

Automatic Differentiation of Functional Programs and its use for Probabilistic Programming

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DE Shaw Research
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Part of this talk covers joint work with Barak A. Pearlmutter.



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Seventh European Workshop on AD



Seventh European Workshop on Automatic Differentiation Monday 24th - Tuesday 25th November 2008 Oxford-Man Institute of Quantitative Finance Oxford University Oxford, UK

This two day workshop is the seventh in a series providing a forum for the presentation of theoretical developments in, and applications of, Automatic Differentiation (AD) and adjoint methods. It will be informal, with no published proceedings, so allowing for the discussion of work in progress and presentations by those new to the subject. The workshop is kindly being hosted by Prof. Mike Giles of the [Oxford-Man Institute of Quantitative Finance](#).

We welcome every researcher who wishes to present their work on any aspect of AD or its applications. However, at this workshop we would encourage contributions in the area of



Applications and Challenges in Economics and Finance

Confirmed speakers include:

Luca Capriotti - CREDIT SUISSE

Oliver Pironneau - Université Paris VI

Luca Guerrieri - US Federal Reserve Bank

Benjamin Skrainka - University College London

If you are interested in giving a 25 minute talk on any aspect of AD, or any applications you think suitable for AD then please contact Dr Martin Buecker at buecker@sc.rwth-aachen.de. We particularly welcome contributions from research students and those new to AD.

URGENT

Accommodation for this workshop has been arranged at Linton Lodge, Oxford which must be booked in the next few days -- see details under [Travel and Accomodation](#)

Contact: webmaster@autodiff.org

Username:

Password:

login



The Essence

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(define (f x) 2x3)
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The Essence

`(define (f x) 2x3)` $\rightsquigarrow$ `(define (f' x) 6x2)`

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need reflective transformation of closure bodies

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need reflective transformation of closure bodies
want transformation done at compile time

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need reflective transformation of closure bodies
want transformation done at compile time
need flow analysis

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need reflective transformation of closure bodies
want transformation done at compile time
need **polyvariant** flow analysis

Nesting

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(sqrt (sqrt x))
```

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`( $\mathcal{D}$  ( $\mathcal{D}$  f))`

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$$\max_x \min_y f(x, y)$$

The Essence of Forward-Mode AD

$$f(c + \varepsilon) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}\varepsilon + \frac{f''(c)}{2!}\varepsilon^2 + \dots + \frac{f^{(i)}(c)}{i!}\varepsilon^i + \dots$$

Taylor, B. (1715). *Methodus Incrementorum Directa et Inversa*. London.

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To compute $\mathcal{D} f c$:

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To compute $\mathcal{D} f c$:

- evaluate f

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To compute $\mathcal{D} f \mathbf{c}$:

- evaluate f at the **term** $\mathbf{c} + \varepsilon$

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To compute $\mathcal{D} f c$:

- evaluate f at the **term** $c + \varepsilon$ to get a **power series**,

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To compute $\mathcal{D} f c$:

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Key idea: Only need output to be a **finite truncated** power series $a + b\varepsilon$.

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$$f(\textcolor{red}{c} + \varepsilon) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}\varepsilon + \frac{f''(c)}{2!}\varepsilon^2 + \dots + \frac{f^{(i)}(c)}{i!}\varepsilon^i + \dots$$

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Can do a *nonstandard interpretation* of f over *truncated power series*.

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Preserves control flow: Augments **original values** with **derivatives**.

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$(\mathcal{D} f)$ is $\mathcal{O}(1)$ relative to f (both space and time).

Arithmetic on Complex Numbers

$$a + bi$$

Hamilton, W. R. (1837). *Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time*. Transactions of the Royal Irish Academy, **17**(1):293–422.

Arithmetic on Complex Numbers

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Clifford, W. K. (1873). *Preliminary Sketch of Bi-quaternions*. Proceedings of the London Mathematical Society, **4**:381–95.

Arithmetic on Dual Numbers

$$x + x'\varepsilon$$

$$\varepsilon^2 = 0, \text{ but } \varepsilon \neq 0$$

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$$\langle x, x' \rangle$$

$$\langle x_1, x'_1 \rangle + \langle x_2, x'_2 \rangle = \langle (x_1 + x_2), (x'_1 + x'_2) \rangle$$

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Dynamic Overloading: SCMUTILS

```
(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define +
  (let ((+ +))
    (lambda (x1 x2)
      (make-bundle (+ (primal x1) (primal x2))
                    (+ (tangent x1) (tangent x2))))))

(define *
  (let ((+ +) (* *))
    (lambda (x1 x2)
      (make-bundle (* (primal x1) (primal x2))
                    (+ (* (primal x1) (tangent x2))
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(define ((D f) x) (tangent (f (make-bundle x 1))))
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Convenient

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Convenient but **slow**

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(define (f x) (* 2 (* x (* x x))))

(D f)
(D (D f))
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Convenient but **slow**

Dynamic Overloading: SCMUTILS

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(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define ((D f) x)
  (fluid-let ((+ (lambda (x1 x2)
                    (make-bundle (+ (primal x1) (primal x2))
                                   (+ (tangent x1) (tangent x2))))))
    (* (lambda (x1 x2)
        (make-bundle (* (primal x1) (primal x2))
                     (+ (* (primal x1) (tangent x2))
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      (tangent (f (make-bundle x 1)))))

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( $\mathcal{D}$  f)
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( $\mathcal{D}$  ( $\mathcal{D}$  f))
( $\mathcal{D}$  (lambda (x) ... ( $\mathcal{D}$  (lambda (y) ...) ...) ...))
```

Convenient but **slow**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

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Fast

Preprocessor: ADIFOR and TAPENADE

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end
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double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast but **inconvenient**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

AD_TOP = f

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast but **inconvenient**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
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AD_TOP = gf
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function f(x)
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AD_TOP = gf
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```
function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
gresult = 6.0d0*x*x*gx
ggresult = 6.0d0*x*x*ggx+12.0d0*x*gx*gx
end
```

Fast but **inconvenient**

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
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AD_TOP = f
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function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
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AD_TOP = f
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function gf(x, gx, gresult)
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gf = 2.0d0*x*x*x
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AD_TOP = gf
AD_IVARS = x, gx
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function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
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Fast but **inconvenient**

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function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
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```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = gf
AD_IVARS = x, gx
AD_DVARS = gf, gresult
AD_PREFIX = h
```

```
function hgf(x, hx, gx, hgx, gresult, hgresult, hresult)
double precision x, hx, gx, hgx, hgf, hresult, gresult, hgresult
hgf = 2.0d0*x*x*x
hresult = 6.0d0*x*x*hx
gresult = 6.0d0*x*x*gx
hgresult = 6.0d0*x*x*hgx+12.0d0*x*gx*hx
end
```

Fast but **inconvenient**

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}  
double x;  
... f(x) ...
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```
F<double> f(F<double> x) {return 2*x*x*x;}  
F<double> x;  
x.diff(0, 1);  
... f(x).d(0) ...
```

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```

```
F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
F<F<double> > x;  
x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
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Slow

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F<double> f(F<double> x) {return 2*x*x*x;}  
F<double> x;  
x.diff(0, 1);  
... f(x).d(0) ...
```

```
F<F<double> > f(F<F<double> > x) {return 2*x*x*x*x;}  
F<F<double> > x;  
x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
... f(x).d(0).d(0) ...
```

Slow

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}  
double x;  
... f(x) ...
```

```
F<double> f(F<double> x) {return 2*x*x*x;}  
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x.diff(0, 1);  
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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
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x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
... f(x).d(0).d(0) ...
```

Slow and **inconvenient**

Static Overloading: FADBAD++

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F<double> f(F<double> x) {return 2*x*x*x;}  
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x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
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Slow and inconvenient

Static Overloading: FADBAD++

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double f(double x) {return 2*x*x*x;}  
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F<double> f(F<double> x) {return 2*x*x*x;}  
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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
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Slow and **inconvenient**

Static Overloading: FADBAD++

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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
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Slow and **inconvenient**

Static Overloading: FADBAD++

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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
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x.diff(0, 1);  
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Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}  
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F<double> f(F<double> x) {return 2*x*x*x;}  
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x.diff(0, 1);  
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```

Slow and **inconvenient**

Static Overloading: FADBAD++

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double f(double x) {return 2*x*x*x;}  
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F<double> f(F<double> x) {return 2*x*x*x;}  
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F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
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```

Slow and **inconvenient**

Static Overloading: FADBAD++

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double f(double x) {return 2*x*x*x;}  
double x;  
... f(x) ...
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```
F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
F<F<double> > x;  
x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
... f(x).d(0).d(0) ...
```

```
template <typename T>  
T f(T x) {return 2*x*x*x;}  
T x;
```

Slow and **inconvenient**

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}  
double x;  
... f(x) ...
```

```
F<double> f(F<double> x) {return 2*x*x*x;}  
F<double> x;  
x.diff(0, 1);  
... f(x).d(0) ...
```

```
F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
F<F<double> > x;  
x.diff(0, 1);  
x.diff(0, 1).diff(0, 1);  
... f(x).d(0).d(0) ...
```

```
template <typename T>  
T f(T x) {return 2*x*x*x;}  
T x;
```

Slow and **inconvenient**

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}  
double x;  
... f(x) ...
```

```
F<double> f(F<double> x) {return 2*x*x*x;}  
F<double> x;  
x.diff(0, 1);  
... f(x).d(0) ...
```

```
F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}  
F<F<double> > x;  
x.diff(0, 1);  
x.diff(0, 1).diff(0,1);  
... f(x).d(0).d(0) ...
```

```
template <typename T>  
T f(T x) {return 2*x*x*x;}  
T x;
```

Slow and **inconvenient**

Our API for Functional Forward AD

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$

Our API for Functional Forward AD

$$\text{bundle} : \mathbb{R}^n \times \overline{\mathbb{R}^h} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$

Our API for Functional Forward AD

`bundle` : $\mathbb{R}^n \times \overline{\mathbb{R}^h} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$

`primal` : $(\textcolor{red}{\mathbb{R}^n} \triangleright \overline{\mathbb{R}^h}) \rightarrow \textcolor{red}{\mathbb{R}^n}$

`tangent` : $(\mathbb{R}^n \triangleright \textcolor{red}{\overline{\mathbb{R}^h}}) \rightarrow \textcolor{red}{\overline{\mathbb{R}^h}}$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$

Our API for Functional Forward AD

$$\text{bundle} : \mathbb{R}^n \times \overline{\mathbb{R}^h} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$$

$$\text{primal} : (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow \mathbb{R}^n$$

$$\text{tangent} : (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow \overline{\mathbb{R}^h}$$

$$j^* : (\mathbb{R}^n \rightarrow \mathbb{R}^m) \rightarrow ((\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow (\mathbb{R}^m \triangleright \overline{\mathbb{R}^h}))$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$
 j^* maps a **function** to its *push forward*

Our API for Functional Forward AD

$$\text{bundle} : \mathbb{R}^n \times \overline{\mathbb{R}^h} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$$

$$\text{primal} : (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow \mathbb{R}^n$$

$$\text{tangent} : (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow \overline{\mathbb{R}^h}$$

$$\mathbf{j}^* : (\mathbb{R}^n \rightarrow \mathbb{R}^m) \rightarrow ((\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow (\mathbb{R}^m \triangleright \overline{\mathbb{R}^m}))$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$
 $\mathbf{j}^*$ maps a function to its *push forward*

Our API for Functional Forward AD

$$\text{bundle} : \mathbb{R}^n \times \overline{\mathbb{R}^h} \rightarrow (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$$

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$$j^* : (\mathbb{R}^n \rightarrow \mathbb{R}^m) \rightarrow ((\mathbb{R}^n \triangleright \overline{\mathbb{R}^h}) \rightarrow (\mathbb{R}^m \triangleright \overline{\mathbb{R}^m}))$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$
 j^* maps a function to its *push forward*

Our API for Functional Forward AD

bundle : $\tau \times \overline{\tau} \rightarrow (\tau \triangleright \overline{\tau})$
primal : $(\tau \triangleright \overline{\tau}) \rightarrow \tau$
tangent : $(\tau \triangleright \overline{\tau}) \rightarrow \overline{\tau}$
 j^* : $(\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \triangleright \overline{\tau}_1) \rightarrow (\tau_2 \triangleright \overline{\tau}_2))$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$
 j^* maps a function to its *push forward*

Generalize to arbitrary types

Our API for Functional Forward AD

`bundle` : $\tau \times \overline{\tau} \rightarrow (\tau \triangleright \overline{\tau})$
`primal` : $(\tau \triangleright \overline{\tau}) \rightarrow \tau$
`tangent` : $(\tau \triangleright \overline{\tau}) \rightarrow \overline{\tau}$
`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \triangleright \overline{\tau}_1) \rightarrow (\tau_2 \triangleright \overline{\tau}_2))$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Our API for Functional Forward AD

$$\begin{aligned}\text{bundle} &: \tau \times \overline{\tau} \rightarrow (\tau \triangleright \overline{\tau}) \\ \text{primal} &: (\tau \triangleright \overline{\tau}) \rightarrow \tau \\ \text{tangent} &: (\tau \triangleright \overline{\tau}) \rightarrow \overline{\tau} \\ \text{j}^* &: (\tau_1 \rightarrow \tau_2) \rightarrow ((\tau_1 \triangleright \overline{\tau}_1') \rightarrow (\tau_2 \triangleright \overline{\tau}_2'))\end{aligned}$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

j^* maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overline{\tau}$ as $\overrightarrow{\tau}$

Our API for Functional Forward AD

$$\begin{aligned}\text{bundle} &: \tau \times \overline{\tau} \rightarrow \overrightarrow{\tau} \\ \text{primal} &: \overrightarrow{\tau} \rightarrow \tau \\ \text{tangent} &: \overrightarrow{\tau} \rightarrow \overline{\tau} \\ \text{j}^* &: (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})\end{aligned}$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

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$$\begin{aligned}\text{bundle} &: \tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau} \\ \text{primal} &: \overrightarrow{\tau} \rightarrow \tau \\ \text{tangent} &: \overrightarrow{\tau} \rightarrow \overrightarrow{\tau} \\ \overrightarrow{\mathcal{J}} &: (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})\end{aligned}$$

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

$\mathbf{j}^*$ maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

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Sometimes write $\mathbf{j}^*$ as $\overrightarrow{\mathcal{J}}$

Our API for Functional Forward AD

$\text{bundle} : \tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$
 $\text{primal} : \overrightarrow{\tau} \rightarrow \tau$
 $\text{tangent} : \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$
 $\text{j}^* : (\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

j^* maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write j^* as $\mathcal{J}$

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`primal` : $\overrightarrow{\tau} \rightarrow \tau$

`tangent` : $\overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

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(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
(D f)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

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Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

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`tangent` : $\overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
(D f)  
(D (D f))
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

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`tangent` : $\overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
(D f)  
(D (D f))  
(D (lambda (x) ... (D (lambda (y) ...) ...) ...) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as `J`

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`primal` : $\overrightarrow{\tau} \rightarrow \tau$

`tangent` : $\overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

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(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
(D f)  
(D (D f))  
(D (lambda (x) ... (D (lambda (y) ...) ...) ...)) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

What is `(j* j*)`?

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`primal` : $\overrightarrow{\tau} \rightarrow \tau$

`tangent` : $\overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
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(D (D f))  
(D (lambda (x) ... (D (lambda (y) ...) ...) ...)) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

What is $(j^* \ j^*)$?

Convenient

Our API for Functional Forward AD

`bundle` : $\tau \times \overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`primal` : $\overrightarrow{\tau} \rightarrow \tau$

`tangent` : $\overrightarrow{\tau} \rightarrow \overrightarrow{\tau}$

`j*` : $(\tau_1 \rightarrow \tau_2) \rightarrow (\overrightarrow{\tau_1} \rightarrow \overrightarrow{\tau_2})$

```
(define ((D f) x) (tangent ((j* f) (bundle x 1))))  
(D f)  
(D (D f))  
(D (lambda (x) ... (D (lambda (y) ...) ...) ...)) ...)
```

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overrightarrow{\mathbb{R}^n}$

`j*` maps a function to its *push forward*

Generalize to arbitrary types

What is the tangent of a discrete value or a function?

Can abbreviate $\tau \triangleright \overrightarrow{\tau}$ as $\overrightarrow{\tau}$

Sometimes write `j*` as $\overrightarrow{\mathcal{J}}$

What is $(j^* \ j^*)$?

Convenient and **fast**

A property

A property

$$x : \mathbb{R}^n$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$((\mathcal{J} f) x)[i, j] = \frac{\partial f(x)[i]}{\partial x[j]}$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$((\mathcal{J} f) x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$$

$$((\mathcal{J} f) x) : \mathbb{R}^{m \times n}$$

A property

$$x : \mathbb{R}^n$$

$$\overline{x'} : \mathbb{R}^n$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$((\mathcal{J} f) x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$$

$$((\mathcal{J} f) x) : \mathbb{R}^{m \times n}$$

$$((\mathcal{J} f) x) : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$$

A property

$$\begin{aligned} x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\ ((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\ (\text{primal } ((j^* f) (\text{bundle } x \overline{x}))) &= (f x) \end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'}\end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'})\end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x)))\end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j]\end{aligned}$$

A property

$$\begin{aligned}
 x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\
 ((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\
 (\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\
 (\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\
 (\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'}) \\
 ((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\
 \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j] \\
 f &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m
 \end{aligned}$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x'} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j] \\f &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\0/1 \text{ matrix, every row has exactly one } &1\end{aligned}$$

A property

$$\begin{aligned}
 x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\
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 (\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= (((\mathcal{J} f) x) \overline{x}) \\
 ((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\
 \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j]
 \end{aligned}$$

$$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$$

0/1 matrix, every row has exactly one 1

$$((\mathcal{J} f) x)$$

A property

$$\begin{aligned}x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\((\mathcal{J} f) x)[i,j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \mathbb{R}^{m \times n} & ((\mathcal{J} f) x) &: \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= ((\mathcal{J} f) x) \times \overline{x} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x}))) &= (((\mathcal{J} f) x) \overline{x}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j]\end{aligned}$$

$$f : \mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$$

0/1 matrix, every row has exactly one 1

$$((\mathcal{J} f) x) = \frac{\partial f(x)[i]}{\partial x[j]}$$

A property

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when f is a rearrangement function

$$((j^* f) x) = (\text{bundle } (f (\text{primal } x)) (f (\text{tangent } x)))$$

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$$\begin{aligned}
 x &: \mathbb{R}^n & \overline{x} &: \mathbb{R}^n & f &: \mathbb{R}^n \rightarrow \mathbb{R}^m \\
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A property

$$\begin{aligned}x &: \tau_1 & \overline{x'} &: \tau_1 & f &: \tau_1 \rightarrow \tau_2 \\((\mathcal{J} f) x)[i, j] &= \frac{\partial f(x)[i]}{\partial x[j]} & ((\mathcal{J} f) x) &: \tau_1 \xrightarrow{L} \tau_2 \\(\text{primal } ((j^* f) (\text{bundle } x \overline{x'}))) &= (f x) \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= ((\mathcal{J} f) x) \times \overline{x'} \\(\text{tangent } ((j^* f) (\text{bundle } x \overline{x'}))) &= (((\mathcal{J} f) x) \overline{x'}) \\((j^* f) x) &= (\text{bundle } (f (\text{primal } x)) (((\mathcal{J} f) (\text{primal } x)) (\text{tangent } x))) \\ \text{rearrangement function: } &(\forall i)(\exists j)(f x)[i] = x[j]\end{aligned}$$

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0/1 matrix, every row has exactly one 1

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when f is a rearrangement function

$$((j^* f) x) = (\text{bundle } (f (\text{primal } x)) (f (\text{tangent } x)))$$

What is the tangent of $\#t$?

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What if we take $\overline{\#t} = \#f$?

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when f is a rearrangement function

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What is the tangent of $\#t$?

What if we take $\overline{\#t} = \#f$?

when f is a rearrangement function

$((j * f) \ x) = (\text{bundle } (f \ (\text{primal } x)) \ (f \ (\text{tangent } x)))$

$f : (\#t \ x \ y) \mapsto (\#t \ x \ y) \quad \text{but} \quad f : (\#f \ x \ y) \mapsto (\#f \ y \ x)$

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f is a rearrangement function

$$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$$

What is the tangent of $\#t$?

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when f is a rearrangement function

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f is a rearrangement function

$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$

$= (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{y} \ \overline{x}))$

What is the tangent of #t?

What if we take $\overline{\#t} = \#f$?

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What is the tangent of #t?

What if we take $\overline{\#t} = \#f$?

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f is a rearrangement function

$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$

$= (\text{bundle } (\#t \ x \ \textcolor{red}{y}) \ (\#f \ \overline{y} \ \textcolor{red}{\overline{x}}))$

What is the tangent of $\#t$?

What if we take $\overline{\#t} = \#f$?

when f is a rearrangement function

$$((j * f) \ x) = (\text{bundle } (f \ (\text{primal } x)) \ (f \ (\text{tangent } x)))$$

$$f : (\#t \ x \ y) \mapsto (\#t \ x \ y) \quad \text{but} \quad f : (\#f \ x \ y) \mapsto (\#f \ y \ x)$$

f is a rearrangement function

$$((j * f) \ (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{x} \ \overline{y}))))$$

$$= (\text{bundle } (\#t \ x \ y) \ (\#f \ \overline{y} \ \overline{x}))$$

Problem avoided if we take $\overline{\#t} = \#t$

What is $(j^* \ j^*)$?

What is $(j^* \circ j^*)$?

when f is a rearrangement function

$$((j^* \circ f) \circ x) = (\text{bundle } (f \circ (\text{primal } x)) \circ (f \circ (\text{tangent } x)))$$

What is $(j^* \circ j^*)$?

when f is a rearrangement function

$$((j^* \circ f) \circ x) = (\text{bundle } (f \circ (\text{primal } x)) \circ (f \circ (\text{tangent } x)))$$

bundle , primal , tangent , and j^* are rearrangement functions

What is $(j^* \ j^*)$?

when f is a rearrangement function

$$((j^* \ f) \ x) = (\text{bundle } (f \ (\text{primal } x)) \ (f \ (\text{tangent } x)))$$

bundle , primal , tangent , and j^* are rearrangement functions

$$((j^* \ \text{bundle}) \ x) = (\text{bundle } (\text{bundle } (\text{primal } x)) \ (\text{bundle } (\text{tangent } x)))$$

$$((j^* \ \text{primal}) \ x) = (\text{bundle } (\text{primal } (\text{primal } x)) \ (\text{primal } (\text{tangent } x)))$$

$$((j^* \ \text{tangent}) \ x) = (\text{bundle } (\text{tangent } (\text{primal } x)) \ (\text{tangent } (\text{tangent } x)))$$

$$((j^* \ j^*) \ x) = (\text{bundle } (j^* \ (\text{primal } x)) \ (j^* \ (\text{tangent } x)))$$

Modularity

$$\nabla f \mathbf{x}$$

$$\triangleq \frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$$

Modularity

$$\begin{array}{lcl} \nabla f \mathbf{x} & \triangleq & \frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n} \\ \\ \text{GRADIENTDESCENT } f \mathbf{x}_0 & \triangleq & \dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \end{array}$$

Modularity

$$\nabla f \mathbf{x} \triangleq \frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$$

$$\text{GRADIENTDESCENT } f \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$$

$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$$

Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$\frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$
<code>GRADIENTDESCENT</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
<code>argmin</code> f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
<code>NEUTRONFLUX</code> r	$\triangleq$	<i>classified</i>

Modularity

$$\nabla f \mathbf{x} \triangleq \frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$$

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$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$\frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n}$
<code>GRADIENTDESCENT</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
<code>argmin</code> f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
<code>NEUTRONFLUX</code> r	$\triangleq$	<i>classified</i>
<code>DEVIATION</code> r	$\triangleq$	$((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$
r^*	$\triangleq$	<code>argmin</code> <code>DEVIATION</code>

Fermi, E. (1946). *The Development of the first chain reaction pile*.
 Proceedings of the American Philosophy Society, **90**:20–4.

Breaking Modularity

$$\nabla f \mathbf{x} \triangleq (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_1), \dots, (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_n)$$

$$\text{GRADIENTDESCENT } f \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$$

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Breaking Modularity

$$\nabla \vec{f} \mathbf{x} \triangleq (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_1'), \dots, (\vec{f} \mathbf{x} \triangleright \vec{\mathbf{e}}_n')$$

$$\text{GRADIENTDESCENT } \vec{f} \mathbf{x}_0 \triangleq \dots \mathbf{x}_{i+1} := \dots \nabla \vec{f} \mathbf{x}_i \dots$$

$$\text{argmin } f \triangleq \dots \text{GRADIENTDESCENT } \vec{f} \mathbf{x}_0 \dots$$

$$\text{NEUTRONFLUX } r \triangleq \boxed{\text{classified}}$$

$$\text{DEVIATION } r \triangleq ((\text{NEUTRONFLUX } r) - \text{NEUTRONFLUX}_{\text{critical}})^2$$

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$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
NEWTONSMETHOD $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
argmin $\overleftarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overline{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\xrightarrow[\rightsquigarrow]{\text{TAPENADE}}$	$\overline{\text{DEVIATION}}$
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GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
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argmin $\overleftarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \mathbf{x}_0 \dots$
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GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
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Breaking Modularity

$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} \overrightarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
NEWTONSMETHOD $\overleftarrow{f} \overrightarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots \mathcal{H} \overrightarrow{f} \mathbf{x}_i \dots$
argmin $\overleftarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \overrightarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\overset{\text{TAPENADE}}{\rightsquigarrow}$	$\overline{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\overset{\text{TAPENADE}}{\rightsquigarrow}$	$\overline{\text{DEVIATION}}$
$\mathbf{r}^*$	$\triangleq$	$\text{argmin } \overline{\text{DEVIATION}}$

Fermi, E. (1946). *The Development of the first chain reaction pile*.
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Breaking Modularity

$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} \overrightarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
NEWTONSMETHOD $\overleftarrow{f} \overrightarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots \mathcal{H} \overrightarrow{f} \mathbf{x}_i \dots$
argmin $\overleftarrow{f} \overrightarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \overrightarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\overset{\text{TAPENADE}}{\rightsquigarrow}$	$\overline{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\overset{\text{TAPENADE}}{\rightsquigarrow}$	$\overline{\text{DEVIATION}}$
$\mathbf{r}^*$	$\triangleq$	$\overline{\text{argmin DEVIATION}}$

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Breaking Modularity

$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} \overrightarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
NEWTONSMETHOD $\overleftarrow{f} \overrightarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots \mathcal{H} \overrightarrow{f} \mathbf{x}_i \dots$
argmin $\overleftarrow{f} \overrightarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD} \overleftarrow{f} \overrightarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\overset{\text{TAPENADE}}{\rightsquigarrow}$	$\overleftarrow{\text{NEUTRONFLUX}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\overset{\text{TAPENADE}}{\rightsquigarrow}$	$\overleftarrow{\text{DEVIATION}}$
$\mathbf{r}^*$	$\triangleq$	argmin $\overleftarrow{\text{DEVIATION}} \overrightarrow{\text{DEVIATION}}$

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Breaking Modularity

$\nabla \overleftarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overleftarrow{f} \mathbf{x} \dots$
$\mathcal{H} \overrightarrow{f} \mathbf{x}$	$\triangleq$	$\dots \overrightarrow{f} \dots \mathbf{x} \dots$
GRADIENTDESCENT $\overleftarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots$
NEWTONSMETHOD $\overleftarrow{f} \overrightarrow{f} \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla \overleftarrow{f} \mathbf{x}_i \dots \mathcal{H} \overrightarrow{f} \mathbf{x}_i \dots$
argmin $\overleftarrow{f} \overrightarrow{f}$	$\triangleq$	$\dots \text{NEWTONSMETHOD } \overleftarrow{f} \overrightarrow{f} \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
NEUTRONFLUX	$\xrightarrow{\text{TAPENADE}}$	$\overleftarrow{\text{NEUTRONFLUX}}$
$\overleftarrow{\text{NEUTRONFLUX}}$	$\xrightarrow{\text{TAPENADE}}$	$\overrightarrow{\overleftarrow{\text{NEUTRONFLUX}}}$
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
DEVIATION	$\xrightarrow{\text{TAPENADE}}$	$\overleftarrow{\text{DEVIATION}}$
$\overleftarrow{\text{DEVIATION}}$	$\xrightarrow{\text{TAPENADE}}$	$\overrightarrow{\overleftarrow{\text{DEVIATION}}}$
$\mathbf{r}^*$	$\triangleq$	argmin $\overleftarrow{\text{DEVIATION}} \overrightarrow{\overleftarrow{\text{DEVIATION}}}$

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Restoring Modularity

$\nabla f \mathbf{x}$	$\triangleq$	
$\mathcal{H} f \mathbf{x}$	$\triangleq$	
GRADIENTDESCENT $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
NEWTONSMETHOD $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
argmin f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	argmin DEVIATION

Fermi, E. (1946). *The Development of the first chain reaction pile*.
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Restoring Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$((\vec{\mathcal{J}} f) \mathbf{x} \triangleright \vec{\mathbf{e}}_1'), \dots, ((\vec{\mathcal{J}} f) \mathbf{x} \triangleright \vec{\mathbf{e}}_n')$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	
GRADIENTDESCENT $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
NEWTONSMETHOD $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
argmin f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
NEUTRONFLUX $\mathbf{r}$	$\triangleq$	classified
DEVIATION $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	argmin DEVIATION

Fermi, E. (1946). *The Development of the first chain reaction pile*.
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Restoring Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$\dots (\overleftarrow{\mathcal{J}} f) \mathbf{x} \dots$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	
<code>GRADIENTDESCENT</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
<code>NEWTONSMETHOD</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
<code>argmin</code> f	$\triangleq$	$\dots \text{GRADIENTDESCENT } f \mathbf{x}_0 \dots$
<code>NEUTRONFLUX</code> $\mathbf{r}$	$\triangleq$	<i>classified</i>
<code>DEVIATION</code> $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	<code>argmin</code> <code>DEVIATION</code>

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Restoring Modularity

$\nabla f \mathbf{x}$	$\triangleq$	$\dots (\overleftarrow{\mathcal{J}} f) \mathbf{x} \dots$
$\mathcal{H} f \mathbf{x}$	$\triangleq$	$\dots (\overrightarrow{\mathcal{J}} (\overleftarrow{\mathcal{J}} f)) \dots \mathbf{x} \dots$
<code>GRADIENTDESCENT</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots$
<code>NEWTONSMETHOD</code> $f \mathbf{x}_0$	$\triangleq$	$\dots \mathbf{x}_{i+1} := \dots \nabla f \mathbf{x}_i \dots \mathcal{H} f \mathbf{x}_i \dots$
<code>argmin</code> f	$\triangleq$	$\dots \text{NEWTONSMETHOD } f \mathbf{x}_0 \dots$
<code>NEUTRONFLUX</code> $\mathbf{r}$	$\triangleq$	classified
<code>DEVIATION</code> $\mathbf{r}$	$\triangleq$	$((\text{NEUTRONFLUX } \mathbf{r}) - \text{NEUTRONFLUX}_{\text{critical}})^2$
$\mathbf{r}^*$	$\triangleq$	<code>argmin</code> <code>DEVIATION</code>

Fermi, E. (1946). *The Development of the first chain reaction pile*.
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Having your cake and eating it too

- Convenient

- Fast

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 - $\mathcal{D}$ formulated as a higher-order function in the language
- Fast

Having your cake and eating it too

- Convenient
 - $\mathcal{D}$ formulated as a higher-order function in the language
 - no arbitrary restrictions
 - applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself

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- $\mathcal{D}$ formulated as a higher-order function in the language
- no arbitrary restrictions
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- higher-order derivatives
 - $(\mathcal{D} (\mathcal{D} f))$

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- higher-order derivatives
 - $(\mathcal{D} (\mathcal{D} f))$
- nesting
 - $(\mathcal{D} (\text{lambda } (...) \dots (\mathcal{D} (\text{lambda } (...) \dots)) \dots))$

- Fast

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applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
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 $(\mathcal{D} (\mathcal{D} f))$
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 $(\mathcal{D} (\text{lambda } (...) \dots (\mathcal{D} (\text{lambda } (...) \dots)) \dots))$

- Fast

- $\mathcal{D}$ implemented by reflective transformation of environments and code associated with closures

Having your cake and eating it too

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applies to all data types and constructs in the language, including code produced by $\mathcal{D}$ and even $\mathcal{D}$ itself
- higher-order derivatives
 $(\mathcal{D} (\mathcal{D} f))$
- nesting
 $(\mathcal{D} (\text{lambda } (...) \dots (\mathcal{D} (\text{lambda } (...) \dots)) \dots))$

- Fast

- $\mathcal{D}$ implemented by reflective transformation of environments and code associated with closures
- compile away reflection with partial evaluation implemented by flow analysis

Monovariant Flow Analysis: 0-CFA

```
(define ( $\mathcal{D}$  f)  
...)
```

Monovariant Flow Analysis: 0-CFA

```
(define ( $\mathcal{D}$  f)  
  ...)
```

```
( $\mathcal{D}$  (lambda (x)  $2x^3$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:  $\lambda x. 2x^3$ )  
  ...)
```

```
(D (lambda (x)  $2x^3$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ ))  
  ...:( $\lambda x\ 6x^2$ )  
  
(D (lambda (x)  $2x^3$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ ))  
  ...:( $\lambda x\ 6x^2$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ ))  
  ...:( $\lambda x\ 6x^2$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

```
(D (lambda (x)  $3x^4$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ )  $\cup$  ( $\lambda x\ 3x^4$ ))  
  ...:( $\lambda x\ 6x^2$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

```
(D (lambda (x)  $3x^4$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ )  $\cup$  ( $\lambda x\ 3x^4$ ))  
  ...:( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

```
(D (lambda (x)  $3x^4$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ )  $\cup$  ( $\lambda x\ 3x^4$ ))  
  ...:( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )
```

```
(D (lambda (x)  $3x^4$ )):( $\lambda x\ 12x^3$ )
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x\ 2x^3$ )  $\cup$  ( $\lambda x\ 3x^4$ ))  
  ...:( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ ))
```

```
(D (lambda (x)  $2x^3$ )):( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ )
```

```
(D (lambda (x)  $3x^4$ )):( $\lambda x\ 6x^2$ )  $\cup$  ( $\lambda x\ 12x^3$ )
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:  $(\lambda x \ 2x^3) \cup (\lambda x \ 3x^4)$ )  
  ...:  $(\lambda x \ 6x^2) \cup (\lambda x \ 12x^3)$ )
```

```
(D (lambda (x)  $2x^3$ )):  $(\lambda x \ 6x^2) \cup (\lambda x \ 12x^3)$ 
```

```
(D (lambda (x)  $3x^4$ )):  $(\lambda x \ 6x^2) \cup (\lambda x \ 12x^3)$ 
```

Monovariant Flow Analysis: 0-CFA

```
(define ( $\mathcal{D}$  f)  
  ...)
```

Monovariant Flow Analysis: 0-CFA

```
(define ( $\mathcal{D}$  f)  
  ...)
```

```
( $\mathcal{D}$  ( $\mathcal{D}$  (lambda (x)  $e^{2x}$ )))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ ))  
  ...)  
  
(D (D (lambda (x)  $e^{2x}$ )))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ )  
  
(D (D (lambda (x)  $e^{2x}$ )))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ ))  
  ...:( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))  
...:( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ ))
```

Monovariant Flow Analysis: 0-CFA

```
(define (D f:( $\lambda x e^{2x}$ )  $\cup$  ( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ )  $\cup$  ...)
...:( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ )  $\cup$  ...)
```

```
(D (D (lambda (x)  $e^{2x}$ )):( $\lambda x 2e^{2x}$ )  $\cup$  ( $\lambda x 4e^{2x}$ )  $\cup$  ...)
```

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

(define ($\mathcal{D}$ f) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}$  f) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define (Dg f:(λx 2x3)) ...:(λx 6x2))
```

```
(define (g ...) ... (D (lambda (x) 2x3)):(λx 6x2) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define (Dg f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define (g ...) ... (D (lambda (x)  $2x^3$ )):( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... (D (lambda (x)  $3x^4$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f : ( $\lambda x$   $2x^3$ )) ... : ( $\lambda x$   $6x^2$ ))
```

```
(define ( $\mathcal{D}_h$  f) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )) : ( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... ( $\mathcal{D}$  (lambda (x)  $3x^4$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define ( $\mathcal{D}_h$  f:( $\lambda x$   $3x^4$ )) ...)
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )):( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... ( $\mathcal{D}$  (lambda (x)  $3x^4$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define ( $\mathcal{D}_h$  f:( $\lambda x$   $3x^4$ )) ...:( $\lambda x$   $12x^3$ ))
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )):( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... ( $\mathcal{D}$  (lambda (x)  $3x^4$ )) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ( $\mathcal{D}_g$  f:( $\lambda x$   $2x^3$ )) ...:( $\lambda x$   $6x^2$ ))
```

```
(define ( $\mathcal{D}_h$  f:( $\lambda x$   $3x^4$ )) ...:( $\lambda x$   $12x^3$ ))
```

```
(define (g ...) ... ( $\mathcal{D}$  (lambda (x)  $2x^3$ )):( $\lambda x$   $6x^2$ ) ...)
```

```
(define (h ...) ... ( $\mathcal{D}$  (lambda (x)  $3x^4$ )):( $\lambda x$   $12x^3$ ) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
  (if (zero? n) x ((compose (- n 1) f) (f x))))
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages or Taming Lambda*. Ph.D. thesis, CMU.

Polyvariant Flow Analysis: k -CFA

with Bounded Context Sensitivity

```
(define ((compose n f) x)
  (if (zero? n) x ((compose (- n 1) f) (f x))))

((compose k  $\mathcal{D}$ ) g)
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```
(define ( $\mathcal{D}_{\text{compose:compose:compose}}$   $f:g''$ ) ...)
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⋮

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$$\mathcal{E} : e \times \sigma \rightarrow v$$

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$$v ::= \#t \mid \#f \mid () \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

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Allows complete unboxing: no allocation, reclamation, indirection

Game Theory

		B			
		b_1	...	b_j	... b_n
A	a_1				
	$\vdots$		$\ddots$	$\vdots$	
	a_i		...	PAYOFF(a_i, b_j)	...
	$\vdots$			$\vdots$	$\ddots$
	a_m				

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Game Theory

		B			
		b_1	$\dots$	b_j	$\dots b_n$
A	a_1				
	$\vdots$		$\ddots$	$\vdots$	
	a_i	$\dots$		$\text{PAYOFF}(a_i, b_j)$	$\dots$
	$\vdots$			$\vdots$	$\ddots$
	a_m				

$$\max_{a \in A} \min_{b \in B} \text{PAYOFF}(a, b)$$

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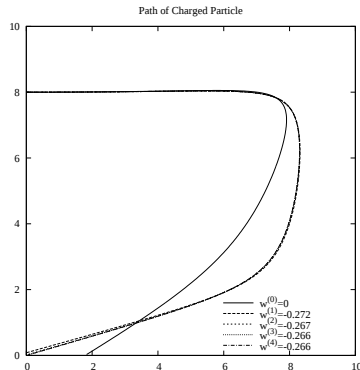
Game Theory

		$\mathbb{R}^n$		
		...	b	...
$\mathbb{R}^m$	a	...	PAYOFF(a , b)	...
		...		...
		...		...

$$\max_{\mathbf{a} \in \mathbb{R}^m} \min_{\mathbf{b} \in \mathbb{R}^n} \text{PAYOFF}(\mathbf{a}, \mathbf{b})$$

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Cathode Ray Tubes



$$\text{potential: } p(\mathbf{x}; w) = \|\mathbf{x} - (10, 10 - w)\|^{-1} + \|\mathbf{x} - (10, 0)\|^{-1}$$

$$\ddot{\mathbf{x}}(t) = -\nabla_{\mathbf{x}} p(\mathbf{x})|_{\mathbf{x}=\mathbf{x}(t)}$$

$$\dot{\mathbf{x}}(t + \Delta t) = \dot{\mathbf{x}}(t) + \Delta t \ddot{\mathbf{x}}(t)$$

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t \dot{\mathbf{x}}(t)$$

$$\text{When: } x_1(t + \Delta t) \leq 0$$

$$\text{let: } \Delta t_f = -x_1(t) / \dot{x}_1(t)$$

$$t_f = t + \Delta t_f$$

$$\mathbf{x}(t_f) = \mathbf{x}(t) + \Delta t_f \dot{\mathbf{x}}(t)$$

$$\text{Error: } E(w) = x_0(t_f)^2$$

$$\text{Find: } \underset{w}{\operatorname{argmin}} E(w)$$

Sprague, C. S. and George, R. H. (1939). *Cathode Ray Deflecting Electrode*. US Patent 2,161,437.

George, R. H. (1940). *Cathode Ray Tube*. US Patent 2,222,942.

Performance Comparison

	particle				saddle			
	FF	FR	RF	RR	FF	FR	RF	RR
STALIN ∇	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
ADIFOR	1.52	■	■	■	2.07	■	■	■
TAPENADE	3.40	■	■	■	2.56	■	■	■
FADBAD++	65.69	■	■	■	22.44	■	■	■
MLTON	53.89	88.88	16.08	28.06	40.39	51.21	1.86	2.67
OCAML	160.50	340.35	147.91	263.66	107.71	156.33	6.75	13.51
SML/NJ	106.21	182.45	105.04	185.15	84.38	106.01	3.55	6.31
GHC	165.22	■	■	■	121.18	■	■	■
BIGLOO	505.90	761.40	104.81	228.56	423.69	440.25	15.77	24.59
CHICKEN	1120.37	2026.31	425.60	1872.85	889.58	1144.65	35.73	68.94
GAMBIT	444.13	752.63	138.34	256.30	362.65	420.48	14.08	23.87
IKARUS	192.07	312.28	61.79	114.87	158.88	205.97	6.75	11.40
LARCENY	726.59	1108.18	144.55	270.14	571.81	613.65	19.14	29.77
MIT SCHEME	1472.26	2500.00	309.66	591.36	1243.26	1428.57	51.36	79.10
MzC	2073.26	3434.64	340.30	655.83	2436.26	1996.40	72.45	150.02
MzSCHEME	2344.70	4076.16	409.95	843.68	2000.89	2332.43	80.78	134.00
SCHEME->C	391.42	605.26	109.77	198.43	324.95	328.84	12.74	18.28
SCMUTILS	3321.20	■	■	■	2800.71	■	■	■
STALIN	208.10	366.08	51.84	91.86	166.96	212.93	7.68	11.40

- not implemented but could implement
- not implemented in existing tool
- can't implement

Gradient-Based Optimization

```
(define (e i n)
  (if (zero? n)
      '()
      (cons (if (zero? i) 1.0 0.0)
              (e (- i 1) (- n 1)))))
```

Gradient-Based Optimization

```
(define (e i n)
  (if (zero? n)
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              (e (- i 1) (- n 1)))))

(define ((gradient f) x)
  (let ((n (length x)))
    (map (lambda (i) (tangent ((j* f) (bundle x (e i n)))))
          (iota n)))))
```

Gradient-Based Optimization

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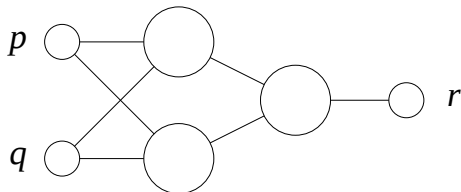
(define (gradient-ascent f x0 n eta)
  (if (zero? n)
      (list x0 (f x0) ((gradient f) x0))
      (gradient-ascent f
                        (zip (lambda (xi gi) (+ xi (* eta gi)))
                            x0
                            ((gradient f) x0))
                        (- n 1)
                        eta)))
```

Gradient-Based Optimization

```
(define ((gradient f) x) (cdr ((cdr ((*j f) (*j x))) 1.0)))
```

```
(define (gradient-ascent f x0 n eta)
  (if (zero? n)
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      (gradient-ascent f
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                        (- n 1)
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```

Neural Networks



p	q	r
0	0	0
0	1	1
1	0	1
1	1	0

Rumelhart, D. E., Hinton, G. E., and Williams, R. J. (1986). *Learning representations by back-propagating errors*. Nature, **323**:533–6.

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
  ((fold + bias) (zip * ws activities)))
```

Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
  ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
  (map (sum-activities activities) ws-layer))
```

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```
(define ((sum-activities activities) bias ws)
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(define (sum-layer activities ws-layer)
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(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))
```

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(define (sum-layer activities ws-layer)
  (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))

(define ((forward-pass ws-layers) in)
  (if (null? ws-layers)
      in
      ((forward-pass (cdr ws-layers))
       (map sigmoid (sum-layer in (car ws-layers))))))
```

Neural Networks in VLAD

```
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      ((forward-pass (cdr ws-layers))
       (map sigmoid (sum-layer in (car ws-layers))))))

(define ((error-on-dataset dataset) ws-layers)
  ((fold + 0)
   (map (lambda ((list in target))
          (* 0.5 (magnitude-squared (v- ((forward-pass ws-layers) in) target))))
        dataset)))
```

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        (* 0.5 (magnitude-squared (v- ((forward-pass ws-layers) in) target))))
        dataset)))

(gradient-descent (error-on-dataset '(((0 0) (0))
                                       ((0 1) (1))
                                       ((1 0) (1))
                                       ((1 1) (0))))
                  '(((0 -0.284227 1.16054) (0 0.617194 1.30467))
                    ((0 -0.084395 0.648461)))
                  1000.0
                  0.3)
```

Performance Comparison

	backprop		
	Fs	Fv	R
STALIN ∇	1.00	■	1.00
ADIFOR	11.84	2.68	■
TAPENADE	11.35	4.33	6.24
ADIC	16.33	3.93	■
ADOL-C	12.34	3.89	35.53
CppAD	42.15	■	23.69
FADBAD++	98.96	33.15	53.03
MLTON	73.94	■	37.94
OCAML	157.75	■	149.14
SML/NJ	142.71	■	94.97
GHC	■	■	■
BIGLOO	577.45	■	306.60
CHICKEN	1391.75	■	971.91
GAMBIT	545.20	■	341.73
IKARUS	216.42	■	147.49
LARCENY	955.98	■	486.64
MIT SCHEME	1900.04	■	1141.22
MzC	2439.93	■	1571.52
MZSCHEME	3477.86	■	1866.28
SCHEME->C	484.24	■	233.75
SCMUTILS	4544.48	■	■
STALIN	832.68	■	367.84

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$P = \text{if } x_0 \text{ then } 0 \text{ else if } x_1 \text{ then } 1 \text{ else } 2$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

$P = \text{if } x_0 \text{ then } 0 \text{ else if } x_1 \text{ then } 1 \text{ else } 2$

$$\Pr(x_0 \mapsto \text{true}) = p_0$$

$$\Pr(x_0 \mapsto \text{false}) = 1 - p_0$$

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$$\Pr(\mathcal{E}(P) = 0 | p_0, p_1) = p_0$$

$$\Pr(\mathcal{E}(P) = 1 | p_0, p_1) = (1 - p_0)p_1$$

$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

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$$\Pr(x_1 \mapsto \mathbf{true}) = p_1$$

$$\Pr(x_1 \mapsto \mathbf{false}) = 1 - p_1$$

$$\Pr(\mathcal{E}(P) = 0 | p_0, p_1) = p_0$$

$$\Pr(\mathcal{E}(P) = 1 | p_0, p_1) = (1 - p_0)p_1$$

$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

$$\prod_{v \in \{0,1,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

$P = \text{if } x_0 \text{ then } 0 \text{ else if } x_1 \text{ then } 1 \text{ else } 2$

$$\Pr(x_0 \mapsto \text{true}) = p_0$$

$$\Pr(x_0 \mapsto \text{false}) = 1 - p_0$$

$$\Pr(x_1 \mapsto \text{true}) = p_1$$

$$\Pr(x_1 \mapsto \text{false}) = 1 - p_1$$

$$\Pr(\mathcal{E}(P) = 0 | p_0, p_1) = p_0$$

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$$\operatorname{argmax}_{p_0, p_1} \prod_{v \in \{0,1,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = \left\langle \frac{1}{4}, \frac{1}{3} \right\rangle$$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Prolog

$p(\theta).$

$p(X) :- q(X).$

$q(1).$

$q(2).$

Probabilistic Prolog

$$\Pr(p(\theta) \text{ .}) = p_0$$

$$\Pr(p(X) : \neg q(X) \text{ .}) = 1 - p_0$$

$$\Pr(q(1) \text{ .}) = p_1$$

$$\Pr(q(2) \text{ .}) = 1 - p_1$$

Probabilistic Prolog

$$\Pr(p(\Theta) \text{ .}) = p_0$$

$$\Pr(p(X) : \neg q(X) \text{ .}) = 1 - p_0$$

$$\Pr(q(1) \text{ .}) = p_1$$

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$$\Pr(? - p(\Theta) \text{ .}) = p_0$$

$$\Pr(? - p(1) \text{ .}) = (1 - p_0)p_1$$

$$\Pr(? - p(2) \text{ .}) = (1 - p_0)(1 - p_1)$$

Probabilistic Prolog

$$\Pr(p(\theta) \text{ .}) = p_0$$

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$$\prod_{q \in \{p(\theta), p(1), p(2), p(2)\}} \Pr(? - q \text{ .}) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

Probabilistic Prolog

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Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
  (cond
    ((constant-expression? expression)
     (singleton-tagged-distribution
      (constant-expression-value expression)))
    ((variable-access-expression? expression)
     (lookup-value
      (variable-access-expression-variable expression) environment))
    ((lambda-expression? expression)
     (singleton-tagged-distribution
      (lambda (tagged-distribution)
        (evaluate
         (lambda-expression-body expression)
         (cons (make-binding (lambda-expression-variable expression)
                             tagged-distribution)
               environment))))))
    (else (let ((tagged-distribution
                  (evaluate (application-argument expression)
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Probabilistic Lambda Calculus

```
(gradient-ascent
 (lambda (p)
  (let ((tagged-distribution
        (evaluate if  $x_0$  then 0 else if  $x_1$  then 1 else 2
                  (list  $\Pr(x_0 \mapsto \mathbf{true}) = p_0$   $\Pr(x_0 \mapsto \mathbf{false}) = 1 - p_0$ 
                         $\Pr(x_1 \mapsto \mathbf{true}) = p_1$   $\Pr(x_1 \mapsto \mathbf{false}) = 1 - p_1$ 
                        ...))))))
 (map-reduce
  *
  1.0
  (lambda (value)
    (likelihood value tagged-distribution))
  '(0 1 2 2))))
'(0.5 0.5)
1000.0
0.1)
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Probabilistic Lambda Calculus

(gradient-ascent

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(let ((tagged-distribution

(evaluate **if** x_0 **then** 0 **else if** x_1 **then** 1 **else** 2

(list $\Pr(x_0 \mapsto \text{true}) = p_0$ $\Pr(x_0 \mapsto \text{false}) = 1 - p_0$

$\Pr(x_1 \mapsto \text{true}) = p_1$ $\Pr(x_1 \mapsto \text{false}) = 1 - p_1$

...))))

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Probabilistic Lambda Calculus

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Probabilistic Prolog

```
(define (proof-distribution term clauses)
  (let ((offset ...))
    (map-reduce
      append
      '()
      (lambda (clause)
        (let ((clause (alpha-rename clause offset)))
          (let loop ((p (clause-p clause))
                     (substitution (unify term (clause-term clause)))
                     (terms (clause-terms clause)))
            (if (boolean? substitution)
                '()
                (if (null? terms)
                    (list (make-double p substitution))
                    (map-reduce
                      append
                      '()
                      (lambda (double)
                        (loop (* p (double-p double))
                              (append substitution (double-substitution double))
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                    (map-reduce
                     append
                     '()
                     (lambda (double)
                       (loop (* p (double-p double))
                             (append substitution (double-substitution double))
                             (rest terms)))
                     (proof-distribution
                      (apply-substitution substitution (first terms)) clauses)))))))
      clauses)))
```

Probabilistic Prolog

```
(define (proof-distribution term clauses)
  (let ((offset ...))
    (map-reduce
      append
      '()
      (lambda (clause)
        (let ((clause (alpha-rename clause offset)))
          (let loop ((p (clause-p clause))
                     (substitution (unify term (clause-term clause)))
                     (terms (clause-terms clause)))
            (if (boolean? substitution)
                '()
                (if (null? terms)
                    (list (make-double p substitution))
                    (map-reduce
                      append
                      '()
                      (lambda (double)
                        (loop (* p (double-p double))
                              (append substitution (double-substitution double))
                              (rest terms)))
                      (proof-distribution
                        (apply-substitution substitution (first terms)) clauses)))))))
      clauses)))
```

Probabilistic Prolog

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(define (proof-distribution term clauses)
  (let ((offset ...))
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      append
      '()
      (lambda (clause)
        (let ((clause (alpha-rename clause offset)))
          (let loop ((p (clause-p clause))
                     (substitution (unify term (clause-term clause)))
                     (terms (clause-terms clause)))
            (if (boolean? substitution)
                '()
                (if (null? terms)
                    (list (make-double p substitution))
                    (map-reduce
                      append
                      '()
                      (lambda (double)
                        (loop (* p (double-p double))
                              (append substitution (double-substitution double))
                              (rest terms))))
                    (proof-distribution
                     (apply-substitution substitution (first terms)) clauses)))))))
    clauses)))
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p0
                        Pr(p(X):-q(X).) = 1 - p0
                        Pr(q(1).) = p1
                        Pr(q(2).) = 1 - p1))))
    (map-reduce
     *
     1.0
     (lambda (query)
      (likelihood (proof-distribution query clauses)))
      '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
1000.0
0.1)
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
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                       Pr(p(X):-q(X).) = 1 - p0
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    *
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    (map-reduce
     *
     1.0
     (lambda (query)
      (likelihood (proof-distribution query clauses)))
      '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
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Probabilistic Prolog

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    (map-reduce
     *
     1.0
     (lambda (query)
      (likelihood (proof-distribution query clauses)))
     '(p(0) p(1) p(2) p(2))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Probabilistic Prolog

```
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 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p0
                       Pr(p(X):-q(X).) = 1 - p0
                       Pr(q(1).) = p1
                       Pr(q(2).) = 1 - p1))))
    (map-reduce
     *
     1.0
     (lambda (query)
      (likelihood (proof-distribution query clauses)))
     '(p(0) p(1) p(2) p(2)))))
'(0.5 0.5)
1000.0
0.1)
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) =  $p_0$ 
                        Pr(p(X):-q(X).) = 1 -  $p_0$ 
                        Pr(q(1).) =  $p_1$ 
                        Pr(q(2).) = 1 -  $p_1$ ))))
    (map-reduce
     *
     1.0
     (lambda (query)
      (likelihood (proof-distribution query clauses)))
      '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p0
                        Pr(p(X):-q(X).) = 1 - p0
                        Pr(q(1).) = p1
                        Pr(q(2).) = 1 - p1))))
    (map-reduce
     *
     1.0
     (lambda (query)
      (likelihood (proof-distribution query clauses)))
      '(p(0) p(1) p(2) p(2)))))
'(0.5 0.5)
1000.0
0.1)
```

Generated Code

```
static void f2679(double a_f2679_0,double a_f2679_1,double a_f2679_2,double a_f2679_3){
    int t272381=((a_f2679_2==0.)?0:1);
    double t272406;
    double t272405;
    double t272404;
    double t272403;
    double t272402;
    if((t272381==0)){
        double t272480=(1.-a_f2679_0);
        double t272572=(1.-a_f2679_1);
        double t273043=(a_f2679_0+0.);
        double t274185=(t272480*a_f2679_1);
        double t274426=(t274185+0.);
        double t275653=(t272480*t272572);
        double t275894=(t275653+0.);
        double t277121=(t272480*t272572);
        double t277362=(t277121+0.);
        double t277431=(t277362*1.);
        double t277436=(t275894*t277431);
        double t277441=(t274426*t277436);
        double t277446=(t273043*t277441);
        ...
        double t1777107=(t1774696+t1715394);
        double t1777194=(0.-t1745420);
        double t1778533=(t1777194+t1419700);
        t272406=a_f2679_0;
        t272405=a_f2679_1;
        t272404=t277446;
        t272403=t1778533;
        t272402=t1777107;}
    else {...}
    r_f2679_0=t272406;
    r_f2679_1=t272405;
    r_f2679_2=t272404;
    r_f2679_3=t272403;
    r_f2679_4=t272402;}
```

Performance Comparison

	probabilistic-lambda-calculus		probabilistic-prolog	
	F	R	F	R
STALIN ∇	1.00	1.00	1.00	1.00
MLTON	106.45	124.95	789.41	483.47
OCAML	215.73	538.68	1207.13	1534.61
SML/NJ	197.75	272.45	2448.02	1471.94
GHC	■	■	■	■
BIGLOO	832.92	1048.11	14422.16	8286.06
CHICKEN	2305.98	3283.00	66948.70	37792.84
GAMBIT	879.88	1153.86	24316.03	13649.81
IKARUS	437.46	531.10	8242.92	4845.86
LARCENY	1651.01	1673.22	25589.62	14833.53
MIT SCHEME	3491.10	4130.19	85819.57	48335.38
MzC	5289.17	5929.14	154206.95	83480.27
MzSCHEME	6235.78	7134.71	166129.12	91630.70
SCHEME->C	682.15	794.31	10530.66	5980.27
SCMUTILS	6456.99	■	80100.23	■
STALIN	1240.73	1137.41	22511.79	10986.43

- not implemented but could implement, including FORTRAN, C, and C++
- not implemented in existing tool
- can't implement

It is, of course, not excluded that the range of arguments or range of values of a function should consist wholly or partly of functions. The derivative, as this notion appears in the elementary differential calculus, is a familiar mathematical example of a function for which both ranges consist of functions.

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(¶4)

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Gottfried Leibniz
|
Jacob Bernoulli
|
Johann Bernoulli
|
Leonhard Euler
|
Joseph Louis Lagrange
|
Simeon Poisson
|
Michel Chasles
|
Hubert Anson Newton
|
Eliakim Hastings Moore
|
Oswald Veblen
|
Alonzo Church