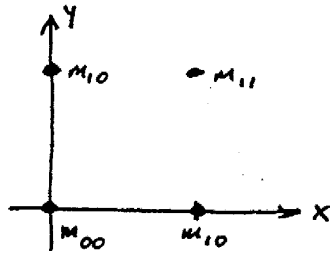


1. Papoulis 6-1:  $X = I_A(\omega)$ , defined on the experiment  $(\Omega, \mathcal{F}, P)$ ,  
 $Y = I_B(\omega)$  where  $A, B \in \mathcal{F}$ .

(a) A sketch of the probability mass functions in the  $xy$ -plane appears as follows:



$$m_{00} = P(\{X=0\} \cap \{Y=0\}) = P(\bar{A} \cap \bar{B})$$

$$m_{10} = P(\{X=1\} \cap \{Y=0\}) = P(A \cap \bar{B})$$

$$m_{01} = P(\{X=0\} \cap \{Y=1\}) = P(\bar{A} \cap B)$$

$$m_{11} = P(\{X=1\} \cap \{Y=1\}) = P(A \cap B)$$

(b) If the events  $A$  and  $B$  are stat. indep., then

$$P(\{X=0\} \cap \{Y=0\}) = P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B}) = P(\{X=0\})P(\{Y=0\})$$

$$P(\{X=1\} \cap \{Y=0\}) = P(A \cap \bar{B}) = P(A)P(\bar{B}) = P(\{X=1\})P(\{Y=0\})$$

$$P(\{X=0\} \cap \{Y=1\}) = P(\bar{A} \cap B) = P(\bar{A})P(B) = P(\{X=0\})P(\{Y=1\})$$

$$P(\{X=1\} \cap \{Y=1\}) = P(A \cap B) = P(A)P(B) = P(\{X=1\})P(\{Y=1\})$$

Thus  $X$  and  $Y$  are statistically independent.

2. Papoulis 6-3:  $Z = X + Y \Rightarrow f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_y(z-x) dx$ ,  $f_y(y) = 1_{(0,1)}(y)$

$$\Rightarrow f_z(z) = \int_{-\infty}^{\infty} f_x(x) 1_{(0,1)}(z-x) dx = \int_{z-1}^z f_x(x) dx = F_x(z) - F_x(z-1)$$

n.b.  $1_{(0,1)}(z-x) = 1_{(z-1, z)}(x)$ .

3. Papoulis 6-4:

(a) If we wish to evaluate  $F(x, y_1)$ ,  $y_1 < g(x)$ , we have

$$F(x, y_1) = P(\{X \leq x\} \cap \{Y \leq y_1\}) = P(\{X \leq x\} \cap \{X \leq g^{-1}(y_1)\})$$

$$= P(\{X \leq \min(x, g^{-1}(y_1))\}) = P(\{Y \leq y_1\}) = F_y(y_1)$$

If we evaluate  $F(x, y_2)$ ,  $y_2 > g(x)$ , we get

$$F(x, y_2) = P(\{X \leq x\} \cap \{X \leq g^{-1}(y_2)\}) = P(\{X \leq x\}) = F_x(x)$$

$$\therefore F_{X,Y}(x, y) = \begin{cases} F_y(y), & y < g(x) \\ F_x(x), & y > g(x) \end{cases}$$

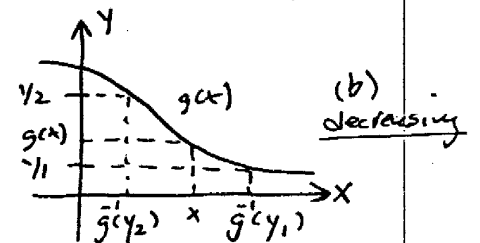
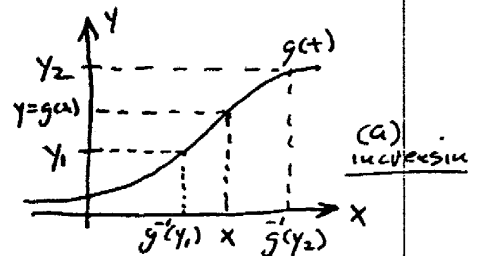
(b) If  $y_1 < g(x)$ , then

$$F(x, y_1) = P(\{X \leq x\} \cap \{X \geq g^{-1}(y_1)\}) = P(\emptyset) = 0$$

If  $y_2 > g(x)$ , then  $F(x, y) = P(\{X \leq x\} \cap \{X \geq g^{-1}(y_2)\})$   
 $= P(\{X \leq x\}) - P(\{Y > y_2\})$

$$= F_x(x) - [1 - F_y(y_2)]$$

$$\therefore F(x, y) = \begin{cases} 0, & y < g(x) \\ F_x(x) - [1 - F_y(y)], & y > g(x) \end{cases}$$



### A More Detailed Solution of Problem 3

6-16 (a) The function  $g(x)$  is monotone increasing with  $Y = g(X)$ . Show that

$$F_{*Y}(x, y) = \begin{cases} F_*(x), & \text{if } y > g(x) \\ F_Y(y), & \text{if } y < g(x) \end{cases}$$

(b) Find  $F_{*Y}(x, y)$  if  $g(x)$  is monotone decreasing

Solution:

$$\begin{aligned} \text{(a)} \quad F_{*Y}(x, y) &= P(\{X \leq x\} \cap \{Y \leq y\}) \\ &= P(\{X \leq x\} \cap \{g(X) \leq y\}) \quad , \text{ because } Y = g(X) \\ &= P(\{g(X) \leq g(x)\} \cap \{g(X) \leq y\}) \quad , \text{ because } g(\cdot) \text{ mono} \uparrow \\ &= \begin{cases} P(\{g(X) \leq g(x)\}), & g(x) < y, \\ P(\{g(X) \leq y\}), & g(x) > y \end{cases} \end{aligned}$$

$$= \begin{cases} P(\{X \leq x\}), & g(x) \leq y \quad , \text{ because } g(\cdot) \text{ mono} \uparrow \\ P(\{Y \leq y\}), & g(x) > y \quad , \text{ because } Y = g(X) \end{cases}$$

$$= \begin{cases} F_*(x), & y > g(x), \\ F_Y(y), & y < g(x). \end{cases}$$

$$\begin{aligned} \text{(b)} \quad F_{*Y}(x, y) &= P(\{X \leq x\} \cap \{Y \leq y\}) \\ &= P(\{X \leq x\} \cap \{g^{-1}(y) \geq g^{-1}(x)\}) \quad , \text{ because } g(\cdot) \text{ mono} \downarrow \\ &\quad \Rightarrow g^{-1}(\cdot) \text{ mono} \downarrow \end{aligned}$$

$$= P(\{X \leq x\} \cap \{X \geq g^{-1}(y)\})$$

$$= \begin{cases} P(\emptyset), & g^{-1}(y) \geq x \iff y \leq g(x) \\ P(\{g^{-1}(y) \leq X \leq x\}), & y > g(x) \end{cases}$$

$$= \begin{cases} 0, & y \leq g(x) \\ P(\{X \leq x\}) - P(\{X \leq g^{-1}(y)\}) = F_*(x) - P(\{Y \geq y\}) = F_*(x) - (1 - F_Y(y)), & y > g(x) \end{cases}$$

$$= \begin{cases} 0, & y < g(x) \\ F_*(x) - (1 - F_Y(y)), & y > g(x). \end{cases}$$

4. Papoulis 6-24 (6-5 in 3rd Edition)

P. 2

$$\text{Let } Z = \max(X, Y)$$

$$W = \min(X, Y)$$

We wish to find

$$F_{ZW}(z, w) = P(\{Z \leq z\} \cap \{W \leq w\})$$

If  $z < w$ , then we have

$$F_{ZW}(z, w) = P(\{Z \leq z\} \cap \{W \leq w\})$$

$$= P(\{Z \leq z\} \cap (\{W \leq z\} \cup \{z < W \leq w\}))$$

$$= P(\{Z \leq z\} \cap \{W \leq z\}) \cup (\{Z \leq z\} \cap \{z < W \leq w\})$$

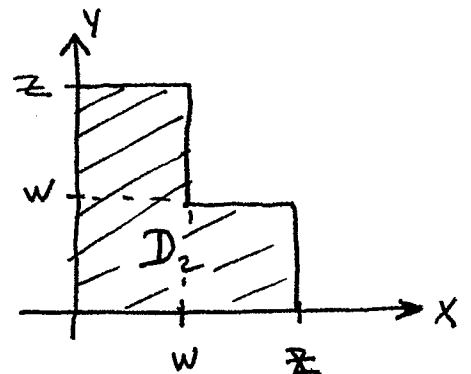
↑ disjoint events ↑

$$= P(\{Z \leq z\} \cap \{W \leq z\}) + P(\{Z \leq z\} \cap \{z < W \leq w\})$$

$$= P(\{X \leq z\} \cap \{Y \leq z\}) + P(\emptyset)$$

$$= F_{XY}(z, z) + 0 = F_{XY}(z, z)$$

If  $z > w$ , we need to find the probability that  $(X, Y)$  falls within the following region in the  $x$ - $y$  plane:



Thus

$$F_{ZW}(z, w) = P(\{(X, Y) \in D_z\})$$

$$= F_{XY}(z, w) + F_{XY}(w, z) - F_{XY}(w, w)$$

Thus we have

$$F_{XY}(x, y) = \begin{cases} F_{XY}(z, z) & , z \leq w \\ F_{XY}(z, w) + F_{XY}(w, z) - F_{XY}(w, w) & \end{cases}$$

5. Papoulis 6-18 (6-7 in 3rd. Edition)

P.3

Let  $Z = XY$  and define the auxiliary RV  $W = X$ .  
Then using the direct pdf method, we have

$$f_{ZW}(z, w) = f_{XY}(x(z, w), y(z, w)) \cdot \left| \frac{\partial(x, y)}{\partial(z, w)} \right|.$$

Here  $x(z, w) = w$  and  $y(z, w) = \frac{z}{x} = \frac{z}{w}$ , and

$$\frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{w} & \frac{\partial y}{\partial w} \end{vmatrix} = -\frac{1}{w}$$

Thus we have

$$f_{ZW}(z, w) = \frac{1}{|w|} f_{XY}(w, z/w) = \frac{1}{|w|} f_X(w) \cdot f_Y(z/w)$$

because  $X \perp Y$

$$= \frac{1}{|w|} \cdot \frac{w}{\alpha^2} e^{-w^2/2\alpha^2} \cdot \mathbb{1}_{[0, \infty)}(w) \cdot \frac{1}{\pi \sqrt{1-(z/w)^2}} \cdot \mathbb{1}_{(-1, 1)}(z/w)$$

(n.b. because the pdf is 0 for negative  $w$ , we can replace  $|w|$  with  $w$ .)

$$= \frac{1}{\alpha^2} e^{-w^2/2\alpha^2} \cdot \frac{1}{\pi \sqrt{1-(z/w)^2}} \cdot \mathbb{1}_{[0, \infty)}(w) \cdot \mathbb{1}_{(-1, 1)}(z/w)$$

We can now integrate w.r.t.  $w$  over the whole real line to get  $f_Z(z)$ . The integrand is non-zero for positive  $w$  when

$$\mathbb{1}_{(-1, 1)}(z/w) = 1 \iff -1 < \frac{z}{w} < 1 \iff -1 < \frac{z}{w} \text{ and } \frac{z}{w} < 1$$

$$\iff -w < z \text{ and } z < w \iff w > -z \text{ and } w > z$$

$$\iff w > |z|$$

Thus we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z, w) dw = \int_{|z|}^{\infty} \frac{e^{-w^2/2\alpha^2}}{\alpha^2 \pi \sqrt{1-(z/w)^2}} dw$$

To simplify evaluation of this integral, we make the change of variable  $s^2 = w^2 - z^2 = s = \sqrt{w^2 - z^2}$

$$\Rightarrow \frac{ds}{dw} = \frac{1}{z} (w^2 - z^2)^{-1/2} \cdot 2w = \frac{w}{\sqrt{w^2 - z^2}}$$

Thus  $dw = \sqrt{w^2 - z^2} ds$ , and with the change P.3B  
of variable, the  $\int_0^w$  integral becomes

$$\begin{aligned}
 f_Z(z) &= \int_0^w \frac{e^{-(s^2+z^2)/2\alpha^2}}{\alpha^2 \pi \sqrt{1-(z/w)^2}} \cdot \frac{\sqrt{w^2-z^2}}{w} ds \\
 &= \int_0^w \frac{e^{-(s^2+z^2)/2\alpha^2}}{\alpha^2 \pi \sqrt{w^2-z^2}} \cdot \sqrt{w^2-z^2} ds \\
 &= \frac{e^{-z^2/2\alpha^2}}{\alpha^2 \pi} \int_0^w e^{-s^2/2\alpha^2} ds = \frac{e^{-z^2/2\alpha^2}}{\alpha \sqrt{\pi}} \sqrt{2} \int_0^w \frac{e^{-s^2/2\alpha^2}}{\sqrt{2\pi} \alpha} ds \\
 &= \frac{e^{-z^2/2\alpha^2}}{\alpha \sqrt{\pi}} \sqrt{2} \cdot \frac{1}{2} \quad \left( \begin{array}{l} \text{Gaussian pdf} \\ \text{integrated over} \\ \text{half its range} \end{array} \right) \\
 &= \frac{1}{\sqrt{2\pi} \alpha} e^{-z^2/2\alpha^2} \quad \frac{1}{2}
 \end{aligned}$$

Which is the pdf of a zero-mean Gaussian RV with variance  $\alpha^2$

i.  $Z$  is a Gaussian RV with mean 0 and variance  $\alpha^2$ . 7

6. Papoulis 6-19 (6-8 in 3rd Edition)

Let  $Z = X/Y$  and  $W = Y$ . Then we have

$$f_{ZW}(z, w) = f_{XY}(X(z, w), Y(z, w)) \left| \frac{\partial(x, y)}{\partial(z, w)} \right|,$$

where  $X(z, w) = zw$  and  $Y(z, w) = w$ .

Evaluating the Jacobian, we have

$$\frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w = w,$$

and thus we have (given that  $f_{XY}(x, y) = f_X(x) f_Y(y)$ )

$$f_{ZW}(z, w) = |w| f_X(zw) \cdot f_Y(w).$$

Now for  $z > 0$ , we have

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} |w| f_X(zw) f_Y(w) dw = \frac{z}{\alpha^2 \beta^2} \int_0^{\infty} w^3 e^{-cw^2} dw \\ &= \frac{z}{2\alpha^2 \beta^2 c^2}, \text{ where } c = \frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2} \end{aligned}$$

Thus we have

$$f_Z(z) = \begin{cases} \frac{2\alpha^2}{\beta^2} \left( \frac{z}{(z^2 + \alpha^2/\beta^2)^2} \right), & z > 0 \\ 0, & \text{elsewhere} \end{cases}$$

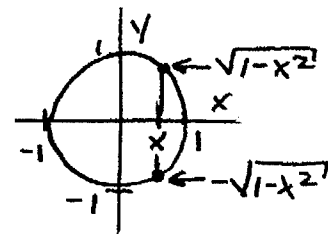
$$\begin{aligned} \text{(b) For } z > 0, F_Z(z) &= \int_0^z f_Z(x) dx = \int_0^z \frac{2\alpha^2 x dx}{\beta^2 (x^2 + \alpha^2/\beta^2)^2} \\ &= \int_{\alpha^2/\beta^2}^{\frac{z^2 + \alpha^2/\beta^2}{2}} \frac{dt}{t^2} = \frac{z^2}{z^2 + \alpha^2/\beta^2} = P(\{Z \leq z\}), \quad z > 0 \\ &= P(\{X \leq zY\}), \text{ for any } z > 0 \end{aligned}$$

$$\therefore \text{For any } k > 0, P(\{X \leq kY\}) = \frac{k^2}{k^2 + \alpha^2/\beta^2}.$$

7. (a)  $f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \leq 1; \\ 0, & \text{elsewhere.} \end{cases}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi} \sqrt{1-x^2} \cdot \mathbb{1}_{(-1,1)}(x)$$



(b)  $R = \sqrt{X^2 + Y^2}$ , where  $(X,Y)$  is uniformly distributed over the unit disk.

$$F_R(r) = P(\{R \leq r\}) = P(\{(X,Y) \in C_r\})$$

where  $C_r = \{(x,y) : x^2 + y^2 \leq r\}$   
 = closed circle of radius  $r$ .

$$\Rightarrow F_R(r) = \begin{cases} \frac{\pi r^2}{\pi 1^2} = r^2, & 0 \leq r \leq 1 \\ 1, & r > 1 \\ 0, & r < 0 \end{cases}$$

$$\Rightarrow f_R(r) = \frac{dF_R(r)}{dr} = 2r \cdot \mathbb{1}_{[0,2]}$$

(c)  $X$  and  $Y$  are not statistically independent, because  $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$ .

(d)  $E\{XY\} = \iint_{C_1} xy \cdot \frac{1}{\pi} dx dy = 0$

$$E\{X\} = \int_{-1}^1 x \cdot \frac{2}{\pi} \sqrt{1-x^2} dx = 0, \text{ and } E\{Y\} = 0 \text{ by symmetry}$$

$\therefore E\{XY\} = E\{X\}E\{Y\} \Rightarrow X$  and  $Y$  are uncorrelated.