{S_i}} \mathrel{+}:= (\mathcal{D}_2\ b_i\ (\overset{\leftharpoonup}{x_{R_i}}, \overset{\leftharpoonup}{x_{S_i}})) \times \overset{\leftharpoondown}{x_{L_i}}; \\ \overset{\leftharpoondown}{x_{L_i}} := 0 \end{array}\right.$$

$$\overset{\leftharpoonup}{x} \equiv x$$

The Functional Reverse-Mode Transformation

$$\begin{aligned} x_{L_1} &:= u_1\ x_{S_1} \\ &\vdots \\ x_{L_n} &:= u_n\ x_{S_n} \end{aligned}$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{lcl}
x_{L_1} & := & u_1\ x_{S_1} \\
 & \vdots & \\
x_{L_n} & := & u_n\ x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{rcl}
x_{L_1} & := & u_1\ x_{S_1} \\
 & \vdots & \\
x_{L_n} & := & u_n\ x_{S_n} \\
\overleftarrow{x_1} & := & 0 \\
 & \vdots & \\
\overleftarrow{x_m} & := & 0 \\
\overleftarrow{x_{S_n}} & +:= & (\mathcal{D}\ u_n\ x_{S_n}) \times \overleftarrow{x_{L_n}} \\
\overleftarrow{x_{L_n}} & := & 0 \\
 & \vdots & \\
\overleftarrow{x_{S_1}} & +:= & (\mathcal{D}\ u_1\ x_{S_1}) \times \overleftarrow{x_{L_1}} \\
\overleftarrow{x_{L_1}} & := & = 0
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{rcl}
x_{L_1} & {\color{red}:=} & u_1\ x_{S_1} \\
 & \vdots & \\
x_{L_n} & {\color{red}:=} & u_n\ x_{S_n}
\end{array}\right\}
\leadsto
\left\{\begin{array}{rcl}
x_{L_1} & := & u_1\ x_{S_1} \\
 & \vdots & \\
x_{L_n} & := & u_n\ x_{S_n} \\
\overleftarrow{x_1} & := & 0 \\
 & \vdots & \\
\overleftarrow{x_m} & := & 0 \\
\overleftarrow{x_{S_n}} & +:= & (\mathcal{D}\ u_n\ x_{S_n}) \times \overleftarrow{x_{L_n}} \\
\overleftarrow{x_{L_n}} & := & 0 \\
 & \vdots & \\
\overleftarrow{x_{S_1}} & +:= & (\mathcal{D}\ u_1\ x_{S_1}) \times \overleftarrow{x_{L_1}} \\
\overleftarrow{x_{L_1}} & := & = 0
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{lcl}
x_{L_1} & := & u_1\ x_{S_1} \\
 & \vdots & \\
x_{L_n} & := & u_n\ x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{lcl}
x_{L_1} & := & u_1\ x_{S_1} \\
 & \vdots & \\
x_{L_n} & := & u_n\ x_{S_n} \\
\grave{\overline{x_1}} & := & 0 \\
 & \vdots & \\
\grave{\overline{x_m}} & := & 0 \\
\grave{\overline{x_{S_n}}} & +:= & (\mathcal{D}\ u_n\ x_{S_n}) \times \grave{\overline{x_{L_n}}} \\
\grave{\overline{x_{L_n}}} & := & 0 \\
 & \vdots & \\
\grave{\overline{x_{S_1}}} & +:= & (\mathcal{D}\ u_1\ x_{S_1}) \times \grave{\overline{x_{L_1}}} \\
\grave{\overline{x_{L_1}}} & := & = 0
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{lcl}
x_{L_1} & := & u_1\ x_{S_1}\\
 & \vdots & \\
x_{L_n} & := & u_n\ x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{lcl}
x_{L_1} & := & u_1\ x_{S_1}\\
 & \vdots & \\
x_{L_n} & := & u_n\ x_{S_n}\\
\overleftarrow{x_1} & := & 0\\
 & \vdots & \\
\overleftarrow{x_m} & := & 0\\
\overleftarrow{x_{S_n}} & +:= & (\mathcal{D}\ u_n\ x_{S_n}) \times \overleftarrow{x_{L_n}}\\
\color{red}{\overleftarrow{x_{L_n}}} & \color{red}{:=} & \color{red}{0}\\
 & \vdots & \\
\overleftarrow{x_{S_1}} & +:= & (\mathcal{D}\ u_1\ x_{S_1}) \times \overleftarrow{x_{L_1}}\\
\color{red}{\overleftarrow{x_{L_1}}} & \color{red}{:=} & \color{red}{=0}
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{rcl}
x_1 & = & u_1\ x_{S_1} \\
 & \vdots & \\
x_n & = & u_n\ x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{rcl}
x_1 & = & u_1\ x_{S_1} \\
 & \vdots & \\
x_n & = & u_n\ x_{S_n} \\
\overleftarrow{x_0} & := & 0 \\
 & \vdots & \\
\overleftarrow{x_{n-1}} & := & 0 \\
\overleftarrow{x_{S_n}} & +:= & (\mathcal{D}\ u_n\ x_{S_n}) \times \overleftarrow{x_n} \\
 & \vdots & \\
\overleftarrow{x_{S_1}} & +:= & (\mathcal{D}\ u_1\ x_{S_1}) \times \overleftarrow{x_1}
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$\left.\begin{array}{rcl} x_1 & = & u_1\ x_{S_1} \\ & \vdots & \\ x_n & = & u_n\ x_{S_n} \end{array}\right\} \leadsto \left\{\begin{array}{rcl} x_1 & = & u_1\ x_{S_1} \\ & \vdots & \\ x_n & = & u_n\ x_{S_n} \\ \overleftarrow{x_0} & := & 0 \\ & \vdots & \\ \overleftarrow{x_{n-1}} & := & 0 \\ \overleftarrow{x_{S_n}} & +:= & (\mathcal{D}\ u_n\ x_{S_n}) \times \overleftarrow{x_n} \\ & \vdots & \\ \overleftarrow{x_{S_1}} & +:= & (\mathcal{D}\ u_1\ x_{S_1}) \times \overleftarrow{x_1} \end{array}\right.$$

The Functional Reverse-Mode Transformation

$$\overline{u_i} \triangleq \lambda\, \overleftarrow{x}\ (\mathcal{D}\ u_i\ x_{S_i}) \times \overleftarrow{x}$$

$$\left.\begin{array}{rcl} x_1 & = & u_1\ x_{S_1} \\ & \vdots & \\ x_n & = & u_n\ x_{S_n} \end{array}\right\} \rightsquigarrow \left\{\begin{array}{rcl} x_1 & = & u_1\ x_{S_1} \\ & \vdots & \\ x_n & = & u_n\ x_{S_n} \\ \overleftarrow{x_0} & := & 0 \\ & \vdots & \\ \overleftarrow{x_{n-1}} & := & 0 \\ \overleftarrow{x_{S_n}} & +:= & \overline{u_n}\ \overleftarrow{x_n} \\ & \vdots & \\ \overleftarrow{x_{S_1}} & +:= & \overline{u_1}\ \overleftarrow{x_1} \end{array}\right.$$

The Functional Reverse-Mode Transformation

$$\overleftarrow{u}\ x \triangleq ((u\ x), (\lambda\ \grave{x}\ (\mathcal{D}\ u\ x) \times \grave{x}))$$

$$\left.\begin{array}{rcl} x_1 & = & u_1\ x_{S_1} \\ & \vdots & \\ x_n & = & u_n\ x_{S_n} \end{array}\right\} \rightsquigarrow \left\{\begin{array}{rcl} (x_1, \overline{x_1}) & = & \overleftarrow{u_1}\ x_{S_1} \\ & \vdots & \\ (x_n, \overline{x_n}) & = & \overleftarrow{u_n}\ x_{S_n} \\ \grave{x_0} & := & 0 \\ & \vdots & \\ \grave{x_{n-1}} & := & 0 \\ \grave{x_{S_n}} & +:= & \overline{x_n}\ \grave{x_n} \\ & \vdots & \\ \grave{x_{S_1}} & +:= & \overline{x_1}\ \grave{x_1} \end{array}\right.$$

The Functional Reverse-Mode Transformation

$$\overleftarrow{u}\; x \triangleq ((u\; x), (\lambda\, \grave{x}\, (\mathcal{D}\; u\; x) \times \grave{x}))$$

$$\left.\begin{array}{rcl} x_1 & = & {\color{red}u_1}\; x_{S_1} \\ & \vdots & \\ x_n & = & {\color{red}u_n}\; x_{S_n} \end{array}\right\} \leadsto \left\{\begin{array}{rcl} (x_1, \overline{x_1}) & = & \overleftarrow{u_1}\; x_{S_1} \\ & \vdots & \\ (x_n, \overline{x_n}) & = & \overleftarrow{u_n}\; x_{S_n} \\ \grave{x_0} & := & 0 \\ & \vdots & \\ \grave{x}_{n-1} & := & 0 \\ \grave{x}_{S_n} & +:= & \overline{x_n}\; \grave{x_n} \\ & \vdots & \\ \grave{x}_{S_1} & +:= & \overline{x_1}\; \grave{x_1} \end{array}\right.$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{rcl}
x_1 & = & x_{R_1}\; x_{S_1} \\
 & \vdots & \\
x_n & = & x_{R_n}\; x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{rcl}
(\overleftarrow{x_1}, \overline{x_1}) & = & \overleftarrow{x_{R_1}}\; \overleftarrow{x_{S_1}} \\
 & \vdots & \\
(\overleftarrow{x_n}, \overline{x_n}) & = & \overleftarrow{x_{R_n}}\; \overleftarrow{x_{S_n}} \\
\grave{x_0} & := & 0 \\
 & \vdots & \\
\grave{x_{n-1}} & := & 0 \\
\grave{x_{S_n}} & +:= & \overline{x_n}\; \grave{x_n} \\
 & \vdots & \\
\grave{x_{S_1}} & +:= & \overline{x_1}\; \grave{x_1}
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{rcl}
x_1 & = & x_{R_1}\; x_{S_1} \\
 & \vdots & \\
x_n & = & x_{R_n}\; x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{rcl}
(\overleftarrow{x_1}, \overline{x_1}) & = & \overleftarrow{x_{R_1}}\; \overleftarrow{x_{S_1}} \\
 & \vdots & \\
(\overleftarrow{x_n}, \overline{x_n}) & = & \overleftarrow{x_{R_n}}\; \overleftarrow{x_{S_n}} \\
\grave{x_0} & := & 0 \\
 & \vdots & \\
\grave{x_{n-1}} & := & 0 \\
\grave{x_{S_n}} & +:= & \overline{x_n}\; \grave{x_n} \\
 & \vdots & \\
\grave{x_{S_1}} & +:= & \overline{x_1}\; \grave{x_1}
\end{array}\right.
$$

$$
\left.\begin{array}{rcl}
x_1 & = & x_{R_1}\ x_{S_1} \\
 & \vdots & \\
x_n & = & x_{R_n}\ x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{rcl}
(\overleftarrow{x_1}, \overline{x_1}) & = & \overleftarrow{x_{R_1}}\ \overleftarrow{x_{S_1}} \\
 & \vdots & \\
(\overleftarrow{x_n}, \overline{x_n}) & = & \overleftarrow{x_{R_n}}\ \overleftarrow{x_{S_n}} \\
\grave{\overline{x_0}} & := & \color{red}{0} \\
 & \vdots & \\
\grave{\overline{x_{n-1}}} & := & \color{red}{0} \\
\grave{\overline{x_{S_n}}} & +:= & \overline{x_n}\ \grave{\overline{x_n}} \\
 & \vdots & \\
\grave{\overline{x_{S_1}}} & +:= & \overline{x_1}\ \grave{\overline{x_1}}
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{rcl}
x_1 & = & x_{R_1}\; x_{S_1}\\
& \vdots & \\
x_n & = & x_{R_n}\; x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{rcl}
(\overleftarrow{x_1}, \overline{x_1}) & = & \overleftarrow{x_{R_1}}\; \overleftarrow{x_{S_1}}\\
& \vdots & \\
(\overleftarrow{x_n}, \overline{x_n}) & = & \overleftarrow{x_{R_n}}\; \overleftarrow{x_{S_n}}\\
\grave{x_0} & := & 0\\
& \vdots & \\
\grave{x_{n-1}} & := & 0\\
\grave{x_{S_n}} & \color{red}{+:=} & \overline{x_n}\; \grave{x_n}\\
& \vdots & \\
\grave{x_{S_1}} & \color{red}{+:=} & \overline{x_1}\; \grave{x_1}
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{rcl}
x_1 & = & x_{R_1}\ x_{S_1}\\
& \vdots & \\
x_n & = & x_{R_n}\ x_{S_n}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{rcl}
(\overleftarrow{x_1}, \overline{x_1}) & = & \overleftarrow{x_{R_1}}\ \overleftarrow{x_{S_1}}\\
& \vdots & \\
(\overleftarrow{x_n}, \overline{x_n}) & = & \overleftarrow{x_{R_n}}\ \overleftarrow{x_{S_n}}\\
\grave{x_0} & := & \mathbf{0}\ (\overleftarrow{\mathcal{J}}^{-1}\ \overleftarrow{x_0})\\
& \vdots & \\
\grave{x_{n-1}} & := & \mathbf{0}\ (\overleftarrow{\mathcal{J}}^{-1}\ \overleftarrow{x_{n-1}})\\
\grave{x_{S_n}} & \oplus\!:= & \overline{x_n}\ \grave{x_n}\\
& \vdots & \\
\grave{x_{S_1}} & \oplus\!:= & \overline{x_1}\ \grave{x_1}
\end{array}\right.
$$

The Functional Reverse-Mode Transformation

$$
\left.\begin{array}{l}
\lambda x_0\ \mathbf{let}\ x_1 \triangleq x_{R_1}\ x_{S_1};\\
\qquad\vdots\\
\qquad x_n \triangleq x_{R_n}\ x_{S_n}\\
\quad \mathbf{in}\ x_n\ \mathbf{end}
\end{array}\right\}
\leadsto
\left\{\begin{array}{l}
\lambda \overleftarrow{x_0}\ \mathbf{let}\ (\overleftarrow{x_1}, \overline{x_1}) \triangleq \overleftarrow{x_{R_1}}\ \overleftarrow{x_{S_1}};\\
\qquad\qquad\vdots\\
\qquad\qquad (\overleftarrow{x_n}, \overline{x_n}) \triangleq \overleftarrow{x_{R_n}}\ \overleftarrow{x_{S_n}}\\
\quad \mathbf{in}\ (\overleftarrow{x_n}, (\lambda \grave{x_n}\ \mathbf{let}\ \grave{x_0} \triangleq \mathbf{0}\ (\overleftarrow{\mathcal{J}}^{-1}\ \overleftarrow{x_0});\\
\qquad\qquad\qquad\qquad\vdots\\
\qquad\qquad\qquad\qquad \grave{x_{n-1}} \triangleq \mathbf{0}\ (\overleftarrow{\mathcal{J}}^{-1}\ \overleftarrow{x_{n-1}});\\
\qquad\qquad\qquad\qquad \grave{x_{S_n}} \oplus\!\triangleq \overline{x_n}\ \grave{x_n};\\
\qquad\qquad\qquad\qquad\vdots\\
\qquad\qquad\qquad\qquad \grave{x_{S_1}} \oplus\!\triangleq \overline{x_1}\ \grave{x_1}\\
\qquad\qquad\qquad \mathbf{in}\ \grave{x_0}\ \mathbf{end}))\ \mathbf{end}
\end{array}\right.
$$

$$
\left.\begin{array}{l}
\lambda x_0\ \textbf{let}\ x_1 \triangleq x_{R_1}\ x_{S_1};\\
\qquad\vdots\\
\qquad x_n \triangleq x_{R_n}\ x_{S_n}\\
\quad \textbf{in}\ x_n\ \textbf{end}
\end{array}\right\}
\rightsquigarrow
\left\{\begin{array}{l}
\lambda \overleftarrow{x_0}\ \textbf{let}\ (\overleftarrow{x_1}, \overline{x_1}) \triangleq \overleftarrow{x_{R_1}}\ \overleftarrow{x_{S_1}};\\
\qquad\qquad\vdots\\
\qquad\qquad (\overleftarrow{x_n}, \overline{x_n}) \triangleq \overleftarrow{x_{R_n}}\ \overleftarrow{x_{S_n}}\\
\qquad \textbf{in}\ (\overleftarrow{x_n}, (\lambda\, \grave{\overline{x_n}}\ \textbf{let}\ \grave{\overline{x_0}} \triangleq \mathbf{0}\ (\overleftarrow{\mathcal{J}}^{-1}\ \overleftarrow{x_0});\\
\qquad\qquad\qquad\qquad\vdots\\
\qquad\qquad\qquad\qquad \grave{\overline{x_{n-1}}} \triangleq \mathbf{0}\ (\overleftarrow{\mathcal{J}}^{-1}\ \overleftarrow{x_{n-1}});\\
\qquad\qquad\qquad\qquad \grave{\overline{x_{S_n}}} \oplus\!\triangleq \overline{x_n}\ \grave{\overline{x_n}};\\
\qquad\qquad\qquad\qquad\vdots\\
\qquad\qquad\qquad\qquad \grave{\overline{x_{S_1}}} \oplus\!\triangleq \overline{x_1}\ \grave{\overline{x_1}}\\
\qquad\qquad\qquad \textbf{in}\ \grave{\overline{x_0}}\ \textbf{end}))\ \textbf{end}
\end{array}\right.
$$

The above is a white lie. The truth is a lot more complicated.

Modularity

$$\nabla\ f\ \mathbf{x} \quad \triangleq \quad \frac{\partial f(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_n}$$

Modularity

$$\nabla\ f\ \mathbf{x} \quad \triangleq \quad \frac{\partial f(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_n}$$

$$\textsc{GradientDescent}\ f\ \mathbf{x}_0 \quad \triangleq \quad \ldots \mathbf{x}_{i+1} := \ldots \nabla\ f\ \mathbf{x}_i \ldots$$

Modularity

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$$\text{argmin}\ \ f \quad \triangleq \quad \ldots \textsc{GradientDescent}\ f\ \mathbf{x}_0 \ldots$$

Modularity

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$$\textsc{NeutronFlux}\ r \triangleq \boxed{\textit{classified}}$$

Modularity

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$$\textsc{NeutronFlux}\ r \triangleq \boxed{\textit{classified}}$$

$$\textsc{Deviation}\ r \triangleq ((\textsc{NeutronFlux}\ r) - \textsc{NeutronFlux}_{\text{critical}})^2$$

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$$r^* \quad \triangleq \quad \text{argmin}\ \textsc{Deviation}$$

Fermi, E. (1946). *The Development of the first chain reaction pile*. Proceedings of the American Philosophy Society, **90**:20–4.

Breaking Modularity

$$\nabla\ f\ \mathbf{x} \triangleq (\overrightarrow{f}\ \ \mathbf{x} \rhd \overrightarrow{\mathbf{e}_1}), \ldots, (\overrightarrow{f}\ \ \mathbf{x} \rhd \overrightarrow{\mathbf{e}_n})$$

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Breaking Modularity

$\nabla\ \overrightarrow{f}\ \mathbf{x}$	$\triangleq$	$(\overrightarrow{f}\ \mathbf{x} \triangleright \overleftarrow{\mathbf{e}_1}), \ldots, (\overrightarrow{f}\ \mathbf{x} \triangleright \overleftarrow{\mathbf{e}_n})$
$\textsc{GradientDescent}\ f\ \mathbf{x}_0$	$\triangleq$	$\ldots \mathbf{x}_{i+1} := \ldots \nabla\ \overrightarrow{f}\ \mathbf{x}_i \ldots$
$\text{argmin}\ f$	$\triangleq$	$\ldots \textsc{GradientDescent}\ f\ \mathbf{x}_0 \ldots$
$\textsc{NeutronFlux}\ r$	$\triangleq$	$\boxed{\textit{classified}}$
$\textsc{Deviation}\ r$	$\triangleq$	$((\textsc{NeutronFlux}\ r) - \textsc{NeutronFlux}_{\text{critical}})^2$
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Breaking Modularity

$$\nabla\ \overrightarrow{f}\ \mathbf{x} \quad \stackrel{\triangle}{=} \quad (\overrightarrow{f}\ \mathbf{x} \rhd \overrightarrow{\mathbf{e}_1}), \ldots, (\overrightarrow{f}\ \mathbf{x} \rhd \overrightarrow{\mathbf{e}_n})$$

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$$\textsc{NeutronFlux}\ r \quad \stackrel{\triangle}{=} \quad \boxed{\textit{classified}}$$

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$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \overset{\text{ADIFOR}}{\rightsquigarrow} \overrightarrow{\textsc{NeutronFlux}}$$

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Breaking Modularity

$$\nabla\ \overleftarrow{f}\ \mathbf{x} \quad \triangleq \quad \ldots\ \overleftarrow{f}\ \mathbf{x}\ldots$$

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$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \quad\triangleq\quad \ldots \mathbf{x}_{i+1} := \ldots \nabla\ \overleftarrow{f}\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ \overleftarrow{f} \quad\triangleq\quad \ldots \textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \quad\triangleq\quad \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \quad\stackrel{\text{ADIFOR}}{\rightsquigarrow}\quad \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \quad\triangleq\quad ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \quad\stackrel{\text{ADIFOR}}{\rightsquigarrow}\quad \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \quad\triangleq\quad \text{argmin}\ \overleftarrow{\textsc{Deviation}}$$

Fermi, E. (1946). *The Development of the first chain reaction pile.* Proceedings of the American Philosophy Society, **90**:20–4.

Breaking Modularity

$$\nabla\ \overleftarrow{f}\ \mathbf{x} \quad\triangleq\quad \ldots\ \overleftarrow{f}\ \mathbf{x}\ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \quad\triangleq\quad \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots$$

$$\operatorname{argmin}\ \overleftarrow{f} \quad\triangleq\quad \ldots\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0\ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \quad\triangleq\quad \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \quad\overset{\textsc{Tapenade}}{\rightsquigarrow}\quad \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \quad\triangleq\quad ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \quad\overset{\textsc{Tapenade}}{\rightsquigarrow}\quad \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \quad\triangleq\quad \operatorname{argmin}\ \overleftarrow{\textsc{Deviation}}$$

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Breaking Modularity

$$\nabla \overleftarrow{f}\ \mathbf{x} \triangleq \ldots \overleftarrow{f}\ \mathbf{x} \ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots \mathcal{H}\, f\ \mathbf{x}_i \ldots$$

$$\operatorname{argmin} \overleftarrow{f} \triangleq \ldots \textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \triangleq \operatorname{argmin} \overleftarrow{\textsc{Deviation}}$$

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Breaking Modularity

$$\nabla\ \overleftarrow{f}\ \mathbf{x} \quad \triangleq \quad \ldots\ \overleftarrow{f}\ \mathbf{x}\ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \quad \triangleq \quad \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 \quad \triangleq \quad \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots\mathcal{H}\ f\ \mathbf{x}_i\ldots$$

$$\text{argmin}\ \overleftarrow{f} \quad \triangleq \quad \ldots\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0\ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \quad \triangleq \quad \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \quad \overset{\textsc{Tapenade}}{\rightsquigarrow} \quad \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \quad \triangleq \quad ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \quad \overset{\textsc{Tapenade}}{\rightsquigarrow} \quad \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \quad \triangleq \quad \text{argmin}\ \overleftarrow{\textsc{Deviation}}$$

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Breaking Modularity

$$\nabla \overleftarrow{f}\ \mathbf{x} \triangleq \ldots \overleftarrow{f}\ \mathbf{x} \ldots$$

$$\mathcal{H}\, f\ \mathbf{x} \triangleq$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots \mathcal{H}\, f\ \mathbf{x}_i \ldots$$

$$\operatorname{argmin} \overleftarrow{f} \triangleq \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \triangleq \operatorname{argmin} \overleftarrow{\textsc{Deviation}}$$

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Breaking Modularity

$$\nabla\ \overleftarrow{f}\ \mathbf{x} \quad \triangleq \quad \ldots\ \overleftarrow{f}\ \mathbf{x}\ldots$$

$$\mathcal{H}\ f\ \mathbf{x} \quad \triangleq \quad \ldots\ \overleftrightarrow{f}\ \ldots\mathbf{x}\ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \quad \triangleq \quad \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 \quad \triangleq \quad \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots\mathcal{H}\ f\ \mathbf{x}_i\ldots$$

$$\operatorname{argmin}\ \overleftarrow{f} \quad \triangleq \quad \ldots\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0\ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \quad \triangleq \quad \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \quad \overset{\textsc{Tapenade}}{\rightsquigarrow} \quad \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \quad \triangleq \quad ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \quad \overset{\textsc{Tapenade}}{\rightsquigarrow} \quad \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \quad \triangleq \quad \operatorname{argmin}\ \overleftarrow{\textsc{Deviation}}$$

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Breaking Modularity

$$\nabla \overleftarrow{f}\ \mathbf{x} \triangleq \ldots \overleftarrow{f}\ \mathbf{x} \ldots$$

$$\mathcal{H} \overleftrightarrow{f}\ \mathbf{x} \triangleq \ldots \overleftrightarrow{f}\ \ldots \mathbf{x} \ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots \mathcal{H}\, f\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ \overleftarrow{f} \triangleq \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \triangleq \text{argmin}\ \overleftarrow{\textsc{Deviation}}$$

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Breaking Modularity

$$
\begin{aligned}
&\nabla\ \overleftarrow{f}\ \mathbf{x} &&\triangleq&& \ldots\ \overleftarrow{f}\ \mathbf{x}\ldots\\
&\mathcal{H}\ \overset{\rightleftarrows}{f}\ \mathbf{x} &&\triangleq&& \ldots\ \overset{\rightleftarrows}{f}\ \ldots\mathbf{x}\ldots\\
&\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 &&\triangleq&& \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots\\
&\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 &&\triangleq&& \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots\mathcal{H}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_i\ldots\\
&\text{argmin}\ \overleftarrow{f} &&\triangleq&& \ldots\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0\ldots\\
&\textsc{NeutronFlux}\ \mathbf{r} &&\triangleq&& \boxed{\textit{classified}}\\
&\textsc{NeutronFlux} &&\overset{\textsc{Tapenade}}{\rightsquigarrow}&& \overleftarrow{\textsc{NeutronFlux}}\\
\\
&\textsc{Deviation}\ \mathbf{r} &&\triangleq&& ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2\\
&\textsc{Deviation} &&\overset{\textsc{Tapenade}}{\rightsquigarrow}&& \overleftarrow{\textsc{Deviation}}\\
\\
&\mathbf{r}^* &&\triangleq&& \text{argmin}\ \overleftarrow{\textsc{Deviation}}
\end{aligned}
$$

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Breaking Modularity

$$
\begin{array}{lcl}
\nabla\ \overleftarrow{f}\ \mathbf{x} & \triangleq & \ldots\ \overleftarrow{f}\ \mathbf{x}\ldots \\[1em]
\mathcal{H}\ \overset{\rightleftarrows}{f}\ \mathbf{x} & \triangleq & \ldots\ \overset{\rightleftarrows}{f}\ \ldots\mathbf{x}\ldots \\
\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 & \triangleq & \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots \\
\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_0 & \triangleq & \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots\mathcal{H}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_i\ldots \\
\mathrm{argmin}\ \overleftarrow{f} & \triangleq & \ldots\textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0\ldots \\
\textsc{NeutronFlux}\ \mathbf{r} & \triangleq & \boxed{\textit{classified}} \\
\textsc{NeutronFlux} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overleftarrow{\textsc{NeutronFlux}} \\[2em]
\textsc{Deviation}\ \mathbf{r} & \triangleq & ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2 \\
\textsc{Deviation} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overleftarrow{\textsc{Deviation}} \\[2em]
\mathbf{r}^* & \triangleq & \mathrm{argmin}\ \overleftarrow{\textsc{Deviation}}
\end{array}
$$

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Breaking Modularity

$$\nabla \overleftarrow{f}\ \mathbf{x} \triangleq \ldots \overleftarrow{f}\ \mathbf{x} \ldots$$

$$\mathcal{H} \overleftrightarrows{f}\ \mathbf{x} \triangleq \ldots \overleftrightarrows{f} \ldots \mathbf{x} \ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrows{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots \mathcal{H} \overleftrightarrows{f}\ \mathbf{x}_i \ldots$$

$$\operatorname{argmin} \overleftarrow{f} \triangleq \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrows{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \triangleq \operatorname{argmin} \overleftarrow{\textsc{Deviation}}$$

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Breaking Modularity

$$\nabla\ \overleftarrow{f}\ \mathbf{x} \quad \triangleq \quad \ldots\ \overleftarrow{f}\ \mathbf{x}\ldots$$

$$\mathcal{H}\ \overleftrightarrow{f}\ \mathbf{x} \quad \triangleq \quad \ldots\ \overleftrightarrow{f}\ \ldots\mathbf{x}\ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \quad \triangleq \quad \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrow{f}\ \mathbf{x}_0 \quad \triangleq \quad \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots\mathcal{H}\ \overleftrightarrow{f}\ \mathbf{x}_i\ldots$$

$$\text{argmin}\ \overleftarrow{f}\ \overleftrightarrow{f} \quad \triangleq \quad \ldots\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrow{f}\ \mathbf{x}_0\ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \quad \triangleq \quad \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \quad \overset{\textsc{Tapenade}}{\rightsquigarrow} \quad \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \quad \triangleq \quad ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \quad \overset{\textsc{Tapenade}}{\rightsquigarrow} \quad \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \quad \triangleq \quad \text{argmin}\ \overleftarrow{\textsc{Deviation}}$$

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Breaking Modularity

$$\nabla\ \overleftarrow{f}\ \mathbf{x} \quad \triangleq \quad \ldots\ \overleftarrow{f}\ \mathbf{x} \ldots$$

$$\mathcal{H}\ \overset{\rightleftarrows}{f}\ \mathbf{x} \quad \triangleq \quad \ldots\ \overset{\rightleftarrows}{f}\ \ldots \mathbf{x} \ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \quad \triangleq \quad \ldots \mathbf{x}_{i+1} := \ldots \nabla\ \overleftarrow{f}\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_0 \quad \triangleq \quad \ldots \mathbf{x}_{i+1} := \ldots \nabla\ \overleftarrow{f}\ \mathbf{x}_i \ldots \mathcal{H}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ \overleftarrow{f}\ \overset{\rightleftarrows}{f} \quad \triangleq \quad \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \quad \triangleq \quad \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \quad \overset{\textsc{Tapenade}}{\rightsquigarrow} \quad \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \quad \triangleq \quad ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \quad \overset{\textsc{Tapenade}}{\rightsquigarrow} \quad \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \quad \triangleq \quad \text{argmin}\ \overleftarrow{\textsc{Deviation}}\ \overset{\rightleftarrows}{\textsc{Deviation}}$$

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Breaking Modularity

$$
\begin{array}{lcl}
\nabla\ \overleftarrow{f}\ \mathbf{x} & \triangleq & \ldots\ \overleftarrow{f}\ \mathbf{x}\ldots \\
\mathcal{H}\ \overset{\rightleftarrows}{f}\ \mathbf{x} & \triangleq & \ldots\ \overset{\rightleftarrows}{f}\ \ldots\mathbf{x}\ldots \\
\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 & \triangleq & \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots \\
\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_0 & \triangleq & \ldots\mathbf{x}_{i+1} := \ldots\nabla\ \overleftarrow{f}\ \mathbf{x}_i\ldots\mathcal{H}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_i\ldots \\
\mathrm{argmin}\ \overleftarrow{f}\ \overset{\rightleftarrows}{f} & \triangleq & \ldots\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overset{\rightleftarrows}{f}\ \mathbf{x}_0\ldots \\
\textsc{NeutronFlux}\ \mathbf{r} & \triangleq & \boxed{\textit{classified}} \\
\textsc{NeutronFlux} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overleftarrow{\textsc{NeutronFlux}} \\
\overleftarrow{\textsc{NeutronFlux}} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overset{\rightleftarrows}{\textsc{NeutronFlux}} \\
\textsc{Deviation}\ \mathbf{r} & \triangleq & ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2 \\
\textsc{Deviation} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overleftarrow{\textsc{Deviation}} \\
\overleftarrow{\textsc{Deviation}} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overset{\rightleftarrows}{\textsc{Deviation}} \\
\mathbf{r}^* & \triangleq & \mathrm{argmin}\ \overleftarrow{\textsc{Deviation}}\ \overset{\rightleftarrows}{\textsc{Deviation}}
\end{array}
$$

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Restoring Modularity

$$\nabla\ f\ \mathbf{x} \quad \triangleq$$

$$\mathcal{H}\ f\ \mathbf{x} \quad \triangleq$$

$$\textsc{GradientDescent}\ f\ \mathbf{x}_0 \quad \triangleq \quad \ldots \mathbf{x}_{i+1} := \ldots \nabla\ f\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ f\ \mathbf{x}_0 \quad \triangleq \quad \ldots \mathbf{x}_{i+1} := \ldots \nabla\ f\ \mathbf{x}_i \ldots \mathcal{H}\ f\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ f \quad \triangleq \quad \ldots \textsc{GradientDescent}\ f\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \quad \triangleq \quad \boxed{\textit{classified}}$$

$$\textsc{Deviation}\ \mathbf{r} \quad \triangleq \quad ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\mathbf{r}^* \quad \triangleq \quad \text{argmin}\ \textsc{Deviation}$$

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Restoring Modularity

$$\nabla\ f\ \mathbf{x} \triangleq ((\overrightarrow{\mathcal{J}}\ f)\ \mathbf{x} \rhd \overrightarrow{\mathbf{e}_1}), \ldots, ((\overrightarrow{\mathcal{J}}\ f)\ \mathbf{x} \rhd \overrightarrow{\mathbf{e}_n})$$

$$\mathcal{H}\ f\ \mathbf{x} \triangleq$$

$$\textsc{GradientDescent}\ f\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla\ f\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ f\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla\ f\ \mathbf{x}_i \ldots \mathcal{H}\ f\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ f \triangleq \ldots \textsc{GradientDescent}\ f\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\mathbf{r}^* \triangleq \text{argmin}\ \textsc{Deviation}$$

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Restoring Modularity

$$\nabla\ f\ \mathbf{x} \triangleq \ldots(\overleftarrow{\mathcal{J}}\ f)\ \mathbf{x}\ldots$$

$$\mathcal{H}\ f\ \mathbf{x} \triangleq$$

$$\textsc{GradientDescent}\ f\ \mathbf{x}_0 \triangleq \ldots\mathbf{x}_{i+1} := \ldots\nabla\ f\ \mathbf{x}_i\ldots$$

$$\textsc{NewtonsMethod}\ f\ \mathbf{x}_0 \triangleq \ldots\mathbf{x}_{i+1} := \ldots\nabla\ f\ \mathbf{x}_i\ldots\mathcal{H}\ f\ \mathbf{x}_i\ldots$$

$$\operatorname{argmin}\ f \triangleq \ldots\textsc{GradientDescent}\ f\ \mathbf{x}_0\ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\mathbf{r}^* \triangleq \operatorname{argmin}\ \textsc{Deviation}$$

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Restoring Modularity

$\nabla\ f\ \mathbf{x}$	$\triangleq$	$\ldots(\overleftarrow{\mathcal{J}}\ f)\ \mathbf{x}\ldots$
$\mathcal{H}\ f\ \mathbf{x}$	$\triangleq$	$\ldots(\overrightarrow{\mathcal{J}}\ (\overleftarrow{\mathcal{J}}\ f))\ldots\mathbf{x}\ldots$
$\textsc{GradientDescent}\ f\ \mathbf{x}_0$	$\triangleq$	$\ldots\mathbf{x}_{i+1} := \ldots\nabla\ f\ \mathbf{x}_i\ldots$
$\textsc{NewtonsMethod}\ f\ \mathbf{x}_0$	$\triangleq$	$\ldots\mathbf{x}_{i+1} := \ldots\nabla\ f\ \mathbf{x}_i\ldots\mathcal{H}\ f\ \mathbf{x}_i\ldots$
$\operatorname{argmin}\ f$	$\triangleq$	$\ldots\textsc{NewtonsMethod}\ f\ \mathbf{x}_0\ldots$
$\textsc{NeutronFlux}\ \mathbf{r}$	$\triangleq$	$\boxed{\textit{classified}}$
$\textsc{Deviation}\ \mathbf{r}$	$\triangleq$	$((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	$\operatorname{argmin}\ \textsc{Deviation}$

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Having your cake and eating it too

- Convenient
- Fast

Having your cake and eating it too

- Convenient
  - $\mathcal{D}$ formulated as a higher-order function in the language
- Fast

Having your cake and eating it too

- Convenient
  - $\mathcal{D}$ formulated as a higher-order function in the language
  - no arbitrary restrictions
    applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
- Fast

Having your cake and eating it too

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  - $\mathcal{D}$ formulated as a higher-order function in the language
  - no arbitrary restrictions
    applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
  - higher-order derivatives
    `(`$\mathcal{D}$ `(`$\mathcal{D}$ `f))`

- Fast

Having your cake and eating it too

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  - $\mathcal{D}$ formulated as a higher-order function in the language
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    applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
  - higher-order derivatives
    `(`$\mathcal{D}$` (`$\mathcal{D}$` f))`
  - nesting
    `(`$\mathcal{D}$` (lambda (...) ... (`$\mathcal{D}$` (lambda (...) ...)) ...))`
- Fast

Having your cake and eating it too

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  - $\mathcal{D}$ formulated as a higher-order function in the language
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    applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
  - higher-order derivatives
    `(`$\mathcal{D}$ `(`$\mathcal{D}$ `f))`
  - nesting
    `(`$\mathcal{D}$ `(lambda (...) ... (`$\mathcal{D}$ `(lambda (...) ...)) ...))`
- Fast
  - $\mathcal{D}$ implemented by reflective transformation of environments and code associated with closures

Having your cake and eating it too

- Convenient
  - $\mathcal{D}$ formulated as a higher-order function in the language
  - no arbitrary restrictions
    applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
  - higher-order derivatives
    ($\mathcal{D}$ ($\mathcal{D}$ `f`))
  - nesting
    ($\mathcal{D}$ `(lambda (...) ...` ($\mathcal{D}$ `(lambda (...) ...)) ...))`
- Fast
  - $\mathcal{D}$ implemented by reflective transformation of environments and code associated with closures
  - compile away reflection with partial evaluation implemented by flow analysis

Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f)
 ...)
```

Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f)
 ...)

(𝒟 (lambda (x) 2x³))
```

Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3)$)
 ...)

($\mathcal{D}$ (lambda (x) $2x^3$))

Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3)$)
 ...:$(\lambda x\ 6x^2)$)

($\mathcal{D}$ (lambda (x) $2x^3$))

Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\; 2x^3)$)
 ...:$(\lambda x\; 6x^2)$)

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\; 6x^2)$

Monovariant Flow Analysis: 0-CFA

`(define (`$\mathcal{D}$` f:`$(\lambda x\ 2x^3)$`)`
` `$\ldots$`:`$(\lambda x\ 6x^2)$`)`

`(`$\mathcal{D}$` (lambda (x) `$2x^3$`)):`$(\lambda x\ 6x^2)$

`(`$\mathcal{D}$` (lambda (x) `$3x^4$`))`

Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$)
 ...:$(\lambda x\ 6x^2)$)

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$

($\mathcal{D}$ (lambda (x) $3x^4$))

Monovariant Flow Analysis: 0-CFA

`(define (`$\mathcal{D}$ `f:`$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$`)`
`...:`$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$`)`

`(`$\mathcal{D}$ `(lambda (x)` $2x^3$`)):`$(\lambda x\ 6x^2)$

`(`$\mathcal{D}$ `(lambda (x)` $3x^4$`))`

Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f:(λx 2x³) ∪ (λx 3x⁴))
  ...:(λx 6x²) ∪ (λx 12x³))
```

$(\text{define } (\mathcal{D}\ \texttt{f}{:}(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4))$
$\ldots{:}(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3))$

$(\mathcal{D}\ (\texttt{lambda}\ (\texttt{x})\ 2x^3)){:}(\lambda x\ 6x^2)$

$(\mathcal{D}\ (\texttt{lambda}\ (\texttt{x})\ 3x^4)){:}(\lambda x\ 12x^3)$

Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$)
 ...:$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$)

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$

($\mathcal{D}$ (lambda (x) $3x^4$)):$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$

Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$)
 ...:$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$))

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$

($\mathcal{D}$ (lambda (x) $3x^4$)):$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$

Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f)
 ...)
```

Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f)
 ...)

(𝒟 (𝒟 (lambda (x) e^{2x})))
```

Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f:(λx e^{2x}))
 ...)

(𝒟 (𝒟 (lambda (x) e^{2x})))
```

Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ e^{2x})$)
 $\ldots$:$(\lambda x\ 2e^{2x})$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) e^{2x})))

Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f:(λx e^{2x}))
 ...:(λx 2e^{2x}))

(𝒟 (𝒟 (lambda (x) e^{2x})):(λx 2e^{2x}))
```

(define ($\mathcal{D}$ f:$(\lambda x\; e^{2x})$)
 $\ldots$:$(\lambda x\; 2e^{2x})$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) e^{2x})):$(\lambda x\; 2e^{2x})$)

Monovariant Flow Analysis: 0-CFA

`(define (`$\mathcal{D}$` f:`$(\lambda x\; e^{2x}) \cup (\lambda x\; 2e^{2x})$`)`
` `$\ldots$`:`$(\lambda x\; 2e^{2x})$`)`

`(`$\mathcal{D}$` (`$\mathcal{D}$` (lambda (x) `e^{2x}`)):`$(\lambda x\; 2e^{2x})$`)`

Monovariant Flow Analysis: 0-CFA

`(define (`$\mathcal{D}$` f:`$(\lambda x\, e^{2x}) \cup (\lambda x\, 2e^{2x})$`)`
` ...:`$(\lambda x\, 2e^{2x}) \cup (\lambda x\, 4e^{2x})$`)`

`(`$\mathcal{D}$` (`$\mathcal{D}$` (lambda (x) `e^{2x}`)):`$(\lambda x\, 2e^{2x})$`)`

Monovariant Flow Analysis: 0-CFA

$\texttt{(define (}\mathcal{D}\texttt{ f:}(\lambda x\, e^{2x}) \cup (\lambda x\, 2e^{2x})\texttt{)}$
$\quad \texttt{...:}(\lambda x\, 2e^{2x}) \cup (\lambda x\, 4e^{2x})\texttt{)}$

$\texttt{(}\mathcal{D}\texttt{ (}\mathcal{D}\texttt{ (lambda (x) } e^{2x}\texttt{)):}(\lambda x\, 2e^{2x}) \cup (\lambda x\, 4e^{2x})\texttt{)}$

Monovariant Flow Analysis: 0-CFA

$\texttt{(define (}\mathcal{D}\texttt{ f:}(\lambda x\, e^{2x}) \cup (\lambda x\, 2e^{2x}) \cup (\lambda x\, 4e^{2x})\texttt{)}$
$\quad \ldots\texttt{:}(\lambda x\, 2e^{2x}) \cup (\lambda x\, 4e^{2x})\texttt{)}$

$\texttt{(}\mathcal{D}\texttt{ (}\mathcal{D}\texttt{ (lambda (x) } e^{2x}\texttt{)):}(\lambda x\, 2e^{2x}) \cup (\lambda x\, 4e^{2x})\texttt{)}$

Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\; e^{2x}) \cup (\lambda x\; 2e^{2x}) \cup (\lambda x\; 4e^{2x}) \cup \ldots$)
 $\ldots$:$(\lambda x\; 2e^{2x}) \cup (\lambda x\; 4e^{2x}) \cup \ldots$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) e^{2x}))):$(\lambda x\; 2e^{2x}) \cup (\lambda x\; 4e^{2x}) \cup \ldots$)

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

```
(define (𝒟 f) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages **or** Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: *k*-CFA

with Bounded Context Sensitivity

```
(define (𝒟 f) ...)
```

```
(define (g ...) ... (𝒟 (lambda (x) 2x³)) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}_{\texttt{g}}$ f) ...)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages **or** Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}_{\texttt{g}}$ f:$(\lambda x\, 2x^3)$) ...)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}_{\texttt{g}}$ f:$(\lambda x\ 2x^3)$) ...:$(\lambda x\ 6x^2)$)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

```
(define (𝒟g f:(λx 2x³)) ...:(λx 6x²))
```

$$\texttt{(define (}\mathcal{D}_{\texttt{g}}\ \texttt{f:}(\lambda x\ 2x^3)\texttt{)}\ \ldots\texttt{:}(\lambda x\ 6x^2)\texttt{)}$$

$$\texttt{(define (g}\ \ldots\texttt{)}\ \ldots\ \texttt{(}\mathcal{D}\ \texttt{(lambda (x)}\ 2x^3\texttt{)):}(\lambda x\ 6x^2)\ \ldots\texttt{)}$$

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

`(define (`$\mathcal{D}_{\mathtt{g}}$` f:`$(\lambda x\, 2x^3)$`) ...:`$(\lambda x\, 6x^2)$`)`

`(define (g ...) ... (`$\mathcal{D}$` (lambda (x) `$2x^3$`)):`$(\lambda x\, 6x^2)$` ...)`

`(define (h ...) ... (`$\mathcal{D}$` (lambda (x) `$3x^4$`)) ...)`

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}$g f:$(\lambda x\ 2x^3)$) ...:$(\lambda x\ 6x^2)$)

(define ($\mathcal{D}$h f) ...)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$ ...)

(define (h ...) ... ($\mathcal{D}$ (lambda (x) $3x^4$)) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}_{\texttt{g}}$ f:$(\lambda x\ 2x^3)$) ...:$(\lambda x\ 6x^2)$)

(define ($\mathcal{D}_{\texttt{h}}$ f:$(\lambda x\ 3x^4)$) ...)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$ ...)

(define (h ...) ... ($\mathcal{D}$ (lambda (x) $3x^4$)) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages **or** Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}_\mathtt{g}$ f:$(\lambda x\, 2x^3)$) ...:$(\lambda x\, 6x^2)$)

(define ($\mathcal{D}_\mathtt{h}$ f:$(\lambda x\, 3x^4)$) ...:$(\lambda x\, 12x^3)$)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\, 6x^2)$ ...)

(define (h ...) ... ($\mathcal{D}$ (lambda (x) $3x^4$)) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k-CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}_{\mathtt{g}}$ f:$(\lambda x\; 2x^3)$) ...:$(\lambda x\; 6x^2)$)

(define ($\mathcal{D}_{\mathtt{h}}$ f:$(\lambda x\; 3x^4)$) ...:$(\lambda x\; 12x^3)$)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\; 6x^2)$ ...)

(define (h ...) ... ($\mathcal{D}$ (lambda (x) $3x^4$)):$(\lambda x\; 12x^3)$ ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: *k*-CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
 (if (zero? n) x ((compose (- n 1) f) (f x))))
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: *k*-CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
 (if (zero? n) x ((compose (- n 1) f) (f x))))

((compose k 𝒟) g)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages **or** Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: *k*-CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
 (if (zero? n) x ((compose (- n 1) f) (f x))))
```

`((compose k` $\mathcal{D}$`) g)`

`(define (`$\mathcal{D}_{\texttt{compose}}$ `f:`g`) ...)`

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages **or** Taming Lambda*. Ph.D. thesis, CMU.

```
(define ((compose n f) x)
 (if (zero? n) x ((compose (- n 1) f) (f x))))
```

((compose k $\mathcal{D}$) g)

(define ($\mathcal{D}_{\texttt{compose}}$ f:g) ...)

(define ($\mathcal{D}_{\texttt{compose:compose}}$ f:g') ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages **or** Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: *k*-CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
 (if (zero? n) x ((compose (- n 1) f) (f x))))
```

$\texttt{((compose k } \mathcal{D}\texttt{) g)}$

$\texttt{(define (}\mathcal{D}_{\texttt{compose}}\ \texttt{f:}g\texttt{) }\ldots\texttt{)}$

$\texttt{(define (}\mathcal{D}_{\texttt{compose:compose}}\ \texttt{f:}g'\texttt{) }\ldots\texttt{)}$

$\texttt{(define (}\mathcal{D}_{\texttt{compose:compose:compose}}\ \texttt{f:}g''\texttt{) }\ldots\texttt{)}$

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages **or** Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: *k*-CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
 (if (zero? n) x ((compose (- n 1) f) (f x))))
```

(compose k $\mathcal{D}$) g)

(define ($\mathcal{D}_{\texttt{compose}}$ f:g) . . .)

(define ($\mathcal{D}_{\texttt{compose:compose}}$ f:g') . . .)

(define ($\mathcal{D}_{\texttt{compose:compose:compose}}$ f:g'') . . .)

$\vdots$

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\overline{\sigma}$.

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\overline{\sigma}$.
Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\overline{\sigma}$.
Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.
Is suitable for FORTRAN-like programs.

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\overline{\sigma}$.
Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.
Is suitable for FORTRAN-like programs.
Necessary for migrating reflective source-code transformation to compile time.

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\overline{\sigma}$.
Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.
Is suitable for FORTRAN-like programs.
Necessary for migrating reflective source-code transformation to compile time.
Side benefit: union-free

Polyvariant Flow Analysis
with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\overline{\sigma}$.
Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.
Is suitable for FORTRAN-like programs.
Necessary for migrating reflective source-code transformation to compile time.
Side benefit: union-free
No tags, tag checking, tag dispatching, indirect calls

Polyvariant Flow Analysis

with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\overline{\sigma}$.
Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.
Is suitable for FORTRAN-like programs.
Necessary for migrating reflective source-code transformation to compile time.
Side benefits: union-free, no cyclic abstract values
No tags, tag checking, tag dispatching, indirect calls

$$\mathcal{E} : e \times \sigma \rightarrow v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

$$\overline{\mathcal{E}} : e \times \overline{\sigma} \rightarrow \overline{v}$$

$$\overline{v} ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (\overline{v_1}, \overline{v_2}) \mid \langle \overline{\sigma}, e \rangle \mid \overline{\mathbb{R}}$$

$$\overline{\sigma} ::= \{x_1 \mapsto \overline{v_1}, \ldots\}$$

Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) e and $\overline{\sigma}$.
Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.
*I*s suitable for FORTRAN-like programs.
Necessary for migrating reflective source-code transformation to compile time.
Side benefits: union-free, no cyclic abstract values
No tags, tag checking, tag dispatching, indirect calls
Allows complete unboxing: no allocation, reclamation, indirection

Game Theory

$$
\begin{array}{cc|ccccc}
 & & & & B & & \\
 & & b_1 & \ldots & b_j & \ldots & b_n \\
\hline
 & a_1 & & & & & \\
 & \vdots & & \ddots & \vdots & & \\
A & a_i & & \ldots & \textsc{Payoff}(a_i, b_j) & \ldots & \\
 & \vdots & & & \vdots & \ddots & \\
 & a_m & & & & &
\end{array}
$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Game Theory

$$
\begin{array}{cc|ccccc}
 & & & & B & & \\
 & & b_1 & \ldots & b_j & \ldots & b_n \\
\hline
 & a_1 & & & & & \\
 & \vdots & & \ddots & \vdots & & \\
A & a_i & & \ldots & \textsc{Payoff}(a_i, b_j) & \ldots & \\
 & \vdots & & & \vdots & \ddots & \\
 & a_m & & & & &
\end{array}
$$

$$\max_{a \in A} \min_{b \in B} \textsc{Payoff}(a, b)$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

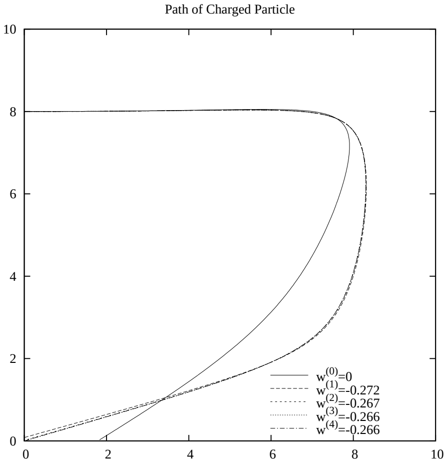
Game Theory

$$
\begin{array}{cc|ccc}
 & & & \mathbb{R}^n & \\
 & & \ldots & \mathbf{b} & \ldots \\
\hline
 & \vdots & \ddots & \vdots & \\
\mathbb{R}^m & \mathbf{a} & \ldots & \textsc{Payoff}(\mathbf{a}, \mathbf{b}) & \ldots \\
 & \vdots & & \vdots & \ddots
\end{array}
$$

$$\max_{\mathbf{a} \in \mathbb{R}^m} \min_{\mathbf{b} \in \mathbb{R}^n} \textsc{Payoff}(\mathbf{a}, \mathbf{b})$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Cathode Ray Tubes



Sprague, C. S. and George, R. H. (1939). *Cathode Ray Deflecting Electrode*. US Patent 2,161,437.

George, R. H. (1940). *Cathode Ray Tube*. US Patent 2,222,942.

Performance Comparison

	saddle				particle			
	FF	FR	RF	RR	FF	FR	RF	RR
STALIN∇	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
IKARUS	150.12	194.49	22.35	33.17	197.32	331.95	243.78	329.86
STALIN	162.20	202.23	26.15	36.54	222.87	380.89	212.39	269.96
SCHEME->C	298.53	312.39	41.52	54.33	394.65	583.13	424.48	537.91
CHICKEN	847.28	1088.64	124.56	203.88	1117.53	2030.66	1565.15	5291.80
BIGLOO	385.67	394.14	55.89	69.72	541.27	764.74	431.03	607.71
GAMBIT	349.43	404.47	49.70	71.16	435.53	748.82	532.35	712.63
LARCENY	537.53	543.88	66.47	85.35	696.14	1163.91	564.50	747.98
MZC	2234.68	2456.97	319.20	428.20	2765.03	3406.98	1311.92	1843.73
MZSCHEME	1874.02	2229.77	273.29	397.18	2314.17	4059.10	1651.69	2416.09
SCMUTILS	1141.06	1351.11	156.71	236.22	1428.85	2354.06	1163.29	1660.56
MLTON	42.42				69.48			
SML/NJ	63.20				84.59			
OCAML	79.65				121.60			
GHC	117.74				157.47			
FADBAD++	21.64				74.01			
ADIFOR	1.96				1.46			
TAPENADE	2.47				3.27			

Gradient-Based Optimization

```
(define (e i n)
 (if (zero? n)
     '()
     (cons (if (zero? i) 1.0 0.0)
           (e (- i 1) (- n 1)))))
```

Gradient-Based Optimization

```
(define (e i n)
 (if (zero? n)
     '()
     (cons (if (zero? i) 1.0 0.0)
           (e (- i 1) (- n 1)))))

(define ((gradient f) x)
 (let ((n (length x)))
  (map (lambda (i) (tangent ((j* f) (bundle x (e i n)))))
       (iota n))))
```

Gradient-Based Optimization

```
(define (e i n)
 (if (zero? n)
     '()
     (cons (if (zero? i) 1.0 0.0)
           (e (- i 1) (- n 1)))))

(define ((gradient f) x)
 (let ((n (length x)))
  (map (lambda (i) (tangent ((j* f) (bundle x (e i n)))))
       (iota n))))

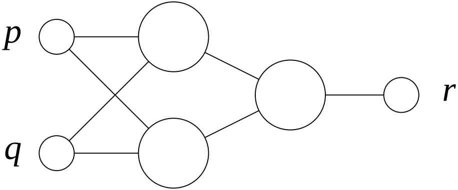
(define (gradient-ascent f x0 n eta)
 (if (zero? n)
     (list x0 (f x0) ((gradient f) x0))
     (gradient-ascent f
                      (zip (lambda (xi gi) (+ xi (* eta gi)))
                           x0
                           ((gradient f) x0))
                      (- n 1)
                      eta)))
```

Gradient-Based Optimization

```
(define ((gradient f) x) (cdr ((cdr ((*j f) (*j x))) 1.0)))
```

```
(define (gradient-ascent f x0 n eta)
 (if (zero? n)
     (list x0 (f x0) ((gradient f) x0))
     (gradient-ascent f
                      (zip (lambda (xi gi) (+ xi (* eta gi)))
                           x0
                           ((gradient f) x0))
                      (- n 1)
                      eta)))
```

Neural Networks



p	q	r
0	0	0
0	1	1
1	0	1
1	1	0

Rumelhart, D. E., Hinton, G. E., and Williams, R. J. (1986). *Learning representations by back-propagating errors*. Nature, **323**:533–6.

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))

(define ((forward-pass ws-layers) in)
 (if (null? ws-layers)
     in
     ((forward-pass (cdr ws-layers))
      (map sigmoid (sum-layer in (car ws-layers))))))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))

(define ((forward-pass ws-layers) in)
 (if (null? ws-layers)
     in
     ((forward-pass (cdr ws-layers))
      (map sigmoid (sum-layer in (car ws-layers))))))

(define ((error-on-dataset dataset) ws-layers)
 ((fold + 0)
  (map (lambda ((list in target))
        (* 0.5 (magnitude-squared (v- ((forward-pass ws-layers) in) target))))
       dataset)))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))

(define ((forward-pass ws-layers) in)
 (if (null? ws-layers)
     in
     ((forward-pass (cdr ws-layers))
      (map sigmoid (sum-layer in (car ws-layers))))))

(define ((error-on-dataset dataset) ws-layers)
 ((fold + 0)
  (map (lambda ((list in target))
        (* 0.5 (magnitude-squared (v- ((forward-pass ws-layers) in) target))))
       dataset)))

(gradient-descent (error-on-dataset '((((0 0) (0))
                                       ((0 1) (1))
                                       ((1 0) (1))
                                       ((1 1) (0))))
                  '((((0 -0.284227 1.16054) (0 0.617194 1.30467))
                    ((0 -0.084395 0.648461)))
                  1000.0
                  0.3)
```

Performance Comparison

	forward scalar	forward vector	reverse
STALIN∇	1.39		1.00
FADBAD++	103.65	35.38	52.40
ADOL–C	12.80	4.07	32.55
CPPAD	44.10		22.20
ADIC	17.50	4.13	
ADIFOR	12.38	2.79	
TAPENADE	11.86	4.54	5.80

Probabilistic Lambda Calculus

$P = \textbf{if } x_0 \textbf{ then } 0 \textbf{ else if } x_1 \textbf{ then } 1 \textbf{ else } 2$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

$P = \textbf{if } x_0 \textbf{ then } 0 \textbf{ else if } x_1 \textbf{ then } 1 \textbf{ else } 2$

$\Pr(x_0 \mapsto \textbf{true}) = p_0$ $\quad$ $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$

$\Pr(x_1 \mapsto \textbf{true}) = p_1$ $\quad$ $\Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

$P = \textbf{if } x_0 \textbf{ then } 0 \textbf{ else if } x_1 \textbf{ then } 1 \textbf{ else } 2$

$\Pr(x_0 \mapsto \textbf{true}) = p_0$ $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$
$\Pr(x_1 \mapsto \textbf{true}) = p_1$ $\Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$

$$\Pr(\mathcal{E}(P) = 0 | p_0, p_1) = p_0$$
$$\Pr(\mathcal{E}(P) = 1 | p_0, p_1) = (1 - p_0)p_1$$
$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

$P = \textbf{if } x_0 \textbf{ then } 0 \textbf{ else if } x_1 \textbf{ then } 1 \textbf{ else } 2$

$\Pr(x_0 \mapsto \textbf{true}) = p_0 \qquad \Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$

$\Pr(x_1 \mapsto \textbf{true}) = p_1 \qquad \Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$

$$\Pr(\mathcal{E}(P) = 0|p_0, p_1) = p_0$$
$$\Pr(\mathcal{E}(P) = 1|p_0, p_1) = (1 - p_0)p_1$$
$$\Pr(\mathcal{E}(P) = 2|p_0, p_1) = (1 - p_0)(1 - p_1)$$

$$\prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v|p_0, p_1) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

$P = \textbf{if } x_0 \textbf{ then } 0 \textbf{ else if } x_1 \textbf{ then } 1 \textbf{ else } 2$

$\Pr(x_0 \mapsto \textbf{true}) = p_0$ $\qquad$ $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$

$\Pr(x_1 \mapsto \textbf{true}) = p_1$ $\qquad$ $\Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$

$$\Pr(\mathcal{E}(P) = 0|p_0, p_1) = p_0$$
$$\Pr(\mathcal{E}(P) = 1|p_0, p_1) = (1 - p_0)p_1$$
$$\Pr(\mathcal{E}(P) = 2|p_0, p_1) = (1 - p_0)(1 - p_1)$$

$$\prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v|p_0, p_1) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

$$\operatorname*{argmax}_{p_0, p_1} \prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v|p_0, p_1) = \left\langle \frac{1}{4}, \frac{1}{3} \right\rangle$$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Prolog

```
p(0).
p(X):-q(X).
q(1).
q(2).
```

Probabilistic Prolog

$\Pr(\texttt{p(0).}) = p_0$
$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$
$\Pr(\texttt{q(1).}) = p_1$
$\Pr(\texttt{q(2).}) = 1 - p_1$

Probabilistic Prolog

$\Pr(\texttt{p(0).}) = p_0$
$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$
$\Pr(\texttt{q(1).}) = p_1$
$\Pr(\texttt{q(2).}) = 1 - p_1$

$\Pr(\texttt{?-p(0).}) = p_0$
$\Pr(\texttt{?-p(1).}) = (1 - p_0)p_1$
$\Pr(\texttt{?-p(2).}) = (1 - p_0)(1 - p_1)$

Probabilistic Prolog

$$\Pr(\texttt{p(0).}) = p_0$$
$$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$$
$$\Pr(\texttt{q(1).}) = p_1$$
$$\Pr(\texttt{q(2).}) = 1 - p_1$$

$$\Pr(\texttt{?-p(0).}) = p_0$$
$$\Pr(\texttt{?-p(1).}) = (1 - p_0)p_1$$
$$\Pr(\texttt{?-p(2).}) = (1 - p_0)(1 - p_1)$$

$$\prod_{q \in \{\texttt{p(0)},\texttt{p(1)},\texttt{p(2)},\texttt{p(2)}\}} \Pr(\texttt{?-}q\texttt{.}) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

Probabilistic Prolog

$$\Pr(\texttt{p(0).}) = p_0$$
$$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$$
$$\Pr(\texttt{q(1).}) = p_1$$
$$\Pr(\texttt{q(2).}) = 1 - p_1$$

$$\Pr(\texttt{?-p(0).}) = p_0$$
$$\Pr(\texttt{?-p(1).}) = (1 - p_0)p_1$$
$$\Pr(\texttt{?-p(2).}) = (1 - p_0)(1 - p_1)$$

$$\prod_{q \in \{\texttt{p(0)},\texttt{p(1)},\texttt{p(2)},\texttt{p(2)}\}} \Pr(\texttt{?-}q\texttt{.}) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

$$\operatorname*{argmax}_{p_0,p_1} \prod_{q \in \{\texttt{p(0)},\texttt{p(1)},\texttt{p(2)},\texttt{p(2)}\}} \Pr(\texttt{?-}q\texttt{.}) = \left\langle \frac{1}{4}, \frac{1}{3} \right\rangle$$

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
         (map-tagged-distribution
          (lambda (value) (value tagged-distribution))
          (evaluate (application-callee expression) environment))))))
```

```
(gradient-ascent
 (lambda (p)
  (let ((tagged-distribution
         (evaluate if x0 then 0 else if x1 then 1 else 2
                   (list Pr(x0 ↦ true) = p0   Pr(x0 ↦ false) = 1 − p0
                         Pr(x1 ↦ true) = p1   Pr(x1 ↦ false) = 1 − p1
                         ...))))
   (map-reduce
    *
    1.0
    (lambda (value)
     (likelihood value tagged-distribution))
    '(0 1 2 2))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Probabilistic Lambda Calculus

```
(gradient-ascent
 (lambda (p)
  (let ((tagged-distribution
         (evaluate if x0 then 0 else if x1 then 1 else 2
                   (list Pr(x0 ↦ true) = p0   Pr(x0 ↦ false) = 1 − p0
                         Pr(x1 ↦ true) = p1   Pr(x1 ↦ false) = 1 − p1
                         ...))))
   (map-reduce
    *
    1.0
    (lambda (value)
     (likelihood value tagged-distribution))
    '(0 1 2 2))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Here $\textbf{if}\ x_0\ \textbf{then}\ 0\ \textbf{else}\ \textbf{if}\ x_1\ \textbf{then}\ 1\ \textbf{else}\ 2$, $\Pr(x_0 \mapsto \textbf{true}) = p_0$, $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$, $\Pr(x_1 \mapsto \textbf{true}) = p_1$, $\Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$.

Probabilistic Lambda Calculus

```
(gradient-ascent
 (lambda (p)
  (let ((tagged-distribution
         (evaluate if x0 then 0 else if x1 then 1 else 2
                   (list Pr(x0 ↦ true) = p0   Pr(x0 ↦ false) = 1 − p0
                         Pr(x1 ↦ true) = p1   Pr(x1 ↦ false) = 1 − p1
                         ...))))
   (map-reduce
    *
    1.0
    (lambda (value)
     (likelihood value tagged-distribution))
    '(0 1 2 2))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Probabilistic Lambda Calculus

```
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                   (list Pr(x0 ↦ true) = p0   Pr(x0 ↦ false) = 1 − p0
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                         ...))))
   (map-reduce
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     (likelihood value tagged-distribution))
    '(0 1 2 2))))
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Probabilistic Prolog

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(define (proof-distribution term clauses)
 (let ((offset ...))
  (map-reduce
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   (lambda (clause)
    (let ((clause (alpha-rename clause offset)))
     (let loop ((p (clause-p clause))
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      (if (boolean? substitution)
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   (map-reduce
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    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
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Generated Code

```
static void f2679(double a_f2679_0,double a_f2679_1,double a_f2679_2,double a_f2679_3){
  int t272381=((a_f2679_2==0.)?0:1);
  double t272406;
  double t272405;
  double t272404;
  double t272403;
  double t272402;
  if((t272381==0)){
    double t272480=(1.-a_f2679_0);
    double t272572=(1.-a_f2679_1);
    double t273043=(a_f2679_0+0.);
    double t274185=(t272480*a_f2679_1);
    double t274426=(t274185+0.);
    double t275653=(t272480*t272572);
    double t275894=(t275653+0.);
    double t277121=(t272480*t272572);
    double t277362=(t277121+0.);
    double t277431=(t277362*1.);
    double t277436=(t275894*t277431);
    double t277441=(t274426*t277436);
    double t277446=(t273043*t277441);
    ...
    double t1777107=(t1774696+t1715394);
    double t1777194=(0.-t1745420);
    double t1778533=(t1777194+t1419700);
    t272406=a_f2679_0;
    t272405=a_f2679_1;
    t272404=t277446;
    t272403=t1778533;
    t272402=t1777107;}
  else {...}
  r_f2679_0=t272406;
  r_f2679_1=t272405;
  r_f2679_2=t272404;
  r_f2679_3=t272403;
  r_f2679_4=t272402;}
```

Performance Comparison

	probabilistic-lambda-calculus		probabilistic-prolog	
	forward	reverse	forward	reverse
STALIN∇	1.00	1.00	1.00	1.00
IKARUS	499.13	419.37	10,384.61	4,347.82
SCHEME->C	934.34	660.69	11,394.23	5,838.50
BIGLOO	1,367.10	967.60	14,531.25	7,701.86
GAMBIT	1,155.64	1,035.55	24,831.73	12,931.67
LARCENY	2,294.07	1,412.75	24,471.15	12,906.83
STALIN	1,313.00	1,925.84	22,524.03	14,633.54
CHICKEN	2,168.67	2,659.16	53,320.31	28,434.78
SCMUTILS	6,448.24	3,449.89	85,697.11	43,416.14
MZC	5,227.44	5,325.36	144,765.62	118,192.54
MZSCHEME	8,667.32	6,370.29	157,965.74	124,975.15

Where You Can Read About This Work

Where You Can Read About This Work

- AD 2004(a,b,c,d), 2008(a,b)
- POPL 2005, 2006, 2008, 2009
- PLDI 2006, 2008, 2009
- ICFP 2007(a,b), 2008

Where You Can Read About This Work

- AD 200[illegible](a,b,c,d), 2008(a,b)
- POPL 200[illegible], 2006, 2008, 2009
- PLDI 2006, 20[illegible], 2009
- ICFP 2007(a,b), 20[illegible]8

```
                  Evaluation:  C. Weak paper, but it will not be an
                                 embarrassment to have it in POPL.
                  Confidence:  Z. I am an informed outsider and tried
                                 my best to understand the paper.

                       ===== Summary =====

Shows how to optimize a functional language with a built-in automatic
differentiation operator.

                   ===== Detailed Comments =====

The results look useful, but I wonder whether POPL is the right place to
present them. Yes, the development involves functional programming. But it
also involves a lot of concepts from scientific computing that may be
unfamiliar to many and that are explained only minimally or not at all.
```

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|
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|
Leonhard Euler
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Joseph Louis Lagrange
|
Simeon Poisson
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Michel Chasles
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Hubert Anson Newton
|
Eliakim Hastings Moore
|
Oswald Veblen
|
Alonzo Church