ECE608 APPENDIX 1 PROBLEMS

1) A.1-1
2) A.1-6
3) A.1-7
4) A.2-4
5) A.2-5
6) A-1

(1) CLR A.1-1

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = \frac{2n(n+1)}{2} - n = n(n+1) - n = n^2$$

(2) CLR A.1-6

Prove that $\sum_{k=1}^{n} O(f_k(n)) = O(\sum_{k=1}^{n} f_k(n))$.

Proof

Let $A = \sum_{k=1}^{n} O(f_k(n))$, and $B = O(\sum_{k=1}^{n} f_k(n))$.
Because A and B are sets of functions, we have to show $A \subseteq B$ and $B \subseteq A$.

(1) $A \subseteq B$

Let $g_k(n) = O(f_k(n))$ for $k \in \{1, 2, \ldots, n\}$. Then, there exist $c_k > 0$ and $n_k > 0$ such that

$$0 \le g_k(n) \le c_k f_k(n) \qquad \forall n \ge n_k$$

Choose $c = \max_{k \in \{1,2,\ldots,n\}}\{c_k\}$ and $n_0 = \max_{k \in \{1,2,\ldots,n\}}\{n_k\}$. Then, we have

$$0 \le g_k(n) \le c_k f_k(n) \le c f_k(n) \qquad \forall n \ge n_0$$

By the linear property of the summation,

$$0 \leq \sum_{k=1}^{n} g_k(n) \leq c \sum_{k=1}^{n} f_k(n) \qquad \forall n \geq n_0$$

Hence, $\sum_{k=1}^{n} g_k(n) = O(\sum_{k=1}^{n} f_k(n))$.
Because the above equation holds for an arbitrary $g_k(n)$, we have:

$$\sum_{k=1}^{n} O(f_k(n)) \subseteq O(\sum_{k=1}^{n} f_k(n))$$

.

(2) $(B \subseteq A)$
Let $g(n) = O(\sum_{k=1}^{n} f_k(n))$. Then, there exist $c > 0$ and $n_0 > 0$ such that

$$0 \leq g(n) \leq c \sum_{k=1}^{n} f_k(n) \qquad \forall n \geq n_0$$

By the linear property of summations,

$$0 \leq g(n) \leq \sum_{k=1}^{n} [c f_k(n)] \qquad \forall n \geq n_0$$

If we choose $c_k = c$ and $n_k = n_0$ for $k \in \{1, 2, \cdots, n\}$ in the above inequality, we obtain that there exist $c_k > 0$ and $n_k > 0$ for $k \in \{1, 2, \cdots, n\}$, such that $0 \leq g(n) \leq c_1 f_1(n) + c_2 f_2(n) + \cdots + c_n f_n(n)$ whenever $n \geq n_1, n \geq n_2, \cdots, n \geq n_n$.
This means that $g(n) = \sum_{k=1}^{n} O(f_k(n))$.
Therefore, $O(\sum_{k=1}^{n} f_k(n)) \subseteq \sum_{k=1}^{n} O(f_k(n))$.

From (1) and (2), we have $A = B$.

(3) CLR A.1-7

One method of solution uses a logarithm to convert the product to a sum:

$$\begin{aligned}
\prod_{k=1}^{n} 2 \cdot 4^k &= 2^n \prod_{k=1}^{n} 4^k \\
\lg(2^n \prod_{k=1}^{n} 4^k) &= \lg(2^n) + \lg(\prod_{k=1}^{n} 4^k) \\
&= \lg(2^n) + \sum_{k=1}^{n} \lg(4^k) \\
&= \lg(2^n) + \lg(4) \sum_{k=1}^{n} k
\end{aligned}$$

$$
\begin{aligned}
&= \lg(2^n) + 2\lg(2)\frac{n(n+1)}{2} \\
&= n\lg(2) + n(n+1)\lg(2) \\
&= n(n+2)\lg(2) \\
&= \lg(2^{n(n+2)})
\end{aligned}
$$

$$
\prod_{k=1}^{n} 2 \cdot 4^k = 2^{n(n+2)}.
$$

An alternative method of solution uses product and exponent rules:

$$
\begin{aligned}
\prod_{k=1}^{n} 2 \cdot 4^k &= 2^n \cdot 4^{\sum_{k=1}^{n} k} \\
&= 2^n \cdot 4^{\frac{n(n+1)}{2}} \\
&= 2^n \cdot 2^{n(n+1)} \\
&= 2^{n+n(n+1)} \\
&= 2^{n^2+2n} \\
&= 2^{n(n+2)}
\end{aligned}
$$

(4) CLR A.2-4

Evaluate $\sum_{k=1}^{n} k^3$

Since k^3 is a monotonically increasing function,

$$
\int_0^n x^3 dx \le \sum_{k=1}^{n} k^3 \le \int_1^{n+1} x^3 dx \le \int_0^{n+1} x^3 dx
$$

Hence:

$$
\frac{x^4}{4}\Big|_0^n \le \sum_{k=1}^{n} k^3 \le \frac{x^4}{4}\Big|_0^{n+1}
$$

Giving:

$$
\tfrac{n^4}{4} \le \sum_{k=1}^{n} k^3 \le \tfrac{(n+1)^4}{4}
$$

Hence, $\sum_{k=1}^{n} k^3 = \Theta(n^4)$

(5) CLR A.2-5

If you bound the summation using an integral directly (the integral bounds are from 0 to n) the final solutions becomes $[\mathit{lnn} - \mathit{ln0}]$.

However, $\mathit{ln0}$ is negative infinity.

So instead we break up the summation

$\sum_{k=1}^{n} 1/k$ and rewrite it as $\sum_{k=2}^{n} 1/k + 1$.

(6) CLR A-1

(a) Use the integral method of bounding the summation:

$$\int_0^n x^r \, dx \leq \sum_{k=1}^{n} k^r \leq \int_1^{n+1} x^r \, dx$$

$$\frac{1}{r+1} x^{r+1}|_0^n \leq \sum_{k=1}^{n} k^r \leq \frac{1}{r+1} x^{r+1}|_1^{n+1}$$

$$\frac{n^{r+1}}{r+1} \leq \sum_{k=1}^{n} k^r \leq \frac{(n+1)^{r+1} - 1}{r+1} \leq \frac{(n+1)^{r+1}}{r+1}$$

Therefore,

$$\sum_{k=1}^{n} k^r = \Theta(n^{r+1})$$

(b) On one hand,
$\sum_{k=1}^{n} \lg^s k \leq \sum_{k=1}^{n} \lg^s n = n \lg^s n$.
Clearly, $\sum_{k=1}^{n} \lg^s k = O(n \lg^s n)$.

On the other hand,
$\sum_{k=1}^{n} \lg^s k = \sum_{k=1}^{\lfloor n/2 \rfloor} \lg^s k + \sum_{k=\lfloor n/2 \rfloor + 1}^{n} \lg^s k$
$\geq \sum_{k=1}^{\lfloor n/2 \rfloor} 0 + \sum_{k=\lfloor n/2 \rfloor + 1}^{n} \lg^s (n/2)$
$\geq (n/2) \lg^s (n/2)$.
We can easily prove that $(n/2) \lg^s (n/2) = \theta(n \lg^s n)$.

So $\sum_{k=1}^{n} \lg^s k = \theta(n \lg^s n)$.

(c)

On one hand,

$$\sum_{k=1}^{n} k^r lg^s k \leq \sum_{k=1}^{n} k^r lg^s n = lg^s n \sum_{k=1}^{n} k^r$$

and from part (a)

$$\sum_{k=1}^{n} k^r = \Theta(n^{r+1})$$

so,

$$\sum_{k=1}^{n} k^r lg^s k = O(n^{r+1} lg^s n)$$

On the other hand,

$$\sum_{k=1}^{n} k^r lg^s k = \sum_{k=1}^{n/2} k^r lg^s k + \sum_{k=n/2}^{n} k^r lg^s k$$

$$\sum_{k=1}^{n} k^r lg^s k \geq \sum_{k=1}^{n/2} 0 + \sum_{k=n/2}^{n} (n/2)^r lg^s (n/2)$$

$$\sum_{k=1}^{n} k^r lg^s k \geq (n/2)^{r+1} lg^s (n/2)$$

And then combining the above two results it can be shown that,

$$\sum_{k=1}^{n} k^r lg^s k = \Theta(n^{r+1} lg^s n)$$