

1. Papoulis 5-22 (5-18 in 3rd Edition)

$$(a) E[Y] = E[aX + b] = aE[X] + b = a\mu_X + b$$

$$\begin{aligned} \text{var}[Y] &= E\left[\underbrace{(aX + b)}_Y - \underbrace{(a\mu_X + b)}_{E[Y]}\right]^2 \\ &= E[(aX - a\mu_X)^2] = a^2 E[(X - \mu_X)^2] \\ &= a^2 \cdot \text{Var}[X] = a^2 \sigma_X^2. \end{aligned}$$

$$\Rightarrow \sigma_Y = \sqrt{\text{Var}(Y)} = \sqrt{a^2 \sigma_X^2} = |a| \sigma_X$$

$$(b) Y = \frac{X - \mu_X}{\sigma_X} \Rightarrow E[Y] = E\left[\frac{X - \mu_X}{\sigma_X}\right] = \frac{E[X] - \mu_X}{\sigma_X} = \frac{0}{\sigma_X}$$

$$\begin{aligned} \text{Var}(Y) &= E[Y^2] - \underbrace{\mu_Y^2}_{=0} = E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^2\right] \\ &= \frac{1}{\sigma_X^2} E[(X - \mu_X)^2] = \frac{\sigma_X^2}{\sigma_X^2} = 1. \end{aligned}$$

2. Papoulis 5-23 (5-19 in 3rd Edition):

X is Rayleigh with parameter $d \Rightarrow f_X(x) = \frac{x}{d^2} e^{-x^2/2d^2} \cdot 1_{[0, \infty)}(x)$

The n -th noncentral moment of X is given by

$$\begin{aligned} E[X^n] &= \frac{1}{d^2} \int_0^{\infty} x^{n+1} e^{-x^2/2d^2} dx = \frac{1}{2d^2} \int_{-\infty}^{\infty} |x|^{n+1} e^{-x^2/2d^2} dx \\ &= \frac{1}{d} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} |x|^{n+1} \cdot \frac{1}{\sqrt{2\pi}d} e^{-x^2/2d^2} dx = \begin{cases} 1 \cdot 3 \cdots n d^n \sqrt{\frac{\pi}{2}}, & n \text{ odd} \\ 2^{n/2} (n/2)! d^n, & n \text{ even.} \end{cases} \end{aligned}$$

n -th noncentral moment of zero-mean Gaussian with variance d^2 (see p. 148 of Papoulis)

$$\begin{aligned} \text{It follows that } E[X^2] &= 2d^2 \text{ and } E[X^4] = 8d^4 \\ \text{Thus } E[Y] &= E[b + cX^2] = b + cE[X^2] = b + 2cd^2 \\ \text{and } E[Y^2] &= E[b^2 + 2bcX^2 + c^2X^4] = b^2 + 4d^2bc + 8d^4c^2 \\ \Rightarrow \text{var}[Y] &= E[Y^2] - (E[Y])^2 = b^2 + 4d^2bc + 8d^4c^2 - (b^2 + 4bcd^2 + 4c^2d^4) \\ &= 4d^4c^2 \end{aligned}$$

3. Papoulis 5-33 (5-21 in 3rd Edition):

$$E[|X|] = \int_{-\infty}^{\infty} |x| f_X(x) dx = \int_0^{\infty} x f_X(x) dx - \int_{-\infty}^0 x f_X(x) dx$$

and

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x f_X(x) dx + \int_{-\infty}^0 x f_X(x) dx$$

So

$$\frac{E[|X|] + E[X]}{2} = \int_0^{\infty} x f_X(x) dx = \int_0^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu_X)^2}{2\sigma^2}} dx$$

$$= \int_{-\mu_X}^{\infty} (y + \mu_X) \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy, \text{ letting } y = x - \mu_X \Rightarrow dy = dx$$

$$= \int_{-\mu_X}^{\infty} y \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy + \mu_X \int_{-\mu_X}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy$$

$$= \frac{-\sigma}{\sqrt{2\pi}} \int_{-\mu_X}^{\infty} \left(\frac{-y}{\sigma^2}\right) e^{-y^2/2\sigma^2} dy + \mu_X \left[1 - \Phi\left(-\frac{\mu_X}{\sigma}\right)\right]$$

$$= \frac{-\sigma}{\sqrt{2\pi}} e^{-y^2/2\sigma^2} \Big|_{-\mu_X}^{\infty} + \mu_X \Phi\left(\frac{\mu_X}{\sigma}\right)$$

because $\Phi(d) = 1 - \Phi(-d)$
 $\forall d \in \mathbb{R}$.

$$= \frac{\sigma}{\sqrt{2\pi}} e^{-\mu_X^2/2\sigma^2} + \mu_X \Phi\left(\frac{\mu_X}{\sigma}\right)$$

Thus it follows that

$$E[|X|] = 2 \left[\frac{\sigma}{\sqrt{2\pi}} e^{-\mu_X^2/2\sigma^2} + \mu_X \Phi\left(\frac{\mu_X}{\sigma}\right) \right] - E[X]$$

$$= \sigma \sqrt{\frac{2}{\pi}} e^{-\mu_X^2/2\sigma^2} + 2\mu_X \Phi\left(\frac{\mu_X}{\sigma}\right) - \mu_X$$

4. Papoulis 5-22 (5-18 in 3rd Edition)

P. 3

Because $f_X(x) = \sum_{i=1}^n f(x|A_i) P(A_i)$, we have

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \left(\sum_{i=1}^n f(x|A_i) P(A_i) \right) dx \\ &= \sum_{i=1}^n \left[P(A_i) \int_{-\infty}^{\infty} x f(x|A_i) dx \right] \\ &= \sum_{i=1}^n P(A_i) E[X|A_i]. \end{aligned}$$

5. Although we can apply the definitions of \bar{X} and σ_X^2 directly, in this case it is much easier to use the characteristic function or generating function:

$$\begin{aligned} \phi_X(s) &= E[e^{sX}] = \sum_{k=0}^n e^{sk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} (e^s p)^k (1-p)^{n-k} \\ &= [e^s p + (1-p)]^n = [1 + p(e^s - 1)]^n \end{aligned}$$

$$\begin{aligned} \bar{X} = E[X] &= \left. \frac{\partial \phi_X(s)}{\partial s} \right|_{s=0} = e^{s np} (1 + p(e^s - 1))^{n-1} \Big|_{s=0} \\ &= \boxed{np} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \left. \frac{\partial^2 \phi_X(s)}{\partial s^2} \right|_{s=0} = e^{s np} (1 + p(e^s - 1))^{n-2} (1 + p(n e^s - 1)) \Big|_{s=0} \\ &= np + n(n-1)p^2 \end{aligned}$$

Thus

$$\begin{aligned} \sigma_X^2 = \text{Var}(X) &= E[X^2] - (E[X])^2 = np + n(n-1)p^2 - (np)^2 \\ &= np - np^2 = \boxed{np(1-p)} \end{aligned}$$

Alternatively, one can use the definitions of expectations directly, although this requires a bit more cleverness to direct the sums: P.4

$$E\{X\} = \sum_{k=0}^n k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = np \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{s=0}^{n-1} \frac{n!}{s!(n-s)!} p^s (1-p)^{n-s} = \boxed{np}, \text{ where } s=k-1, m=n-1.$$

Similarly

$$E\{X^2\} = \sum_{k=0}^n k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = np \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-k)!} k p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{s=0}^{n-1} \frac{n!}{s!(n-s)!} (s+1) p^s (1-p)^{n-s}, \text{ where } s=k-1, m=n-1$$

$$= np \left[\sum_{s=0}^{n-1} \frac{n!}{s!(n-s)!} s p^s (1-p)^{n-s} + \sum_{s=0}^{n-1} \frac{n!}{s!(n-s)!} p^s (1-p)^{n-s} \right]$$

$$= np \left[\underbrace{(n-1)p}_m + 1 \right] = n^2 p^2 - np^2 + np$$

$$= n^2 p^2 + np(1-p)$$

$$\therefore \sigma_x^2 = \text{Var}\{X\} = E\{X^2\} - (E\{X\})^2 = n^2 p^2 + np(1-p) - (np)^2 = \boxed{np(1-p)}$$

Yet another way of simplifying the sums involves using derivatives:

$$E\{X\} = \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} = (1-p)^n \sum_{k=0}^n \binom{n}{k} k \left(\frac{p}{1-p}\right)^k$$

$$= p(1-p)^{n-1} \sum_{k=0}^n \binom{n}{k} k \left(\frac{p}{1-p}\right)^{k-1} = p(1-p)^{n-1} (1-p)^{-2} \sum_{k=0}^n \binom{n}{k} \left[k \left(\frac{p}{1-p}\right)^{k-1} \cdot \frac{1}{(1-p)^2} \right]$$

$$= p(1-p)^{n+1} \sum_{k=0}^n \binom{n}{k} \cdot \frac{d}{dp} \left\{ \left(\frac{p}{1-p}\right)^k \right\} = p(1-p)^{n+1} \frac{d}{dp} \left\{ \sum_{k=0}^n \binom{n}{k} \left(\frac{p}{1-p}\right)^k \cdot 1^{n-k} \right\}$$

$$= p(1-p)^{n+1} \frac{d}{dp} \left\{ \left(\frac{p}{1-p} + 1\right)^n \right\} = p(1-p)^{n+1} \cdot \frac{d}{dp} \left\{ (1-p)^{-n} \right\}$$

$$= p(1-p)^{n+1} (-n)(1-p)^{-(n+1)} (-1) = \boxed{np}. \text{ A similar computation can be done for } E\{X^2\}.$$

6. $E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\beta x} dx = \frac{1}{\beta \Gamma(\alpha)} \int_0^{\infty} y^\alpha e^{-y} dy$ P.5
 $\stackrel{\text{let } y = \beta x \Rightarrow dy = \beta dx}{=} \frac{1}{\beta} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\alpha}{\beta}$, since $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$.

Similarly

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{-\infty}^{\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha+1} e^{-\beta x} dx = \frac{1}{\beta^2 \Gamma(\alpha)} \int_0^{\infty} y^{\alpha+1} e^{-y} dy$$

$$= \frac{1}{\beta^2} \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\beta^2}$$

Now $\sigma_x^2 = E\{X^2\} - (E\{X\})^2 = \frac{\alpha(\alpha+1)}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$
 since $\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\Gamma(\alpha)$.

7. $E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx = \frac{1}{b-a} \int_a^b x dx = \frac{x^2}{2(b-a)} \Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{x^3}{3(b-a)} \Big|_a^b = \dots = \frac{a^2 + ab + b^2}{3}$$

The variance σ_x^2 is given by

$$\sigma_x^2 = E\{X^2\} - (E\{X\})^2 = \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$$

8. Papoulis 5-30: The Markov Inequality (Papoulis Eq. (5-58)) states that if X is a random variable with $f_X(x) = 0$ for $x < 0$, then

$$P\{X \geq \alpha\} \leq \frac{E\{X\}}{\alpha}, \quad \alpha > 0. \quad (*)$$

Now note that for any real RV X , when s is a real number $\Psi = e^{sX}$ is a real random variable that is always non-negative. Thus applying $(*)$, we have

$$P\{e^{sX} \geq \alpha\} \leq \frac{E\{e^{sX}\}}{\alpha} = \frac{\phi_X(s)}{\alpha}$$

Now letting $\alpha = e^{sA}$, we get

$$P\{e^{sX} \geq e^{sA}\} = P\{sX \geq sA\} = P\{X \geq A\} \leq \frac{\phi_X(s)}{e^{sA}} = e^{-sA} \phi_X(s), \quad s > 0$$

and $P\{e^{sX} \geq e^{sA}\} = P\{sX \geq sA\} = P\{X \leq A\} \leq e^{-sA} \phi_X(s), \quad s < 0.$

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\alpha!}{(\alpha-1)!} = \alpha$$

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9. Papoulis 5-31:

(a) If $\Phi_X(\omega) = e^{-\alpha|\omega|}$, then

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha|\omega|} e^{-i\omega x} d\omega = \frac{1}{2\pi} \int_0^{\infty} e^{-(\alpha+i\omega)x} d\omega + \frac{1}{2\pi} \int_{-\infty}^0 e^{(\alpha-i\omega)x} d\omega$$

$$= \frac{1}{2\pi} \left\{ \frac{e^{-(\alpha+i\omega)x}}{-(\alpha+i\omega)} \Big|_0^{\infty} + \frac{e^{(\alpha-i\omega)x}}{\alpha-i\omega} \Big|_0^{\infty} \right\} = \frac{1}{2\pi(\alpha+i\omega)} + \frac{1}{2\pi(\alpha-i\omega)}$$

$$= \frac{1}{2\pi} \frac{2\alpha}{\alpha^2 + \omega^2} = \frac{\alpha}{\pi(\alpha^2 + \omega^2)}$$

(b) If $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} \frac{\alpha}{2} e^{-\alpha|x|} e^{i\omega x} dx = \frac{\alpha}{2} \int_0^{\infty} e^{-(\alpha-i\omega)x} dx + \frac{\alpha}{2} \int_{-\infty}^0 e^{(\alpha+i\omega)x} dx$$

$$= \dots = \frac{\alpha^2}{\alpha^2 + \omega^2}$$

similar to part (a)

10. (a) If $Y = aX + b$, then

$$\Phi_Y(\omega) = E\{e^{i\omega Y}\} = E\{e^{i\omega[aX+b]}\} = E\{e^{i\omega b} e^{i\omega a X}\}$$

$$= e^{i\omega b} E\{e^{i\omega a X}\} = e^{i\omega b} \Phi_X(a\omega)$$

(b) $\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - i2\omega x)} dx$

completing the square

$$= e^{-\omega^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-i\omega)^2} dx$$

Formally, we can evaluate this integral by the change of variable by letting $y = x - i\omega \Rightarrow dy = dx$

$$\Rightarrow e^{-\omega^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = \boxed{e^{-\omega^2/2}}$$

Contour integration can be used to justify this formal computation rigorously.

(c) If X is a $N(0,1)$ R.V., $Y = \sigma X + \mu$ has mean μ and variance σ^2
 $\therefore \Phi_Y(\omega) = \Phi_X(\omega\sigma) e^{i\omega\mu} = e^{i\omega\mu} e^{-\sigma^2\omega^2/2} = e^{i\omega\mu - \omega^2\sigma^2/2}$ (from part (a))

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Homework #6 Solutions Addendum

Papoulis 4ed., Problem 5-30:

(a) X is a RV with pdf $f_X(x) = \frac{1}{2} \cdot \frac{1}{[10,12]}$.

A new RV Y is defined by $Y = g(X) = X^3$.
(n.b. $y = g(x) = x^3$ is one-to-one.)

Using the direct pdf method, we find $f_Y(y)$ as follows:

$$f_Y(y) = f_X[X(y)] \left| \frac{dx(y)}{dy} \right|$$

$$\text{Here we have } x(y) = y^{1/3}$$

$$\Rightarrow \frac{dx(y)}{dy} = \frac{1}{3} y^{-2/3} = \frac{1}{3y^{2/3}}$$

Thus it follows that

$$f_Y(y) = \frac{1}{2} \cdot \frac{1}{[10,12]} (y^{1/3}) \cdot \left| \frac{1}{3y^{2/3}} \right|$$

$$= \frac{1}{6} y^{2/3} \cdot \frac{1}{[1000,1728]}$$

$$\begin{aligned} \text{(b) } E[Y] &= E[X^3] = \int_{10}^{12} x^3 \cdot \frac{1}{2} dx = \frac{x^4}{8} \Big|_{10}^{12} \\ &= 1342 \end{aligned}$$

n.b. Eq. (5-86) states $\mu_Y \approx g(\mu_X) + g''(\mu_X) \frac{\sigma_X^2}{2}$
 $g''(x) = 6x$, $\sigma_X^2 = \frac{1}{3}$. Thus by (5-86)

$$E[Y] \approx 11^3 + 6(11) \cdot \frac{1}{6} = 11^3 + 11 = 1342.$$

Papoulis 4ed., Problem 5-31

Using Eq. (5-86), we have

$$E[Y] \approx g(\eta_x) + g''(\eta_x) \cdot \frac{\sigma_x^2}{2}$$

Here X has mean 100 and $\sigma_x^2 = 9^2 = 81$

Thus we have

$$g(x) = \frac{1}{x} \Rightarrow g''(x) = \frac{2}{x^3}$$

Thus

$$E[Y] \approx \frac{1}{100} + \frac{2}{(100)^3} \cdot \frac{81}{2} = 0.010081$$

A numerical computation of the integral giving the expectation yielded 0.010083