ECE608 appendix B problems

1) B.1-2
2) B.1-4
3) B.1-5
4) B.2-3
5) B.4-1
6) B.4-3
7) B.5-4
(B.1-2) The first identity can be proved using induction. The base case is when $n = 2$.

$$A_1 \cap A_2 = \overline{A_1} \cup \overline{A_2}$$

To prove the base case, we need to prove that the sets on the left hand side and the
right hand side are subsets of each other.

Let $a \in A_1 \cap A_2$. Then we have that $a$ is not contained in atleast one of $A_1$ or $A_2$. If
this is the case, then $a$ is contained in either $\overline{A_1}$ or $\overline{A_2}$. So, $A_1 \cap A_2 \subseteq \overline{A_1} \cup \overline{A_2}$.

Now, to argue the other way around, let $b \in \overline{A_1} \cup \overline{A_2}$. Then we have that $b$ is lies
outside of either $A_1$ or $A_2$, which means it is not contained in their intersection. As a
result, $b \in A_1 \cap A_2$. Combining these two arguments, we have $A_1 \cap A_2 = \overline{A_1} \cup \overline{A_2}$ and
we have shown that the base case is true.

Let us now assume that the relationship is true for some $n = k$ i.e,

$$A_1 \cap \cdots \cap A_k = \overline{A_1} \cup \cdots \cup \overline{A_k}$$

and prove that $A_1 \cap \cdots \cap A_{k+1} = \overline{A_1} \cup \cdots \cup \overline{A_k} \cup \overline{A_{k+1}}$. This
can be accomplished in the following steps:

$$A_1 \cap \cdots \cap A_{k+1} = A_1 \cap \cdots \cap A_k \cup A_{k+1} = A_1 \cup \cdots \cup A_k \cup A_{k+1}$$

The first equality is from the base case and the second equality is from the induction
hypothesis. So, the first identity is true by the principle of mathematical induction. The
second part can be proved in an identical fashion.

(B.1-4) The set of odd natural numbers are countable if they can be put in one-to-one
correspondence with the set of natural numbers. $\{1, 3, 5, \cdots \}$ is the set of odd natural
numbers, and $\{(1 + 1)/2, (3 + 1)/2, (5 + 1)/2, \cdots \}$ is the set of natural numbers. In
general any odd number $o_n$ can be put in correspondence with the natural number
$(o_n + 1)/2$. Hence, the set of odd natural numbers is countable.

(B.1-5) Assume that $S = \{s_1, \cdots, s_{|S|}\}$. We can represent any subset of $V$ of $S$ using a
$|S|$ bit binary number such that the $k^{th}$ bit is 1 if $s_k \in V$ and 0 otherwise. Therefore, the
number of subsets of $S$ is the number of $|S|$ bit binary numbers which is equal to $2^{|S|}$.

(B.2-3) A binary relation on set $A$ can be represented as a subset of $A \times A$.
(a) We can construct an example with $A = \{a, b, c\}$ with all three elements being
distinct. Let $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$. This is reflexive and symmetric but
not transitive.

(b) Define $A = \{a, b, c\}$ with three distinct elements and
$R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. The relation $R$ is both reflexive and
transitive but not symmetric.

(c) Similar to the preceding parts, let $A = \{a, b\}$ with $a$ and $b$ distinct. Let
$R = \{(a, a)\}$. This relation is symmetric and transitive. But it is not reflexive since
$(b, b)$ is not in $R$. Reflexivity requires that $(s, s) \in R$ for all $s \in A$.

(B.4-1) Let each professor be represented using a node and the handshake between two
professors be denoted using an edge between the corresponding two nodes in the graph \( G = (V, E) \). The degree of each node is the number of edges that are incident on that node. The number of times each professor shakes hands is equal to the degree of the corresponding node. The sum of all the degrees is twice the number of edges because we count each edge twice. So we have \( \sum_{v \in V} \text{degree}(v) = 2|E| \). Since the sum of all degrees is equal to the sum of the number of times each professor shook hands, the sum is even.

(B.4-3) Let there be \(|V| = 2\) nodes. If this graph has to be connected, then there has to be atleast one edge. Therefore, \(|E| \geq |V| - 1\) is satisfied for the base case of two nodes. Assume that the inequality is satisfied by a general connected graph with \(|E| - 1\) edges and \(|V| - 1\) nodes. For a larger \( G = (V, E) \), assume the contrary that \( G \) is connected and \(|E| < |V| - 1\). From the problem B.4-1, we have \( \sum_{v \in V} \text{degree}(v) = 2|E| < 2|V| - 2 \).

So there is at least one vertex \( u \), whose degree is less than or equal to 1. If the degree of \( u \) is zero, then the graph is disconnected and we have a contradiction. If the degree of \( u \) is 1, then consider the graph \( \bar{G} \) obtained by removing \( u \) and the incident edge. \( \bar{G} \) has \(|E| - 1 < |V| - 2\). By applying the induction hypothesis to the smaller graph \( \bar{G} \), we say that \( \bar{G} \) is not connected. If \( \bar{G} \) is not connected, then adding \( u \) with just one incident edge will still not make \( G \) connected. So, there is again a contradiction from which we conclude that \(|E| \geq |V| - 1\). So we have shown the result through induction.

(B.5-4) We first prove that the number of nodes at depth \( d \) cannot exceed \( 2^d \). For \( d = 1 \), there can be a maximum of 2 nodes and therefore, the condition holds in this case. Assume that for depth \( k \), the number of nodes at that depth does not exceed \( 2^k \). The number of nodes at depth \( k + 1 \) can therefore not exceed \( 2^{k+1} \) because each of the possible \( 2^k \) nodes can only have two children. Adding up all the nodes in a tree of height \( h \), we have the bound \( n \leq 2^h - 1 \), which can be obtained by summing a finite geometric series. From that bound we have \( h \geq \lg(n + 1) \geq \lg(n) \geq [\lg(n)] \).