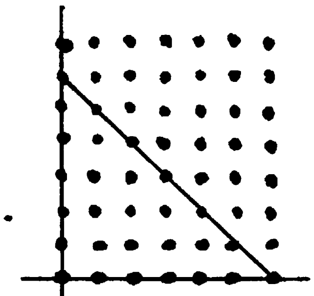
EE 600 Homework #8 Solutions Bell

1. Papoulis 6-14: A plot of the point masses of the joint pmf appears as follows:

We wish to find the probability that $\mathbf{z} = \mathbf{x} + \mathbf{y} = n$. This implies that $\mathbf{x} = k$ and $\mathbf{y} = n-k$, where k can take on values $0, 1, 2, \ldots, n$. The possible $(k, n-k)$-pairs are $(0,n), (1,n-1), \ldots, (n,0)$.



The probability of a particular $(k,n-k)$ pair is

$$P(\{\mathbf{x}=k\} \cap \{\mathbf{y}=n-k\}) = P(\{\mathbf{x}=k\})P(\{\mathbf{y}=n-k\}) = a_k b_{n-k}.$$

Thus upon summing over all valid pairs for a given $\mathbf{z} = \mathbf{x} + \mathbf{y} = n$, we get

$$P(\{\mathbf{z}=n\}) = \sum_{k=0}^{n} P(\{\mathbf{x}=k\} \cap \{\mathbf{y}=n-k\}) = \sum_{k=0}^{n} a_k b_{n-k}.$$

2. Let us first look at the special case of $\eta_X = \eta$ and $\eta_Y = 0$. Thus we have X and Y are independent RVs having pdfs

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\eta)^2}{2\sigma^2}\right\}.$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{y^2}{2\sigma^2}\right\}.$$

If we now define $R = \sqrt{X^2+Y^2}$ and $\Theta = \tan^{-1}(Y, X)$ we can write

$$f_{R\Theta}(r,\theta) = f_{XY}(x(r,\theta), y(r,\theta)) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$$

Now $x(r,\theta) = r\cos\theta$,
$y(r,\theta) = r\sin\theta$,

and

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Thus we have

$$f_{R\Theta}(r,\theta) = f_X(r\cos\theta)\cdot f_Y(r\sin\theta)\cdot|r|$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(r\cos\theta-\eta)^2}{2\sigma^2}\right\}\cdot\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(r\sin\theta)^2}{2\sigma^2}\right\}$$

$$= \frac{r}{2\pi\sigma^2} \exp\left\{-\frac{(r^2\cos^2 + r^2\sin^2\theta - 2\eta r\cos\theta + \eta^2)}{2\sigma^2}\right\}\cdot 1_{(0,\infty)}(r)\cdot 1_{[0,2\pi)}(\theta)$$

$$= \frac{r}{2\pi\sigma^2} \exp\left\{-\frac{(r^2 - 2r\eta\cos\theta + \eta^2)}{2\sigma^2}\right\}\cdot 1_{(0,\infty)}(r)\cdot 1_{[0,2\pi)}(\theta)$$

Thus $f_R(r) = \int_{-\infty}^{\infty} f_{R\Theta}(r,\theta)\,d\theta = \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{(r^2+\eta^2)}{2\sigma^2}\right\}\int_0^{2\pi} e^{+\frac{r\eta\cos\theta}{\sigma^2}}\,d\theta \cdot 1_{(0,\infty)}(r)$

Now $I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\,d\theta$, as given in the problem.

Thus we have

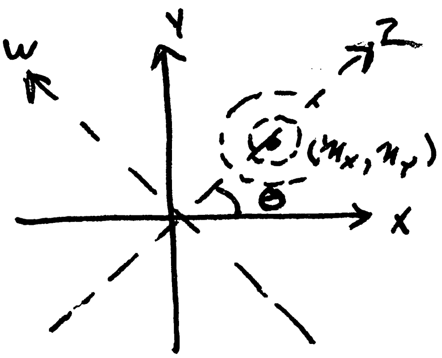
$$f_{\mathbb{R}}(r) = \frac{r}{\sigma^2} \exp\left\{\frac{-(r^2+\eta^2)}{2\sigma^2}\right\} \cdot 1_{(0,\infty)}(r) \cdot \underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{\left(\frac{r\eta}{\sigma^2}\right)\cos\theta}\, d\theta}_{I_0\left(\frac{r\eta}{\sigma^2}\right)}$$

$$= \frac{r}{\sigma^2} \exp\left\{\frac{-(r^2+\eta^2)}{2\sigma^2}\right\} I_0\left(\frac{r\eta}{\sigma^2}\right) \cdot 1_{[0,\infty)}(r)$$

Thus we have that the RV $\mathbb{R}$ has a Rician p.d.f.:

$$f_{\mathbb{R}}(r) = \frac{r}{\sigma^2} \exp\left\{-\frac{(r^2+\eta^2)}{2\sigma^2}\right\} I_0\left(\frac{r\eta}{\sigma^2}\right) \cdot 1_{(0,\infty)}(r) .$$

Now let's look at the more general case where $\eta_y \neq 0$. Here we have a situation that appears as follows:



We have zero mean circular Gaussian noise added to (η_x, η_y)

However if we rotate the axes clockwise by an angle $\theta = \tan^{-1}(\eta_y, \eta_x)$, we get two new random variables

$$\mathbb{Z} = \mathbb{X}\cos\theta + \mathbb{Y}\sin\theta$$
$$\mathbb{W} = -\mathbb{X}\sin\theta + \mathbb{Y}\cos\theta$$

We can write

$$f_{\mathbb{Z}\mathbb{W}}(z,w) = f_{\mathbb{X}\mathbb{Y}}(x(z,w), y(z,w)) \left|\frac{\partial(x,y)}{\partial(z,w)}\right| .$$

We note that

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}$$

$$\therefore \frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

Thus the circularly symmetric Gaussian noise with its means removed has the same pdf before and after rotating the axes. Thus we have if

$$\mathbb{V} = \sqrt{\mathbb{Z}^2 + \mathbb{W}^2}$$

we have the same problem we had before with $\mathbb{R} = \sqrt{\mathbb{X}^2 + \mathbb{Y}^2}$, but now $\eta_{\mathbb{Z}} = \sqrt{\eta_X^2 + \eta_Y^2}$ and $\eta_{\mathbb{W}} = 0$.

Thus we have

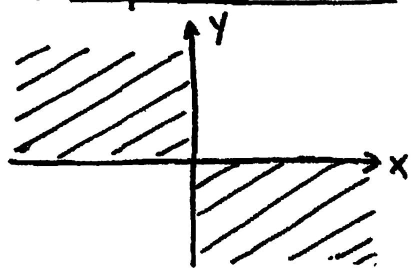
$$f_{\mathbb{V}}(v) = \frac{v}{\sigma^2} \exp\left\{\frac{-(v^2 + (\eta_X^2 + \eta_Y^2))}{2\sigma^2}\right\} I_0\left(\frac{v\sqrt{\eta_X^2 + \eta_Y^2}}{\sigma^2}\right) \cdot 1_{(0,\infty)}(v)$$

of course $\mathbb{V} = \mathbb{R}$, so we can rewrite this as

$$f_{\mathbb{R}}(r) = \frac{r}{\sigma^2} \exp\left\{-\frac{(r^2 + \eta_X^2 + \eta_Y^2)}{2\sigma^2}\right\} I_0\left(\frac{r\sqrt{\sigma_X^2 + \sigma_Y^2}}{\sigma^2}\right) \cdot 1_{(0,\infty)}(r).$$

This approach of rotating circularly symmetric Gaussian noise is widely used in problems of detection, estimation, and communications theory.

3. Papoulis 6-18: If $XY < 0$, then (X, Y) must fall in one of the two shaded quadrants shown (excluding axes). Equivalently, we can write this as



$$\{XY<0\} = \underbrace{\underbrace{(\{X<0\}\cap\{Y>0\})}\cup\underbrace{(\{X>0\}\cap\{Y<0\})}}_{\text{disjoint events.}}$$

$$\begin{aligned} P(\{XY<0\}) &= P((\{X<0\}\cap\{Y>0\})\cup(\{X>0\}\cap\{Y<0\})) \\ &= P(\{X<0\}\cap\{Y>0\}) + P(\{X>0\}\cap\{Y<0\}) \\ &= P(\{X<0\})P(\{Y>0\}) + P(\{X>0\})P(\{Y<0\}), \quad X \perp\!\!\!\perp Y \end{aligned}$$

(3 - Continued)

$$= F_X(0)[1-F_Y(0)] + [1-F_X(0)]F_Y(0)$$

$$= \Phi\left(-\frac{\mu_x}{\sigma_x}\right)\left[1-\Phi\left(-\frac{\mu_y}{\sigma_y}\right)\right] + \left[1-\Phi\left(-\frac{\mu_x}{\sigma_x}\right)\right]\Phi\left(-\frac{\mu_y}{\sigma_y}\right)$$

$$= \left[1-\Phi\left(\frac{\mu_x}{\sigma_x}\right)\right]\Phi\left(\frac{\mu_y}{\sigma_y}\right) + \Phi\left(\frac{\mu_x}{\sigma_x}\right)\left[1-\Phi\left(\frac{\mu_y}{\sigma_y}\right)\right]$$

$$= \Phi\left(\frac{\mu_y}{\sigma_y}\right) - \Phi\left(\frac{\mu_x}{\sigma_x}\right)\Phi\left(\frac{\mu_y}{\sigma_y}\right) + \Phi\left(\frac{\mu_x}{\sigma_x}\right) - \Phi\left(\frac{\mu_x}{\sigma_x}\right)\Phi\left(\frac{\mu_y}{\sigma_y}\right)$$

$$= \Phi\left(\frac{\mu_x}{\sigma_x}\right) + \Phi\left(\frac{\mu_y}{\sigma_y}\right) - 2\Phi\left(\frac{\mu_x}{\sigma_y}\right)\Phi\left(\frac{\mu_y}{\sigma_y}\right)$$

<u>n.b.</u> In Papoulis's notation, $G(\cdot) = \Phi(\cdot)$, which is the cdf of a zero-mean, unit-variance Gaussian RV.

<u>4.</u> <u>Papoulis 7-1:</u> If $W = X - Y$, then $E\{W\} = E\{X\} - E\{Y\} = 0$.

and $\sigma_W^2 = E\{W^2\} = E\{X^2 - 2XY + Y^2\}$

$$= E\{X^2\} - 2\cancel{E\{X\}}^{0}E\{Y\} + E\{Y^2\}$$

$$= \sigma_x^2 + \sigma_y^2 = 2\sigma^2.$$

Thus $W \sim N[0, \sigma\sqrt{2}]$. $\quad Z \triangleq |X - Y| = |W|$

$E\{Z\} = E\{|W|\} = \sigma\sqrt{2}\sqrt{\frac{2}{\pi}} = \frac{2\sigma}{\sqrt{\pi}}$ (See Eq. (5-45) of Papoulis).

$E\{Z^2\} = E\{|W|^2\} = E\{W^2\} = 2\sigma^2.$

<u>5.</u> <u>Papoulis 7-2:</u> $f_X(x) = e^{-x}1_{[0,\infty)}(x)$, $f_Y(y) = e^{-y}1_{[0,\infty)}(y)$

Define $Z \triangleq (X-Y)^+ = (X-Y)\,1_{[0,\infty)}(X-Y) = g(X,Y)$

Show that $E\{Z\} = \frac{1}{2}$.

$$E\{Z\} = E\{g(X,Y)\} = \iint_{\mathbb{R}^2} g(x,y) f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty (x-y)e^{-x}e^{-y}1_{[0,\infty)}(x-y)\,dx\,dy = \int_0^\infty\int_y^\infty (x-y)e^{-x}e^{-y}\,dx\,dy$$

$$= \int_0^\infty \left[-(x-y)e^{-x} - e^{-x}\right]_y^\infty e^{-y}\,dy = \int_0^\infty e^{-2y}\,dy = \left.\frac{e^{-2y}}{-2}\right|_0^\infty = \frac{1}{2}$$

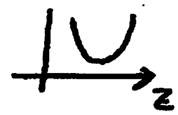
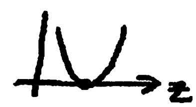
6. <u>Papoulis 7-3:</u> For real or complex $\mathbf{X}$ and $\mathbf{Y}$, show

(a) $|E\{\mathbf{X}\mathbf{Y}\}|^2 \le E\{|\mathbf{X}|^2\}\, E\{|\mathbf{Y}|^2\}$:

Because $|E\{\mathbf{X}\mathbf{Y}\}| \le E\{|\mathbf{X}||\mathbf{Y}|\}$, we can assume $\mathbf{X}$ and $\mathbf{Y}$ are real. Then for any number z, we have that

$$E\{[z\mathbf{X}-\mathbf{Y}]^2\} = z^2 E\{\mathbf{X}^2\} - 2z E\{\mathbf{X}\mathbf{Y}\} + E\{\mathbf{Y}^2\} \ge 0$$

This is a non-negative quadratic, having either no real roots, or one real root:


or


$\Rightarrow$ Its discriminate is nonpositive.

$$\Rightarrow 4|E\{\mathbf{X}\mathbf{Y}\}|^2 - 4E\{|\mathbf{X}|^2\}E\{|\mathbf{Y}|^2\} \le 0$$

$$\Rightarrow |E\{\mathbf{X}\mathbf{Y}\}|^2 \le E\{|\mathbf{X}|^2\}E\{|\mathbf{Y}|^2\}$$

(b) Using the result of part (a), we have

$$\begin{aligned} E\{\mathbf{X}^2\} + E\{\mathbf{Y}^2\} + 2\sqrt{E\{\mathbf{X}^2\}E\{\mathbf{Y}^2\}} &\ge E\{\mathbf{X}^2\} + E\{\mathbf{Y}^2\} \\ &\quad + 2E\{\mathbf{X}\mathbf{Y}\} \\ &= E\{(\mathbf{X}+\mathbf{Y})^2\}. \end{aligned}$$

7. <u>Papoulis 7-4</u>: Show that if $r_{xy} = 1$, then $\mathbf{Y} = a\mathbf{X} + b$

Recall that $r_{xy} = \dfrac{E\{(\mathbf{X}-\eta_x)(\mathbf{Y}-\eta_y)\}}{\sqrt{E\{(\mathbf{X}-\eta_x)^2\}\cdot E\{(\mathbf{Y}-\eta_y)^2}} = \dfrac{\operatorname{cov}(\mathbf{X},\mathbf{Y})}{\sigma_x \sigma_y}$

If $r_{xy} = 1$, then this implies that

$$E^2\{(\mathbf{X}-\eta_x)(\mathbf{Y}-\eta_y)\} = E\{(\mathbf{X}-\eta_x)^2\}E\{(\mathbf{Y}-\eta_y)^2\},$$

that is, the Schwarz inequality is satisfied with equality for the RVs $\tilde{\mathbf{X}} = \mathbf{X} - \eta_x$ and $\tilde{\mathbf{Y}} = \mathbf{Y} - \eta_y$. This implies that the discriminate of the quadratic...

(7 - continued)

$$E\{[z\tilde{X}-\tilde{Y}]^2\} = E\{[z(X-\eta_x)-(Y-\eta_y)]^2\}$$
$$= z^2 E\{(X-\eta_x)^2\} - 2zE\{(X-\eta_x)(Y-\eta_y)\} + E\{(Y-\eta_y)^2\} \quad (*)$$

is equal to zero.

This is true because the discriminate is given by

$$4E^2\{(X-\eta_x)(Y-\eta_y)\} - 4E\{(X-\eta_x)^2\}E\{(Y-\eta_y)^2\}$$

If the discriminate is zero (as it is), the quadratic (*) is zero for some real $z=z_0$ (the unique real root).

$$\Rightarrow z_0(X-\eta_x)-(Y-\eta_y)=0$$
$$\Rightarrow Y-\eta_y = z_0(X-\eta_x)$$
$$\Rightarrow Y = \underbrace{z_0}_{a} X + \underbrace{(\eta_Y - z_0\eta_x)}_{b} = aX+b, \quad a = z_0, \ b = \eta_y - z_0\eta_x$$

<u>8.</u> <u>Papoulis 7-7:</u> X has Cauchy density $f_X(x) = \dfrac{\alpha}{\pi(\alpha^2+x^2)}$,

$$\Rightarrow \Phi_X(\omega) = E\{e^{i\omega X}\} = e^{-\alpha|\omega|} \quad \left(\text{See problem 9 of Homework \#6}\right).$$

Now note that if $W = kX$, $k=0,1,2,\ldots$, then

$$\Phi_W(\omega) = E\{e^{i\omega kX}\} = e^{-\alpha|\omega|k}.$$

Now $Z = NX$, where N is an independent Poisson RV taking on non-negative integer values. So we have

$$\Phi_Z(\omega) = E\{e^{i\omega Z}\} = E\{e^{i\omega NX}\}$$
$$= E\{E\{e^{i\omega kX} \mid N=k\}\} = \sum_{k=0}^{\infty} E\{e^{i\omega kX}\}\cdot \frac{e^{-\lambda}\lambda^k}{k!}$$
$$= e^{-\lambda}\sum_{k=0}^{\infty} \frac{[e^{-\alpha|\omega|}\lambda]^k}{k!} = e^{-\lambda}e^{+\lambda e^{-\alpha|\omega|}}$$
$$= \exp\{\lambda e^{-\alpha|\omega|} - \lambda\}.$$

9. Papoulis 7-10: $f_{XY}(x,y) = \frac{1}{4} 1_{[-1,1]}(x) \cdot 1_{[-1,1]}(y)$.

Hence if $\mathbb{R} = \sqrt{X^2+Y^2}$, $P(\{\mathbb{R} \le 1\}) = \frac{\pi 1^2}{4} = \frac{\pi}{4}$

and

$$P(\{\mathbb{R} \le r\}) = \frac{\pi r^2}{4}, \text{ for } 0 \le r \le 1.$$

So
$$P(\{\mathbb{R}\le r\} \cap \{\mathbb{R} \le 1\}) = \begin{cases} P(\mathbb{R} \le r), & r \le 1 \\ P(\mathbb{R} \le 1), & r > 1 \end{cases}$$
$$= \begin{cases} \pi r^2/4, & r \le 1 \\ \pi/4, & r > 1 \end{cases}$$

Hence if $M = \{\mathbb{R} \le 1\}$,

$$F_{\mathbb{R}}(r|M) = \frac{P(\{\mathbb{R}\le r\}\cap M)}{P(M)} = \begin{cases} r^2, & r \le 1 \\ 1, & r > 1 \end{cases}$$

$$\Rightarrow f_{\mathbb{R}}(r|M) = \frac{d F_{\mathbb{R}}(r|M)}{dr} = 2r \cdot 1_{[0,1]}(r).$$

10. Papoulis 7-11: Let C_r = r-th coin selected

A_k = k heads occur in n tossings.

From the random selection and independent tossings, it follows that $P(C_r) = \frac{1}{m}$, $r = 1, \dots, m$.

$$P(A_k|C_r) = \binom{n}{k} p_r^k (1-p_r)^{n-k}$$

We wish to find $P(C_r|A_k)$. Since $\{C_1, \dots, C_m\}$ form a partition of the sample space, Bayes' Theorem yields

$$P(C_r|A_k) = \frac{P(A_k|C_r)P(C_r)}{\sum_{i=1}^{m} P(C_i)P(A_k|C_i)} = \frac{\binom{n}{k} p_r^k (1-p_r)^{n-k} \cdot \frac{1}{m}}{\sum_{i=1}^{m} \frac{1}{m}\binom{n}{k} p_i^k (1-p_i)^{n-k}}$$
$$= \left(p_r^k (1-p_r)^{n-k}\right) \Big/ \left(\sum_{i=1}^{m} p_i^k (1-p_i)^{n-k}\right)$$

11. <u>Papoulis 7-14:</u> Recall that the Markov inequality states that for any nonnegative RV $\mathbf{X}$ and any $\alpha > 0$

$$P(\{\mathbf{X} > \alpha\}) \leq \frac{E\{\mathbf{X}\}}{\alpha}$$

Now taking $|\mathbf{Z}|^2 = |\mathbf{X} - \mathbf{Y}|^2$ and $\alpha = \varepsilon^2 > 0$, we get

$$P(\{|\mathbf{Z}|^2 > \varepsilon^2\}) \leq \frac{E\{|\mathbf{Z}|^2\}}{\varepsilon^2} = \frac{E\{|\mathbf{X} - \mathbf{Y}|^2\}}{\varepsilon^2}.$$

Thus

$$P(\{|\mathbf{X} - \mathbf{Y}| > \varepsilon\}) \leq \frac{E\{|\mathbf{X} - \mathbf{Y}|^2\}}{\varepsilon^2}$$