

1. Papoulis 7-15: n.b.  $U(a-X) = \mathbb{1}_{(-\infty, a]}(X)$  and  $U(b-Y) = \mathbb{1}_{(-\infty, b]}(Y)$

$$E\{U(a-X)\} = \int_{-\infty}^{\infty} \mathbb{1}_{(-\infty, a]}(x) f_X(x) dx = \int_{-\infty}^a f_X(x) dx = F_X(a). \quad \text{Similarly } E\{U(b-Y)\} = F_Y(b).$$

$$E\{U(a-X)U(b-Y)\} = \iint_{\mathbb{R}^2} \mathbb{1}_{(-\infty, a]}(x) \cdot \mathbb{1}_{(-\infty, b]}(y) dx dy = \iint_{-\infty}^a \int_{-\infty}^b f_{XY}(x,y) dx dy = F_{XY}(a,b)$$

If  $X$  and  $Y$  are stat indep  $\Rightarrow f_{XY}(a,b) = f_X(a)f_Y(b) \Rightarrow F_{XY}(a,b) = F_X(a)F_Y(b)$

$$\Rightarrow E\{U(a-X)U(b-Y)\} = E\{U(a-X)\} \cdot E\{U(b-Y)\}$$

conversely,  $E\{U(a-X)U(b-Y)\} = E\{U(a-X)\} \cdot E\{U(b-Y)\} \Rightarrow F_{XY}(a,b) = F_X(a)F_Y(b)$

$$\Rightarrow X \text{ and } Y \text{ stat. indep.}$$

2. Papoulis 7-17:  $Z = X + Y$ . Let  $W = X$  (change of variables) with aux. variable  $W$

Then  $f_{XZ}(x,z) = f_{WZ}(w,z) = f_{XY}(x, z(x,y)) \left| \frac{\partial(x,y)}{\partial(x,z)} \right|$ ,  $\left| \frac{\partial(x,y)}{\partial(x,z)} \right| = 1$

$$\Rightarrow f_{XZ}(x,z) = f_{XY}(x, z-x)$$

If  $X$  and  $Y$  are stat. indep., then  $f_{XY}(x,y) = f_X(x)f_Y(y)$

$$\Rightarrow f_{XZ}(x,z) = f_X(x)f_Y(z-x)$$

$$\therefore f_Z(z|x) = \frac{f_{XZ}(x,z)}{f_X(x)} = \frac{f_X(x)f_Y(z-x)}{f_X(x)} = f_Y(z-x)$$

3. We need to find  $f_{XY}(x,y) = f(x|y)$ . Note that  $f_{XY}(x,y)$  is a jointly Gaussian  $f_{XY}(x,y)$ . You can show this using char. fens. (n.b.  $\Phi_{XY}(w_1, w_2) = \Phi_{XN}(w_1, w_1+w_2) = \Phi_X(w_1) \Phi_N(w_1+w_2)$ )

$$\bar{X} = 0, \bar{Y} = 0, \text{var}(X) = \sigma_X^2, \text{var}(Y) = \sigma_X^2 + \sigma_N^2$$

$$\Gamma_{XY} = \frac{E\{XY\}}{\sigma_X \sigma_Y} = \frac{E\{X(X+N)\}}{\sigma_X \sigma_Y} = \frac{\sigma_X^2}{\sigma_X \sigma_Y} = \frac{\sigma_X}{\sigma_Y} = \frac{\sigma_X}{\sqrt{\sigma_X^2 + \sigma_N^2}}$$

Thus we have

$$f_{XY}(x,y) = \frac{1}{2\pi \sigma_X \sqrt{\sigma_X^2 + \sigma_N^2} \sqrt{1 - \left(\frac{\sigma_X}{\sqrt{\sigma_X^2 + \sigma_N^2}}\right)^2}} \exp \left\{ -\frac{1}{2 \left(1 - \frac{\sigma_X^2}{\sigma_X^2 + \sigma_N^2}\right)} \left[ \frac{x^2}{\sigma_X^2} - \frac{2\sigma_X x y}{(\sigma_X^2 + \sigma_N^2)\sigma_X} + \frac{y^2}{\sigma_X^2 + \sigma_N^2} \right] \right\}$$

$$= \frac{1}{2\pi \sigma_X \sigma_N} \exp \left\{ -\frac{(\sigma_X^2 + \sigma_N^2)}{2\sigma_N^2} \left[ \frac{x^2}{\sigma_X^2} - \frac{2xy}{\sigma_X^2 + \sigma_N^2} + \frac{y^2}{\sigma_Y^2} \right] \right\}, \quad \text{and } f_Y(y) = \frac{1}{\sqrt{2\pi(\sigma_X^2 + \sigma_N^2)}} \exp \left\{ -\frac{y^2}{2(\sigma_X^2 + \sigma_N^2)} \right\}$$



4. Papoulis 7-24: Recall that  $E\{g(X, Y)\} = E\{E\{g(X, Y) | X=x\}\}$

Thus if we wish to minimize  $E\{[Y - (AX+B)]^2\}$   
 $= E\{E\{[Y - (AX+B)]^2 | X=x\}\}$ , we should take

$$B = E\{Y\} - A E\{X\} = \mu_Y - A \mu_X$$

We then have that

$$\begin{aligned} Y - (AX+B) &= Y - [AX + \mu_Y - A\mu_X] \\ &= (Y - \mu_Y) - A[X - \mu_X] \end{aligned}$$

where  $A = a = \frac{E\{(X - \mu_X)(Y - \mu_Y)\}}{E\{(X - \mu_X)^2\}}$ , see Papoulis Eqs. (7-72) and (7-73)

5. We know that  $\Phi_{X,Y}(\omega_X, \omega_Y) = e^{i[\mu_X \omega_X + \mu_Y \omega_Y]} e^{-\frac{1}{2}[\sigma_X^2 \omega_X^2 + 2r \sigma_X \sigma_Y \omega_X \omega_Y + \sigma_Y^2 \omega_Y^2]}$   
 and if  $W$  and  $V$  are jointly Gaussian it has a char. fun. of the form  
 $\Phi_{V,W}(\omega_V, \omega_W) = e^{i[\mu_V \omega_V + \mu_W \omega_W]} e^{-\frac{1}{2}[\sigma_V^2 \omega_V^2 + 2\rho \sigma_V \sigma_W \omega_V \omega_W + \sigma_W^2 \omega_W^2]}$  (\*)  
 We must show that  $\Phi_{V,W}(\omega_V, \omega_W)$  has this form. By defn.

$$\begin{aligned} \Phi_{V,W}(\omega_V, \omega_W) &= E\{e^{i[\omega_V (aX+bY) + \omega_W (cX+dY)]}\} = E\{e^{i[(a\omega_V + c\omega_W)X + (b\omega_V + d\omega_W)Y]}\} \\ &= e^{i[(a\omega_V + c\omega_W)\mu_X + (b\omega_V + d\omega_W)\mu_Y]} \cdot \exp\left\{-\frac{1}{2}\left[\sigma_X^2 (a\omega_V + c\omega_W)^2 + \sigma_Y^2 (b\omega_V + d\omega_W)^2 + 2r \sigma_X \sigma_Y (a\omega_V + c\omega_W)(b\omega_V + d\omega_W)\right]\right\} \\ &= e^{i[(a\mu_X + b\mu_Y)\omega_V + (c\mu_X + d\mu_Y)\omega_W]} \cdot \exp\left\{-\frac{1}{2}\left[\sigma_X^2 (a^2 \omega_V^2 + 2ac \omega_V \omega_W + c^2 \omega_W^2) + \sigma_Y^2 (b^2 \omega_V^2 + 2bd \omega_V \omega_W + d^2 \omega_W^2) + 2r \sigma_X \sigma_Y (ab \omega_V^2 + (ad+bc)\omega_V \omega_W + cda)\right]\right\} \\ &= e^{i[(a\mu_X + b\mu_Y)\omega_V + (c\mu_X + d\mu_Y)\omega_W]} \cdot \exp\left\{-\frac{1}{2}\left[\omega_V^2 (a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2rab \sigma_X \sigma_Y) + \omega_W^2 (c^2 \sigma_X^2 + d^2 \sigma_Y^2 + 2rcd \sigma_X \sigma_Y) + 2\omega_V \omega_W (ac \sigma_X^2 + bd \sigma_Y^2 + r \sigma_X \sigma_Y (ad+bc))\right]\right\} \end{aligned}$$

which is the char. fun. of a Gaussian (has the form of (\*)).

Here

$$\mu_V = a\mu_X + b\mu_Y, \quad \sigma_V^2 = a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2rab \sigma_X \sigma_Y$$

$$\mu_W = c\mu_X + d\mu_Y, \quad \sigma_W^2 = c^2 \sigma_X^2 + d^2 \sigma_Y^2 + 2rcd \sigma_X \sigma_Y$$

$$\rho = \frac{\rho \sigma_V \sigma_W}{\sqrt{\sigma_V^2 \sigma_W^2}} = \frac{ac \sigma_X^2 + bd \sigma_Y^2 + (ad+bc) r \sigma_X \sigma_Y}{\sqrt{(a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2rab \sigma_X \sigma_Y)(c^2 \sigma_X^2 + d^2 \sigma_Y^2 + 2rcd \sigma_X \sigma_Y)}}$$

NATIONAL BRAND  
 43 390  
 43 392  
 43 393  
 43 394  
 43 395  
 43 396  
 43 397  
 43 398  
 43 399  
 43 400  
 43 401  
 43 402  
 43 403  
 43 404  
 43 405  
 43 406  
 43 407  
 43 408  
 43 409  
 43 410  
 43 411  
 43 412  
 43 413  
 43 414  
 43 415  
 43 416  
 43 417  
 43 418  
 43 419  
 43 420  
 43 421  
 43 422  
 43 423  
 43 424  
 43 425  
 43 426  
 43 427  
 43 428  
 43 429  
 43 430  
 43 431  
 43 432  
 43 433  
 43 434  
 43 435  
 43 436  
 43 437  
 43 438  
 43 439  
 43 440  
 43 441  
 43 442  
 43 443  
 43 444  
 43 445  
 43 446  
 43 447  
 43 448  
 43 449  
 43 450  
 43 451  
 43 452  
 43 453  
 43 454  
 43 455  
 43 456  
 43 457  
 43 458  
 43 459  
 43 460  
 43 461  
 43 462  
 43 463  
 43 464  
 43 465  
 43 466  
 43 467  
 43 468  
 43 469  
 43 470  
 43 471  
 43 472  
 43 473  
 43 474  
 43 475  
 43 476  
 43 477  
 43 478  
 43 479  
 43 480  
 43 481  
 43 482  
 43 483  
 43 484  
 43 485  
 43 486  
 43 487  
 43 488  
 43 489  
 43 490  
 43 491  
 43 492  
 43 493  
 43 494  
 43 495  
 43 496  
 43 497  
 43 498  
 43 499  
 43 500  
 43 501  
 43 502  
 43 503  
 43 504  
 43 505  
 43 506  
 43 507  
 43 508  
 43 509  
 43 510  
 43 511  
 43 512  
 43 513  
 43 514  
 43 515  
 43 516  
 43 517  
 43 518  
 43 519  
 43 520  
 43 521  
 43 522  
 43 523  
 43 524  
 43 525  
 43 526  
 43 527  
 43 528  
 43 529  
 43 530  
 43 531  
 43 532  
 43 533  
 43 534  
 43 535  
 43 536  
 43 537  
 43 538  
 43 539  
 43 540  
 43 541  
 43 542  
 43 543  
 43 544  
 43 545  
 43 546  
 43 547  
 43 548  
 43 549  
 43 550  
 43 551  
 43 552  
 43 553  
 43 554  
 43 555  
 43 556  
 43 557  
 43 558  
 43 559  
 43 560  
 43 561  
 43 562  
 43 563  
 43 564  
 43 565  
 43 566  
 43 567  
 43 568  
 43 569  
 43 570  
 43 571  
 43 572  
 43 573  
 43 574  
 43 575  
 43 576  
 43 577  
 43 578  
 43 579  
 43 580  
 43 581  
 43 582  
 43 583  
 43 584  
 43 585  
 43 586  
 43 587  
 43 588  
 43 589  
 43 590  
 43 591  
 43 592  
 43 593  
 43 594  
 43 595  
 43 596  
 43 597  
 43 598  
 43 599  
 43 600  
 43 601  
 43 602  
 43 603  
 43 604  
 43 605  
 43 606  
 43 607  
 43 608  
 43 609  
 43 610  
 43 611  
 43 612  
 43 613  
 43 614  
 43 615  
 43 616  
 43 617  
 43 618  
 43 619  
 43 620  
 43 621  
 43 622  
 43 623  
 43 624  
 43 625  
 43 626  
 43 627  
 43 628  
 43 629  
 43 630  
 43 631  
 43 632  
 43 633  
 43 634  
 43 635  
 43 636  
 43 637  
 43 638  
 43 639  
 43 640  
 43 641  
 43 642  
 43 643  
 43 644  
 43 645  
 43 646  
 43 647  
 43 648  
 43 649  
 43 650  
 43 651  
 43 652  
 43 653  
 43 654  
 43 655  
 43 656  
 43 657  
 43 658  
 43 659  
 43 660  
 43 661  
 43 662  
 43 663  
 43 664  
 43 665  
 43 666  
 43 667  
 43 668  
 43 669  
 43 670  
 43 671  
 43 672  
 43 673  
 43 674  
 43 675  
 43 676  
 43 677  
 43 678  
 43 679  
 43 680  
 43 681  
 43 682  
 43 683  
 43 684  
 43 685  
 43 686  
 43 687  
 43 688  
 43 689  
 43 690  
 43 691  
 43 692  
 43 693  
 43 694  
 43 695  
 43 696  
 43 697  
 43 698  
 43 699  
 43 700  
 43 701  
 43 702  
 43 703  
 43 704  
 43 705  
 43 706  
 43 707  
 43 708  
 43 709  
 43 710  
 43 711  
 43 712  
 43 713  
 43 714  
 43 715  
 43 716  
 43 717  
 43 718  
 43 719  
 43 720  
 43 721  
 43 722  
 43 723  
 43 724  
 43 725  
 43 726  
 43 727  
 43 728  
 43 729  
 43 730  
 43 731  
 43 732  
 43 733  
 43 734  
 43 735  
 43 736  
 43 737  
 43 738  
 43 739  
 43 740  
 43 741  
 43 742  
 43 743  
 43 744  
 43 745  
 43 746  
 43 747  
 43 748  
 43 749  
 43 750  
 43 751  
 43 752  
 43 753  
 43 754  
 43 755  
 43 756  
 43 757  
 43 758  
 43 759  
 43 760  
 43 761  
 43 762  
 43 763  
 43 764  
 43 765  
 43 766  
 43 767  
 43 768  
 43 769  
 43 770  
 43 771  
 43 772  
 43 773  
 43 774  
 43 775  
 43 776  
 43 777  
 43 778  
 43 779  
 43 780  
 43 781  
 43 782  
 43 783  
 43 784  
 43 785  
 43 786  
 43 787  
 43 788  
 43 789  
 43 790  
 43 791  
 43 792  
 43 793  
 43 794  
 43 795  
 43 796  
 43 797  
 43 798  
 43 799  
 43 800  
 43 801  
 43 802  
 43 803  
 43 804  
 43 805  
 43 806  
 43 807  
 43 808  
 43 809  
 43 810  
 43 811  
 43 812  
 43 813  
 43 814  
 43 815  
 43 816  
 43 817  
 43 818  
 43 819  
 43 820  
 43 821  
 43 822  
 43 823  
 43 824  
 43 825  
 43 826  
 43 827  
 43 828  
 43 829  
 43 830  
 43 831  
 43 832  
 43 833  
 43 834  
 43 835  
 43 836  
 43 837  
 43 838  
 43 839  
 43 840  
 43 841  
 43 842  
 43 843  
 43 844  
 43 845  
 43 846  
 43 847  
 43 848  
 43 849  
 43 850  
 43 851  
 43 852  
 43 853  
 43 854  
 43 855  
 43 856  
 43 857  
 43 858  
 43 859  
 43 860  
 43 861  
 43 862  
 43 863  
 43 864  
 43 865  
 43 866  
 43 867  
 43 868  
 43 869  
 43 870  
 43 871  
 43 872  
 43 873  
 43 874  
 43 875  
 43 876  
 43 877  
 43 878  
 43 879  
 43 880  
 43 881  
 43 882  
 43 883  
 43 884  
 43 885  
 43 886  
 43 887  
 43 888  
 43 889  
 43 890  
 43 891  
 43 892  
 43 893  
 43 894  
 43 895  
 43 896  
 43 897  
 43 898  
 43 899  
 43 900  
 43 901  
 43 902  
 43 903  
 43 904  
 43 905  
 43 906  
 43 907  
 43 908  
 43 909  
 43 910  
 43 911  
 43 912  
 43 913  
 43 914  
 43 915  
 43 916  
 43 917  
 43 918  
 43 919  
 43 920  
 43 921  
 43 922  
 43 923  
 43 924  
 43 925  
 43 926  
 43 927  
 43 928  
 43 929  
 43 930  
 43 931  
 43 932  
 43 933  
 43 934  
 43 935  
 43 936  
 43 937  
 43 938  
 43 939  
 43 940  
 43 941  
 43 942  
 43 943  
 43 944  
 43 945  
 43 946  
 43 947  
 43 948  
 43 949  
 43 950  
 43 951  
 43 952  
 43 953  
 43 954  
 43 955  
 43 956  
 43 957  
 43 958  
 43 959  
 43 960  
 43 961  
 43 962  
 43 963  
 43 964  
 43 965  
 43 966  
 43 967  
 43 968  
 43 969  
 43 970  
 43 971  
 43 972  
 43 973  
 43 974  
 43 975  
 43 976  
 43 977  
 43 978  
 43 979  
 43 980  
 43 981  
 43 982  
 43 983  
 43 984  
 43 985  
 43 986  
 43 987  
 43 988  
 43 989  
 43 990  
 43 991  
 43 992  
 43 993  
 43 994  
 43 995  
 43 996  
 43 997  
 43 998  
 43 999  
 43 1000



Problem 7 Addendum:

From the discussion in section 8-2 of Papoulis, we have that the joint char. fn. of the jointly Gaussian RVs  $X_1, \dots, X_n$  is

$$\Phi_{X_1, \dots, X_n}(w_1, \dots, w_n) = e^{i \sum_{j=1}^n \eta_j w_j} \cdot \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n w_j w_k C_{jk} \right\}$$

$$\text{where } C_{jk} = \text{cov}(X_j, X_k) = r_{X_j X_k} \sigma_{X_j} \sigma_{X_k}$$

$$\eta_j = E\{X_j\}$$

In problem 7 we have three joint Gaussians  $X, Y,$  and  $Z$ . Hence we have (noting that  $r_{XY} = r_{XZ} = r_{YZ} = 0$ )

$$\begin{aligned} \Phi_{XYZ}(w_1, w_2, w_3) &= e^{i(\eta_X w_1 + \eta_Y w_2 + \eta_Z w_3)} \cdot \exp \left\{ -\frac{1}{2} \left[ w_1^2 \sigma_X^2 + w_1 w_2 \cancel{r_{XZ} \sigma_X \sigma_Z} + w_1 w_3 \cancel{r_{XZ} \sigma_X \sigma_Z} \right. \right. \\ &\quad \left. \left. + w_1 w_3 \cancel{r_{XZ} \sigma_X \sigma_Z} + w_2 w_1 \cancel{r_{YZ} \sigma_Y \sigma_X} + w_2^2 \sigma_Y^2 + w_2 w_3 \cancel{r_{YZ} \sigma_Y \sigma_Z} \right. \right. \\ &\quad \left. \left. + w_3 w_1 \cancel{r_{ZX} \sigma_Z \sigma_X} + w_3 w_2 \cancel{r_{ZY} \sigma_Z \sigma_Y} + w_3^2 \sigma_Z^2 \right] \right\} \\ &= e^{i(\eta_X w_1 + \eta_Y w_2 + \eta_Z w_3)} \cdot e^{-\frac{1}{2} [\sigma_X^2 w_1^2 + \sigma_Y^2 w_2^2 + \sigma_Z^2 w_3^2]} \\ &= \left[ e^{i \eta_X w_1} e^{-\frac{1}{2} \sigma_X^2 w_1^2} \right] \cdot \left[ e^{i \eta_Y w_2} e^{-\frac{1}{2} \sigma_Y^2 w_2^2} \right] \cdot \left[ e^{i \eta_Z w_3} e^{-\frac{1}{2} \sigma_Z^2 w_3^2} \right] \\ &= \Phi_X(w_1) \Phi_Y(w_2) \Phi_Z(w_3) \end{aligned}$$

$$\Rightarrow f_{XYZ}(x, y, z) = f_X(x) f_Y(y) f_Z(z)$$

$$\Rightarrow X, Y, \text{ and } Z \text{ are stat. indep.}$$

8. (Papoulis 7-4)

Define  $Z = X_1 + X_2 + X_3$ , where  $X_1, X_2$  and  $X_3$  are i.i.d. RVs with pdf

$$f_{X_i}(x) = \begin{cases} 1 & (x) \\ [-1/2, 1/2] \end{cases}, \quad i=1, 2, 3.$$

The characteristic function for each  $X_i$  is

$$\begin{aligned} \underline{\Phi}_{X_i}(\omega) &= E[e^{i\omega X_i}] = \int f_{X_i}(x) e^{i\omega x} dx \\ &= \int_{-1/2}^{1/2} e^{i\omega x} dx = \left. \frac{e^{i\omega x}}{i\omega} \right|_{-1/2}^{1/2} = \frac{e^{i\omega/2} - e^{-i\omega/2}}{i2(\omega/2)} \\ &= \frac{\sin(\omega/2)}{(\omega/2)} = \frac{2 \sin(\omega/2)}{\omega}, \quad i=1, 2, 3. \end{aligned}$$

Thus it follows that since  $X_1, X_2$  and  $X_3$  are statistically independent

$$\underline{\Phi}_Z(\omega) = \underline{\Phi}_{X_1}(\omega) \cdot \underline{\Phi}_{X_2}(\omega) \cdot \underline{\Phi}_{X_3}(\omega) = \left[ \frac{2 \sin(\omega/2)}{\omega} \right]^3$$

$$\begin{aligned} \text{Now } \underline{\Phi}_Z(\omega) &= E[e^{i\omega Z}] = E\left[ \sum_{k=0}^{\infty} \frac{(i\omega Z)^k}{k!} \right] = \sum_{k=0}^{\infty} \frac{(i\omega)^k}{k!} E[Z^k] \\ &= \sum_{k=0}^{\infty} \frac{i^k \omega^k}{k!} m_k = \sum_{k=0}^{\infty} a_k \omega^k, \end{aligned}$$

where  $m_k = E[X^k]$  is the  $k$ -th moment of  $Z$ ,

and  $a_k = \frac{i^k m_k}{k!}$ . Thus we have  $E[X^k] = m_k = \frac{a_k \cdot k!}{i^k}$ .

Now expanding  $\underline{\Phi}_{X_i}(\omega)$  about  $\omega=0$  as a Taylor series yields

$$\underline{\Phi}_{X_i}(\omega) = \frac{2 \sin(\omega/2)}{\omega} = 1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \dots$$

Thus we have

$$\begin{aligned} \underline{\Phi}_Z(\omega) &= \left[ \frac{2 \sin(\omega/2)}{\omega} \right]^3 = \left[ 1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \dots \right]^3 \\ &= 1 - \frac{\omega^2}{8} + \frac{13}{1920} \omega^4 - \dots \end{aligned}$$

The coefficient on  $\frac{\omega^4}{8}$  in this expansion is  $a_k = \frac{13}{1920}$

$$\therefore E[Z^4] = \frac{a_4 \cdot 4!}{i^4} = \frac{13(4!)}{1920} \cdot \frac{1}{(i)^4} = \boxed{\frac{11}{80}}$$

9. Papoulis 7-7 (8-7 in 2nd edition)

We wish to show that

$$\begin{aligned} E[X_1 \cdot X_2 | X_3] &= E \left[ E[X_1 \cdot X_2 | X_2, X_3] | X_3 \right] \\ &= E \left[ X_2 \cdot E[X_1 | X_2, X_3] | X_3 \right] \end{aligned}$$

We start by noting that

$$E[X_1 \cdot X_2 | X_3 = x_3] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X_1, X_2}(x_1, x_2 | X_3 = x_3) dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} x_1 x_2 f_{X_1}(x_1 | X_2 = x_2, X_3 = x_3) f_{X_2}(x_2 | X_3 = x_3) dx_1 dx_2$$

(because  $f_{X_1, X_2}(x_1, x_2 | X_3 = x_3) = f_{X_1}(x_1 | X_2 = x_2, X_3 = x_3) \cdot f_{X_2}(x_2 | X_3 = x_3)$ )

$$= \int_{-\infty}^{\infty} f_{X_2}(x_2 | X_3 = x_3) \int_{-\infty}^{\infty} x_1 x_2 f_{X_1}(x_1 | X_2 = x_2, X_3 = x_3) dx_1 dx_2 \quad \dots (*)$$

$$= \int_{-\infty}^{\infty} f_{X_2}(x_2 | X_3 = x_3) \cdot E[X_1 \cdot X_2 | X_2 = x_2, X_3 = x_3] dx_2 = \varphi_1(x_3)$$

$$\Rightarrow E[X_1 \cdot X_2 | X_3] = \int_{-\infty}^{\infty} f_{X_2}(x_2 | X_3) \cdot E[X_1 \cdot X_2 | X_2 = x_2, X_3] dx_2$$

$$= \varphi_1(X_3) = E \left[ E[X_1 \cdot X_2 | X_2, X_3] | X_3 \right]$$

(proving first result)

Also from (\*)

$$E[X_1 \cdot X_2 | X_3 = x_3] = \int_{-\infty}^{\infty} x_2 f_{X_2}(x_2 | X_3 = x_3) \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1 | X_2 = x_2, X_3 = x_3) dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} x_2 f_{X_2}(x_2 | X_3 = x_3) \cdot E[X_1 | X_2 = x_2, X_3 = x_3] dx_2$$

$$= E \left[ X_2 \cdot E[X_1 | X_2, X_3 = x_3] | X_3 = x_3 \right] = \varphi_2(x_3)$$

from which it follows from substituting the RV  $X_3$  for  $x_3$ , we have

$$E[X_1 \cdot X_2 | X_3] = \varphi_2(X_3) = E \left[ X_2 \cdot E[X_1 | X_2, X_3] | X_3 \right]$$

(proving second result)