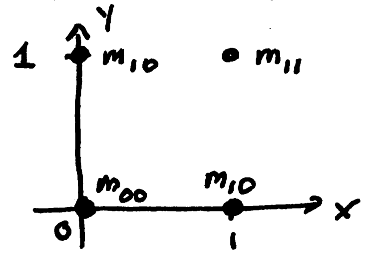
1. Let $X = 1_A(\omega)$, defined on $(S, \mathcal{F}, P)$, $Y = 1_B(\omega)$, where $A, B \in \mathcal{F}$.

(a) A sketch of the probability mass functions in the xy-plane appears as follows:



$$m_{00} = P_{XY}(0,0) = P(\{X=0\} \cap \{Y=0\}) = P(\bar{A} \cap \bar{B})$$
$$m_{10} = P_{XY}(1,0) = P(\{X=1\} \cap \{Y=0\}) = P(A \cap \bar{B})$$
$$m_{01} = P_{XY}(0,1) = P(\{X=0\} \cap \{Y=1\}) = P(\bar{A} \cap B)$$
$$m_{11} = P_{XY}(1,1) = P(\{X=1\} \cap \{Y=1\}) = P(A \cap B)$$

(b) If the events A and B are statistically independent, then

$$P_{XY}(0,0) = P(\{X=0\} \cap \{Y=0\}) = P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B}) = P(\{X=0\}) \cdot P(\{Y=0\}) = p_X(0) \cdot p_Y(0).$$
$$P_{XY}(1,0) = P(\{X=1\} \cap \{Y=0\}) = P(A \cap \bar{B}) = P(A)P(\bar{B}) = P(\{X=1\}) \cdot P(\{Y=0\}) = p_X(1) \cdot p_Y(0)$$
$$P_{XY}(0,1) = P(\{X=0\} \cap \{Y=1\}) = P(\bar{A} \cap B) = P(\bar{A})P(B) = P(\{X=0\}) \cdot P(\{Y=1\}) = p_X(0) \cdot p_Y(1).$$
$$P_{XY}(1,1) = P(\{X=1\} \cap \{Y=1\}) = P(A \cap B) = P(A)P(B) = P(\{X=1\})P(\{Y=1\}) = p_X(1) \cdot p_Y(1).$$

Thus X and Y are statistically independent.

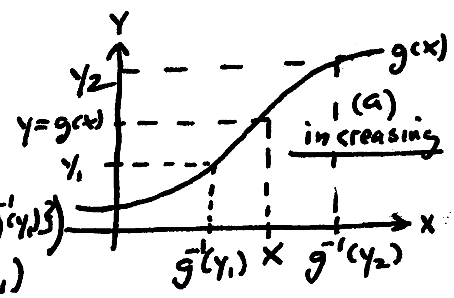
2. (Papoulis 6-15): $Z = X + Y \Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$, since $X \perp\!\!\!\perp Y$.

Here we have $f_Y(y) = 1_{(0,1)}(y)$, so we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot 1_{(0,1)}(z-x)\,dx = \int_{z-1}^{z} f_X(x)\,dx = F_X(z) - F_X(z-1) \blacksquare$$

n.b. $1_{(0,1)}(z-x) = 1_{(-1,0)}(x-z) = 1_{(z-1,z)}(x)$.

3. (Papoulis 6-16): (a) If we wish to evaluate $F_{XY}(x, y_1)$, $y_1 < g(x)$, we have

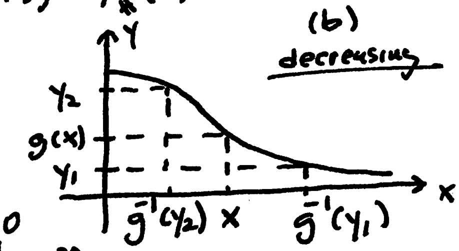


$$F_{XY}(x, y_1) = P(\{X \le x\} \cap \{Y \le y_1\}) = P(\{X \le x\} \cap \{X \le g^{-1}(y_1)\})$$
$$= P(\{X \le g^{-1}(y_1)\}) = P(\{Y \le y_1\}) = F_Y(y_1)$$

If we evaluate $F_{XY}(x, y_2)$, for $y_2 > g(x)$, we have

$$F_{XY}(x, y_2) = P(\{X \le x\} \cap \{X \le g^{-1}(y_2)\}) = P(\{X \le x\}) = F_X(x)$$

$$\therefore F_{XY}(x,y) = \begin{cases} F_Y(y), & y < g(x), \\ F_X(x), & x > g(x). \end{cases}$$



(b) If $y_1 < g(x)$, then

$$F_{XY}(x, y_1) = P(\{X \le x\} \cap \{X \ge g^{-1}(y_1)\}) = P(\phi) = 0$$

If $y_2 > g(x)$, then $F_{XY}(x, y_2) = P(\{X \le x\} \cap \{X \ge g^{-1}(y_2)\})$

$$= P(\{X \le x\}) - P(\{Y > y_2\})$$
$$= F_X(x) - [1 - F_Y(y_2)] \Rightarrow$$

$$\therefore F_{XY}(x,y) = \begin{cases} 0, & y < g(x) \\ F_X(x) - [1 - F_Y(y)], & y > g(x). \end{cases}$$

A More Detailed Solution of Problem 3

6-16 (a) The function $g(x)$ is monotone increasing with $\mathbf{Y} = g(\mathbf{X})$. Show that

$$F_{\mathbf{XY}}(x,y) = \begin{cases} F_{\mathbf{X}}(x), & \text{if } y > g(x) \\ F_{\mathbf{Y}}(y), & \text{if } y < g(x) \end{cases}$$

(b) Find $F_{\mathbf{XY}}(x,y)$ if $g(x)$ is monotone decreasing

Solution:

(a)
$$\begin{aligned} F_{\mathbf{XY}}(x,y) &= P(\{\mathbf{X} \le x\} \cap \{\mathbf{Y} \le y\}) \\ &= P(\{\mathbf{X} \le x\} \cap \{g(\mathbf{X}) \le y\}) \quad \text{, because } \mathbf{Y} = g(\mathbf{X}) \\ &= P(\{g(\mathbf{X}) \le g(x)\} \cap \{g(\mathbf{X}) \le y\}) \text{, because } g(\cdot) \text{ mono} \uparrow \end{aligned}$$

$$= \begin{cases} P(\{g(\mathbf{X}) \le g(x)\}), & g(x) < y, \\ P(\{g(\mathbf{X}) \le y\}), & g(x) > y \end{cases}$$

$$= \begin{cases} P(\{\mathbf{X} \le x\}), & g(x) \le y \quad \text{, because } g(\cdot) \text{ mono} \uparrow \\ P(\{\mathbf{Y} \le y\}), & g(x) > y \quad \text{, because } \mathbf{Y} = g(\mathbf{X}) \end{cases}$$

$$= \begin{cases} F_{\mathbf{X}}(x), & y > g(x), \\ F_{\mathbf{Y}}(y), & y < g(x). \end{cases}$$

(b)
$$\begin{aligned} F_{\mathbf{XY}}(x,y) &= P(\{\mathbf{X} \le x\} \cap \{\mathbf{Y} \le y\}) \\ &= P(\{\mathbf{X} \le x\} \cap \{g^{-1}(\mathbf{Y}) \ge g^{-1}(y)\}) \text{, because } g(\cdot) \text{ mono} \downarrow \Rightarrow g^{-1}(\cdot) \text{ mono} \downarrow \end{aligned}$$

$$= P(\{\mathbf{X} \le x\} \cap \{\mathbf{X} \ge g^{-1}(y)\})$$

$$= \begin{cases} P(\phi), & g^{-1}(y) \ge x \Leftrightarrow y \le g(x) \\ P(\{g^{-1}(y) \le \mathbf{X} \le x\}), & y > g(x) \end{cases}$$

$$= \begin{cases} 0, & y < g(x) \\ P(\{\mathbf{X} \le x\}) - P(\{\mathbf{X} \le g^{-1}(y)\}) = F_{\mathbf{X}}(x) - P(\{\mathbf{Y} \ge y\}) = F_{\mathbf{X}}(x) - (1 - F_{\mathbf{Y}}(y)) & y > g(x) \end{cases}$$

$$= \begin{cases} 0, & y < g(x) \\ F_{\mathbf{X}}(x) - (1 - F_{\mathbf{Y}}(y)), & y > g(x). \end{cases}$$

4. Papoulis 6-24 (6-5 in 3rd Edition)

Let $Z = \max(X, Y)$

$W = \min(X, Y)$

We wish to find

$$F_{ZW}(z,w) = P(\{Z \le z\} \cap \{W \le w\})$$

If $z < w$, then we have

$$F_{ZW}(z,w) = P(\{Z \le z\} \cap \{W \le w\})$$

$$= P(\{Z \le z\} \cap (\{W \le z\} \cup \{z < W \le w\}))$$

$$= P((\{Z \le z\} \cap \{W \le z\}) \cup (\{Z \le z\} \cap \{z < W \le w\}))$$

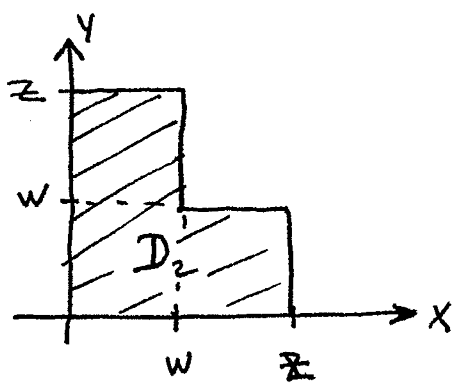
disjoint events

$$= P(\{Z \le z\} \cap \{W \le z\}) + P(\{Z \le z\} \cap \{z < W \le w\})$$

$$= P(\{X \le z\} \cap \{Y \le z\}) + P(\phi)$$

$$= F_{XY}(z,z) + 0 = F_{XY}(z,z)$$

If $z > w$, we need to find the probability that (X, Y) falls within the following region in the x-y plane:



Thus

$$F_{ZW}(z,w) = P(\{(X,Y) \in D_2\})$$

$$= F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(w,w)$$

Thus we have

$$F_{XY}(x,y) = \begin{cases} F_{XY}(z,z), & z \le w \\ F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(w,w) \end{cases}$$

5. Papoulis 6-18 (6-7 in 3rd. Edition)

Let $\mathbb{Z} = \mathbb{X}\mathbb{Y}$ and define the auxiliary RV $\mathbb{W} = \mathbb{X}$. Then using the direct pdf method, we have

$$f_{\mathbb{Z}\mathbb{W}}(z,w) = f_{\mathbb{X}\mathbb{Y}}(x(z,w), y(z,w)) \cdot \left| \frac{\partial(x,y)}{\partial(z,w)} \right| .$$

Here $x(z,w) = w$ and $y(z,w) = \frac{z}{x} = \frac{z}{w}$, and

$$\frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{w} & \frac{\partial y}{\partial w} \end{vmatrix} = -\frac{1}{w}$$

Thus we have

$$f_{\mathbb{Z}\mathbb{W}}(z,w) = \frac{1}{|w|} f_{\mathbb{X}\mathbb{Y}}(w, z/w) = \frac{1}{|w|} f_{\mathbb{X}}(w) \cdot f_{\mathbb{Y}}(z/w)$$

because $\mathbb{X} \perp\!\!\!\perp \mathbb{Y}$

$$= \frac{1}{|w|} \cdot \frac{w}{\alpha^2} e^{-w^2/2\alpha^2} \cdot 1_{[0,\infty)}(w) \cdot \frac{1}{\pi\sqrt{1-(z/w)^2}} \cdot 1_{(-1,1)}(z/w)$$

(n.b. because the pdf is 0 for negative w, we can replace $|w|$ with w.)

$$= \frac{1}{\alpha^2} e^{-w^2/2\alpha^2} \cdot \frac{1}{\pi\sqrt{1-(z/w)^2}} \cdot 1_{[0,\infty)}(w) \cdot 1_{(-1,1)}(z/w)$$

We can now integrate w.r.t. w over the whole real line to get $f_{\mathbb{Z}}(z)$. The integrand is non-zero for positive w when

$$1_{(-1,1)}(z/w) = 1 \Leftrightarrow -1 < \frac{z}{w} < 1 \Leftrightarrow -1 < \frac{z}{w} \ \underline{\text{and}}\ \frac{z}{w} < 1$$

$$\Leftrightarrow -w < z \ \underline{\text{and}}\ z < w \Leftrightarrow w > -z \ \underline{\text{and}}\ w > z$$

$$\Leftrightarrow w > |z|$$

Thus we have

$$f_{\mathbb{Z}}(z) = \int_{-\infty}^{\infty} f_{\mathbb{Z}\mathbb{W}}(z,w)\,dw = \int_{|z|}^{\infty} \frac{e^{-w^2/2\alpha^2}}{\alpha^2\pi\sqrt{1-(z/w)^2}}\,dw$$

To simplify evaluation of this integral, we make the change of variable $s^2 = w^2 - z^2 = s = \sqrt{w^2-z^2}$

$$\Rightarrow \frac{ds}{dw} = \frac{1}{\cancel{2}}(w^2-z^2)^{-\frac{1}{2}} \cdot \cancel{2}w = \frac{w}{\sqrt{w^2-z^2}}$$

Thus $dw = \dfrac{\sqrt{w^2-z^2}}{w}\,ds$, and with the change of variable, the integral becomes

$$f_Z(z) = \int_0^\infty \frac{e^{-(s^2+z^2)/2\alpha^2}}{\alpha^2\pi\sqrt{1-(z/w)^2}} \cdot \frac{\sqrt{w^2-z^2}}{w}\,ds$$

$$= \int_0^\infty \frac{e^{-(s^2+z^2)/2\alpha^2}}{\alpha^2\pi\cancel{\sqrt{w^2-z^2}}} \cdot \cancel{\sqrt{w^2-z^2}}\,ds$$

$$= \frac{e^{-z^2/2\alpha^2}}{\alpha^2\pi}\int_0^\infty e^{-s^2/2\alpha^2}\,ds = \frac{e^{-z^2/2\alpha^2}}{\alpha\sqrt{\pi}}\sqrt{2}\,\underbrace{\int_0^\infty \frac{e^{-s^2/2\alpha^2}}{\sqrt{2\pi}\,\alpha}\,ds}_{\substack{\left(\text{Gaussian pdf integrated over half its range}\right)\\ \frac{1}{2}}}$$

$$= \frac{e^{-z^2/2\alpha^2}}{\alpha\sqrt{\pi}}\sqrt{2}\cdot\frac{1}{2}$$

$$= \frac{1}{\sqrt{2\pi}\,\alpha}\,e^{-z^2/2\alpha^2},$$

which is the pdf of a zero-mean Gaussian RV with variance α^2

$\therefore$ Z is a Gaussian RV with mean 0 and variance α^2.

6. Papoulis 6-19 (6-8 in 3rd Edition)

Let $\mathbb{Z} = \mathbb{X}/\mathbb{Y}$ and $\mathbb{W} = \mathbb{Y}$. Then we have

$$f_{ZW}(z,w) = f_{XY}(x(z,w), y(z,w)) \left| \frac{\partial(x,y)}{\partial(z,w)} \right| ,$$

where $x(z,w) = zw$ and $y(z,w) = w$.

Evaluating the Jacobian, we have

$$\frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} y & \frac{dx}{dw} \\ 0 & 1 \end{vmatrix} = y = w ,$$

and thus we have (given that $f_{XY}(x,y) = f_X(x) f_Y(y)$)

$$f_{ZW}(z,w) = |w| \, f_X(zw) \cdot f_Y(w).$$

Now for $z > 0$, we have

$$f_Z(z) = \int_{-\infty}^{\infty} |w| f_X(zw) f_Y(w) \, dw = \frac{z}{\alpha^2\beta^2} \int_0^{\infty} w^3 e^{-cw^2} dw$$

$$= \frac{z}{2\alpha^2\beta^2 c^2}, \text{ where } c = \frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2}$$

Thus we have

$$f_Z(z) = \begin{cases} \frac{2\alpha^2}{\beta^2} \left(\frac{z}{(z^2 + \alpha^2/\beta^2)^2} \right), & z > 0 \\ 0, \text{elsewhere.} \end{cases}$$

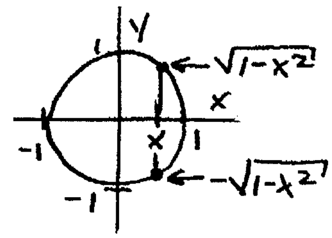
(b) For $z > 0$, $F_Z(z) = \int_0^z f_Z(\gamma) \, d\gamma = \int_0^z \frac{2\alpha^2 \gamma \, d\gamma}{\beta^2(\gamma^2 + \alpha^2/\beta^2)^2}$

$$= \int_{\alpha^2/\beta^2}^{z^2+\alpha^2/\beta^2} \frac{dt}{t^2} = \frac{z^2}{z^2 + \alpha^2/\beta^2} = P(\{\mathbb{Z} \le z\}), \; z > 0$$

$$= P(\{\mathbb{X} \le z\mathbb{Y}\}), \text{ for any } z > 0$$

$\therefore$ For any $k > 0$, $P(\{\mathbb{X} \le k\mathbb{Y}\}) = \dfrac{k^2}{k^2 + \alpha^2/\beta^2}$.

7. (a) $f_{XY}(x,y) = \begin{cases} \frac{1}{\pi}, & x^2+y^2 \le 1; \\ 0, & \text{elsewhere.} \end{cases}$



$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \frac{2}{\pi}\sqrt{1-x^2} \cdot 1_{(-1,1)}(x)$$

(b) $R = \sqrt{X^2+Y^2}$, where (X,Y) is uniformly distributed over the unit disk.

$F_R(r) = P(\{R \le r\}) = P(\{(X,Y) \in C_r\})$

where $C_r = \{(x,y) : x^2+y^2 \le r\}$

= closed circle of radius r.

$$\Rightarrow F_R(r) = \begin{cases} \frac{\pi r^2}{\pi 1^2} = r^2, & 0 \le r \le 1 \\ 1, & r > 1 \\ 0, & r < 0 \end{cases}$$

$$\Rightarrow f_R(r) = \frac{dF_R(r)}{dr} = 2r \cdot 1_{[0,1]}(r)$$

(c) X and Y are <u>not</u> statistically independent, because $f_{XY}(x,y) \ne f_X(x) f_Y(y)$.

(d) $E\{XY\} = \iint_{C_1} xy \cdot \frac{1}{\pi}\,dx\,dy = 0$

$E\{X\} = \int_{-1}^{1} x \cdot \frac{2}{\pi}\sqrt{1-x^2}\,dx = 0$, and $E\{Y\} = 0$ by symmetry

$\therefore E\{XY\} = E\{X\}E\{Y\} \Rightarrow X$ and Y are uncorrelated.