1. Papoulis 7-15: n.b. $U(a-X) = 1_{[-\infty,a]}(X)$ and $U(b-Y) = 1_{[-\infty,b]}(Y)$

$E\{U(a-X)\} = \int_{-\infty}^{\infty} 1_{[-\infty,a]}(x) f_X(x)\,dx = \int_{-\infty}^{a} f_X(x)\,dx = F_X(a)$. Similarly $E\{U(b-Y)\} = F_Y(b)$.

$E\{U(a-X)U(b-Y)\} = \iint_{\mathbb{R}^2} 1_{[-\infty,a]}(x)\cdot 1_{[-\infty,b]}(y)\,dx\,dy = \int_{-\infty}^{b}\int_{-\infty}^{a} f_{XY}(x,y)\,dx\,dy = F_X(a,b)$

If X and Y are stat indep $\Rightarrow f_{XY}(a,b) = f_X(a) f_Y(b) \Rightarrow F_{XY}(a,b) = F_X(a)F_Y(b)$.

$\Rightarrow E\{U(a-X)U(b-Y)\} = E\{U(a-X)\}\cdot E\{U(b-Y)\}$

Conversely, $E\{U(a-X)U(b-Y)\} = E\{U(a-X)\}\cdot E\{U(b-Y)\} \Rightarrow F_{XY}(a,b) = F_X(a)F_Y(b)$

$\Rightarrow$ X and Y stat. indep.

2. Papoulis 7-17: $Z = X + Y$. Let $W = X$ (change of variables with aux. variable W)

Then $f_{XZ}(x,z) = f_{WZ}(w,z) = f_{XY}(x, z(x,y)) \left|\frac{\partial(x,y)}{\partial(x,z)}\right|$, $\left|\frac{\partial(x,y)}{\partial(x,z)}\right| = 1$

$\Rightarrow f_{XZ}(x,z) = f_{XY}(x, z-x)$

If X and Y are stat. indep., then $f_{XY}(x,y) = f_X(x) f_Y(y)$

$\Rightarrow f_{XZ}(x,z) = f_X(x) f_Y(z-x)$

$\therefore f_Z(z|x) = \frac{f_{XZ}(x,z)}{f_X(x)} = \frac{f_X(x) f_Y(z-x)}{f_X(x)} = f_Y(z-x)$

3. We need to find $\frac{f_{XY}(x,y)}{f_Y(y)} = f(x|y)$. Note that $f_{XY}(x,y)$ is a jointly Gaussian pdf. You can show this using char. fcns. (n.b. $\Phi_{XY}(\omega_1,\omega_2) = \Phi_{XN}(\omega_1, \omega_1+\omega_2) = \Phi_X(\omega_1)\Phi_N(\omega_1+\omega_2)$)

$\bar{X} = 0$, $\bar{Y} = 0$, $var(X) = \sigma_X^2$, $var(Y) = \sigma_X^2 + \sigma_N^2$

$r_{XY} = \frac{E\{XY\}}{\sigma_X\sigma_Y} = \frac{E\{X(X+N)\}}{\sigma_X\sigma_Y} = \frac{\sigma_X^2}{\sigma_X\sigma_Y} = \frac{\sigma_X}{\sigma_Y} = \frac{\sigma_X}{\sqrt{\sigma_X^2+\sigma_N^2}}$

Thus we have

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sqrt{\sigma_X^2+\sigma_N^2}\sqrt{1-\underbrace{(\sigma_X/\sigma_Y)^2}_{\sigma_X^2/(\sigma_X^2+\sigma_N^2)}}} \exp\left\{\frac{-1}{2\left(1-\frac{\sigma_X^2}{\sigma_Y^2}\right)}\left[\frac{x^2}{\sigma_X^2} - \frac{2\sigma_X xy}{(\sqrt{\sigma_X^2+\sigma_N^2})^2\sigma_X} + \frac{y^2}{\sigma_X^2+\sigma_N^2}\right]\right\}$$

$$= \frac{1}{2\pi\sigma_X\sigma_N}\exp\left\{-\frac{(\sigma_X^2+\sigma_N^2)}{2\sigma_N^2}\left[\frac{x^2}{\sigma_X^2} - \frac{2xy}{\sigma_X^2+\sigma_N^2} + \frac{y^2}{\sigma_Y^2}\right]\right\}$$ and $$f_Y(y) = \frac{1}{\sqrt{2\pi(\sigma_X^2+\sigma_N^2)}}\exp\left\{\frac{-y^2}{2(\sigma_X^2+\sigma_N^2)}\right\}$$

n.b $f(x|y) = \frac{f(x,y)}{f_Y(y)} = \cdots = \frac{1}{\sqrt{2\pi\left(\frac{\sigma_x^2\sigma_N^2}{\sigma_x^2+\sigma_N^2}\right)}} \exp\left\{-\frac{(\sigma_x^2+\sigma_N^2)}{2\sigma_x^2\sigma_N^2}\left[x-\left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}\right)y\right]^2\right\}$

$f(x|y) \sim \mathcal{N}\left[\frac{\sigma_x^2 y}{\sigma_x^2+\sigma_N^2}, \sqrt{\frac{\sigma_x^2\sigma_N^2}{\sigma_x^2+\sigma_N^2}}\right]$, a Gaussian with the specified mean and standard deviation

We can now find the MMS and MAP estimators in a straightforward manner

(a) The MMS estimator $\hat{x}_{MMS}(y)$ is given by the conditional expectation

$$\hat{x}_{MMS}(y) = E\{X|Y=y\} = \int_{-\infty}^{\infty} x f(x|y)\,dx = \boxed{\frac{\sigma_x^2 y}{\sigma_x^2+\sigma_Y^2}}$$

(b) The MAP estimator is given by

$$\hat{x}_{MAP}(y) = \arg\max_{x\in\mathbb{R}} \{f(x|y)\}$$

n.b. We can write $f(x|y)$ as

$$f(x|y) = K_1 \exp\left\{-K_2\left[x - \frac{\sigma_x^2 y}{\sigma_x^2+\sigma_N^2}\right]^2\right\}$$

where $K_1, K_2 > 0$.

Because the exponential is a monotonically increasing fcn., it takes on its maximum value when its argument takes on its max. value. This happens when $\left[x - \frac{\sigma_x^2 y}{\sigma_x^2+\sigma_N^2}\right]^2 = 0$

which occurs when $x = \frac{\sigma_x^2 y}{\sigma_x^2+\sigma_N^2}$

$$\therefore \quad \hat{x}_{MAP}(y) = \frac{\sigma_x^2 y}{\sigma_x^2+\sigma_N^2}$$

n.b $\hat{x}_{MMS}(y) = \hat{x}_{MAP}(y)$ in this case. Note also that

$\sigma_x^2 >> \sigma_N^2$ (reliable measurement): $\hat{x}(y) \simeq 1\cdot y = y$ (observed value)

$\sigma_x^2 << \sigma_N^2$ (noisy measurement): $\hat{x}(y) \simeq 0 \cdot y = 0$ (prior mean)

4. Papoulis 7-24: Recall that $E\{g(X,Y)\} = E\{E\{g(X,Y)|X=x\}\}$

Thus if we wish to minimize $E\{[Y-(AX+B)]^2\}$

$= E\{E\{[Y-(AX+B)]^2|X=x\}\}$, we should take

$$B = E\{Y\} - AE\{X\} = \eta_Y - A\eta_X$$

We then have that

$$Y-(AX+B) = Y - [AX + \eta_Y - A\eta_X]$$
$$= (Y-\eta_Y) - A[X-\eta_X]$$

where $A = a = \dfrac{E\{(X-\eta_X)(Y-\eta_Y)\}}{E\{(X-\eta_X)^2\}}$, see Papoulis Eqns. (7-72) and (7-7…

5. We know that $\Phi_{XY}(\omega_X,\omega_Y) = e^{i[\eta_X\omega_X+\eta_Y\omega_Y]} e^{-\frac{1}{2}[\sigma_X^2\omega_X^2 + 2r\sigma_X\sigma_Y\omega_X\omega_Y+\sigma_Y^2 \ldots}$

and if W and V are jointly Gaussian it has a char ftn of the form

$$\Phi_{VW}(\omega_V,\omega_W) = e^{i[\eta_V\omega_V+\eta_W\omega_W]} e^{-\frac{1}{2}[\sigma_V^2\omega_V^2+2\rho\sigma_V\sigma_W\omega_V\omega_W+\sigma_W^2\omega_W^2]} \quad (*)$$

We must show that $\Phi_{VW}(\omega_V,\omega_W)$ has this form. By defn.

$$\Phi_{VW}(\omega_V,\omega_W) = E\{e^{i[\omega_V(aX+bY)+\omega_W(cX+dY)]}\} = E\{e^{i[(a\omega_V+c\omega_W)X+(b\omega_V+d\omega_W)Y \ldots}\}$$

$$= e^{i[(a\omega_V+c\omega_W)\eta_X+(b\omega_V+d\omega_W)\eta_Y]}\cdot\exp\left\{-\frac{1}{2}\left[\begin{array}{l}\sigma_X^2(a\omega_V+c\omega_W)^2+\sigma_Y^2(b\omega_V+d\omega_W)^2\\ +2r\sigma_X\sigma_Y(a\omega_V+c\omega_W)(b\omega_V+d\omega_W)\end{array}\right]\right\}$$

$$= e^{i[(a\eta_X+b\eta_Y)\omega_V+(c\eta_X+d\eta_Y)\omega_W]}\cdot\exp\left\{-\frac{1}{2}\left[\begin{array}{l}\sigma_X^2(a^2\omega_V^2+2ac\,\omega_V\omega_W+c^2\omega_W^2)\\ +\sigma_Y^2(b^2\omega_V^2+2bd\,\omega_V\omega_W+d^2\omega_W^2)\\ +2r\sigma_X\sigma_Y(ab\,\omega_V^2+(ad+bc)\omega_V\omega_W+cd\ldots\end{array}\right.\right.$$

$$= e^{i[(a\eta_X+b\eta_Y)\omega_V+(c\eta_X+d\eta_Y)\omega_W]}\cdot\exp\left\{-\frac{1}{2}\left[\begin{array}{l}\omega_V^2(a^2\sigma_X^2+b^2\sigma_Y^2+2rab\,\sigma_X\sigma_Y)\\ +\omega_W^2(c^2\sigma_X^2+d^2\sigma_Y^2+2rcd\,\sigma_X\sigma_Y)\\ +2\omega_V\omega_W(ac\,\sigma_X^2+bd\,\sigma_Y^2+r\sigma_X\sigma_Y(ad\ldots\end{array}\right.\right.$$

which is the char ftn. of a Gaussian (has the form of (*)).

Here
$$\eta_V = a\eta_X+b\eta_Y, \quad \sigma_V^2 = a^2\sigma_X^2+b^2\sigma_Y^2+2rab\,\sigma_X\sigma_Y$$
$$\eta_W = c\eta_X+d\eta_Y, \quad \sigma_W^2 = c^2\sigma_X^2+d^2\sigma_Y^2+2rcd\,\sigma_X\sigma_Y$$
$$\rho = \frac{\rho\sigma_V\sigma_W}{\sqrt{\sigma_V^2\sigma_W^2}} = \frac{ac\,\sigma_X^2+bd\,\sigma_Y^2+(ad+bc)\,r\,\sigma_X\sigma_Y}{\sqrt{(a^2\sigma_X^2+b^2\sigma_Y^2+2rab\,\sigma_X\sigma_Y)(c^2\sigma_X^2+d^2\sigma_Y^2+2rcd\,\sigma_X\sigma_Y)}}$$

6. <u>Papoulis 8-2</u>: Define the zero-one RVs
$X_A = 1_A(\omega)$, $X_B = 1_B(\omega)$, $X_C = 1_C(\omega)$, $\omega \in S$.

$P(\{X_A=1\} \cap \{X_B=1\} \cap \{X_C=1\}) = P(A \cap B \cap C) = \frac{1}{4}$

$$\left.\begin{array}{l} P(\{X_A=1\}) = P(A) = \frac{1}{2} \\ P(\{X_B=1\}) = P(B) = \frac{1}{2} \\ P(\{X_C=1\}) = P(C) = \frac{1}{2} \end{array}\right\} \Rightarrow P(\{X_A=1\})P(\{X_B=1\})P(\{X_C=1\}) = \frac{1}{8}$$

Thus $P(\{X_A=1\} \cap \{X_B=1\} \cap \{X_C=1\}) \neq P(\{X_A=1\})P(\{X_B=1\})P(\{X_C=1\})$

$\Rightarrow$ X_A, X_B, X_C are <u>not</u> stat. indep.

However $P(\{X_A=1\} \cap \{X_B=1\}) = P(A \cap B) = \frac{1}{4} = P(\{X_A=1\})P(\{X_B=1\})$
and this will be true for all combination of the form $P(\{X_Q = b_Q\} \cap \{X_R = b_R\}) = 1/4$, for $Q, R \in \{A, B, C\}$ and $b_Q, b_R \in \{0, 1\}$.

7. <u>Papoulis 8-3</u>: If X, Y, and Z are independent in pairs, then

$$r_{XY} = 0, \quad r_{XZ} = 0, \quad r_{YZ} = 0$$

and the char. fn. $\Phi_{XYZ}(\omega_1, \omega_2, \omega_3)$ is given by

$$\Phi_{XYZ}(\omega_1, \omega_2, \omega_3) = \exp\{i[\omega_1 \eta_x + \omega_2 \eta_y + \omega_3 \eta_z]\} \cdot \exp\{-\tfrac{1}{2}[\sigma_x^2 \omega_1^2 + \sigma_y^2 \omega_2^2 + \sigma_z^2 \omega_3^2]\}$$

See Prob. 7 Addendum on Page 4(b)

$$= \exp\{i\omega_1\eta_x - \tfrac{1}{2}\omega_1^2\sigma_x^2\} \cdot \exp\{i\omega_2\eta_y - \tfrac{1}{2}\omega_2^2\sigma_y^2\} \cdot \exp\{i\omega_3\eta_z - \tfrac{1}{2}\omega_3^2\sigma_z^2\}$$

$$= \Phi_x(\omega_1)\Phi_y(\omega_2)\Phi_z(\omega_3)$$

$$\Rightarrow f_{XYZ}(x, y, z) = f_x(x) f_y(y) f_z(z)$$

$\Rightarrow$ X, Y, and Z are stat. indep. Gaussians,

Problem 7 Addendum:

From the discussion in section 8-2 of Papoulis, we have that the joint char. ftn. of the jointly Gaussian RVs $X_1, \ldots, X_n$ is

$$\Phi_{X_1,\ldots,X_n}(\omega_1,\ldots,\omega_n) = e^{i\sum_{j=1}^{n} \eta_j \omega_j} \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3} \omega_j \omega_k C_{jk}\right\}$$

$$\text{where} \quad C_{jk} = \operatorname{cov}(X_j, X_k) = r_{X_j X_k}\sigma_{X_j}\sigma_{X_k}$$
$$\eta_j = E\{X_j\}$$

In problem 7 we have three joint Gaussians X, Y, and Z. Hence we have (noting that $r_{XY} = r_{XZ} = r_{YZ} = 0$)

$$\Phi_{XYZ}(\omega_1,\omega_2,\omega_3) = e^{i(\eta_x \omega_1 + \eta_y \omega_2 + \eta_z \omega_3)} \cdot \exp\Big\{-\frac{1}{2}\Big[\omega_1^2\sigma_X^2 + \omega_1\omega_2 \cancelto{0}{r_{XY}}\sigma_X\sigma_Y$$
$$+ \omega_1\omega_3 \cancelto{0}{r_{XZ}}\sigma_X\sigma_Z + \omega_2\omega_1 \cancelto{0}{r_{YX}}\sigma_Y\sigma_X + \omega^2\sigma_Y^2 + \omega_2\omega_3 \cancelto{0}{r_{YZ}}\sigma_Y\sigma_Z$$
$$+ \omega_3\omega_1 \cancelto{0}{r_{ZX}}\sigma_Z\sigma_X + \omega_3\omega_2 \cancelto{0}{r_{ZY}}\sigma_Z\sigma_Y + \omega_3^2\sigma_Z^2\Big]\Big\}$$

$$= e^{i(\eta_X\omega_1 + \eta_Y\omega_2 + \eta_Z\omega_3)} \cdot e^{-\frac{1}{2}[\sigma_X^2\omega_1^2 + \sigma_Y^2\omega_2^2 + \sigma_Z^2\omega_3^2]}$$

$$= \left[e^{i\eta_X\omega_1} e^{-\frac{1}{2}\sigma_X^2\omega_1^2}\right]\cdot\left[e^{i\eta_Y\omega_2} e^{-\frac{1}{2}\sigma_Y^2\omega_2^2}\right]\cdot\left[e^{i\eta_Z\omega_3} e^{-\frac{1}{2}\sigma_Z^2\omega_3^2}\right]$$

$$= \Phi_X(\omega_1)\,\Phi_Y(\omega_2)\,\Phi_Z(\omega_3)$$

$$\Rightarrow f_{XYZ}(x,y,z) = f_X(x)\,f_Y(y)\,f_Z(z)$$

$\Rightarrow$ X, Y, and Z are stat. indep.

8. (Papoulis 7-4)

Define $\mathbb{Z} = \mathbb{X}_1 + \mathbb{X}_2 + \mathbb{X}_3$, where $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ are i.i.d. RVs with pdf

$$f_{\mathbb{X}_i}(x) = 1_{[-1/2,1/2]}(x), \quad i = 1, 2, 3.$$

The characteristic function for each $\mathbb{X}_i$ is

$$\Phi_{\mathbb{X}_i}(\omega) = E[e^{i\omega \mathbb{X}_i}] = \int_{\mathbb{R}} f_{\mathbb{X}_i}(x) e^{+i\omega x} dx$$

$$= \int_{-1/2}^{1/2} e^{i\omega x} dx = \frac{e^{i\omega x}}{i\omega}\Bigg|_{-1/2}^{1/2} = \frac{e^{i\omega/2} - e^{-i\omega/2}}{i2(\omega/2)}$$

$$= \frac{\sin(\omega/2)}{(\omega/2)} = \frac{2\sin(\omega/2)}{\omega}, \quad i = 1, 2, 3.$$

Thus it follows that since $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ are statistically independent

$$\Phi_{\mathbb{Z}}(\omega) = \Phi_{\mathbb{X}_1}(\omega) \cdot \Phi_{\mathbb{X}_2}(\omega) \cdot \Phi_{\mathbb{X}_3}(\omega) = \left[\frac{2\sin(\omega/2)}{\omega}\right]^3$$

Now $$\Phi_{\mathbb{Z}}(\omega) = E\left[e^{i\omega \mathbb{Z}}\right] = E\left[\sum_{k=0}^{\infty} \frac{(i\omega \mathbb{Z})^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{(i\omega)^k}{k!} E[\mathbb{Z}^k]$$

$$= \sum_{k=0}^{\infty} \frac{i^k \omega^k}{k!} \cdot m_k = \sum_{k=0}^{\infty} a_k \omega^k,$$

where $m_k = E[\mathbb{X}^k]$ is the k-th moment of $\mathbb{Z}$, and $a_k = \frac{i^k m_k}{k!}$. Thus we have $E[\mathbb{X}^k] = m_k = \frac{a_k \cdot k!}{i^k}$.

Now expanding $\Phi_{\mathbb{X}_i}(\omega)$ about $\omega = 0$ as a Taylor series yields

$$\Phi_{\mathbb{X}_i}(\omega) = \frac{2\sin(\omega/2)}{\omega} = 1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \cdots$$

Thus we have

$$\Phi_{\mathbb{Z}}(\omega) = \left[\frac{2\sin(\omega/2)}{\omega}\right]^3 = \left[1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \cdots\right]^3$$

$$= 1 - \frac{\omega^2}{8} + \frac{13}{1920}\omega^4 - \cdots$$

The coefficient on ω^4 in this expansion is $a_k = \frac{13}{1920}$

$$\therefore E[\mathbb{Z}^4] = \frac{a_4 \cdot 4!}{i^4} = \frac{13(4!)}{1920} \cdot \frac{1}{(1)} = \boxed{\frac{11}{80}}$$

9. Papoulis 7-7 (8-7 in 3rd edition)

We wish to show that

$$E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_3] = E\big[\, E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_2, \mathbb{X}_3] \,\big|\, \mathbb{X}_3 \big]$$

$$= E\big[\, \mathbb{X}_2 \cdot E[\mathbb{X}_1 \mid \mathbb{X}_2, \mathbb{X}_3] \,\big|\, \mathbb{X}_3 \big]$$

We start by noting that

$$E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_3 = x_3] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{\mathbb{X}_1 \mathbb{X}_2}(x_1, x_2 \mid \mathbb{X}_3 = x_3)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} x_1 x_2 f_{\mathbb{X}_1}(x_1 \mid \mathbb{X}_2 = x_2, \mathbb{X}_3 = x_3)\, f_{\mathbb{X}_2}(x_2 \mid \mathbb{X}_3 = x_3)\, dx_1\, dx_2$$

(because $f_{\mathbb{X}_1 \mathbb{X}_2}(x_1, x_2 \mid \mathbb{X}_3 = x_3) = f_{\mathbb{X}_1}(x_1 \mid \mathbb{X}_2 = x_2, \mathbb{X}_3 = x_3) \cdot f_{\mathbb{X}_2}(x_2 \mid \mathbb{X}_3 = x_3)$)

$$= \int_{-\infty}^{\infty} f_{\mathbb{X}_2}(x_2 \mid \mathbb{X}_3 = x_3) \int_{-\infty}^{\infty} x_1 x_2 f_{\mathbb{X}_1}(x_1 \mid \mathbb{X}_2 = x_2, \mathbb{X}_3 = x_3)\, dx_1\, dx_2 \qquad ---(*)$$

$$= \int_{-\infty}^{\infty} f_{\mathbb{X}_2}(x_2 \mid \mathbb{X}_3 = x_3) \cdot E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_2 = x_2, \mathbb{X}_3 = x_3]\, dx_2 = \varphi_1(x_3)$$

$$\Rightarrow E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_3] = \int_{-\infty}^{\infty} f_{\mathbb{X}_2}(x_2 \mid \mathbb{X}_3) \cdot E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_2 = x_2, \mathbb{X}_3]\, dx_2$$

$$= \varphi_1(\mathbb{X}_3) = E\big[\, E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_2, \mathbb{X}_3] \,\big|\, \mathbb{X}_3 \big]$$

(proving first result)

Also from (*)

$$E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_3 = x_3] = \int_{-\infty}^{\infty} x_2 f_{\mathbb{X}_2}(x_2 \mid \mathbb{X}_3 = x_3) \int_{-\infty}^{\infty} x_1 f_{\mathbb{X}_1}(x_1 \mid \mathbb{X}_2 = x_2, \mathbb{X}_3 = x_3)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} x_2 f_{\mathbb{X}_2}(x_2 \mid \mathbb{X}_3 = x_3) \cdot E[\mathbb{X}_1 \mid \mathbb{X}_2 = x_2, \mathbb{X}_3 = x_3]\, dx_2$$

$$= E\big[\, \mathbb{X}_2 \cdot E[\mathbb{X}_1 \mid \mathbb{X}_2, \mathbb{X}_3 = x_3] \,\big|\, \mathbb{X}_3 = x_3 \big] = \varphi_2(x_3)$$

from which it follows from substituting the RV $\mathbb{X}_3$ for x_3, we have

$$E[\mathbb{X}_1 \cdot \mathbb{X}_2 \mid \mathbb{X}_3] = \varphi_2(\mathbb{X}_3) = E\big[\, \mathbb{X}_2 \cdot E[\mathbb{X}_1 \mid \mathbb{X}_2, \mathbb{X}_3] \,\big|\, \mathbb{X}_3 \big]$$

(proving second result)