ECE 600 Homework # 6 Solutions Bell

1. Papoulis 5-22 (5-18 in 3rd Edition)

(a) $E[Y] = E[aX+b] = a\,E[X]+b = a\eta_X + b$

$$\mathrm{var}[Y] = E[(\underbrace{aX+b}_{Y} - \underbrace{(a\eta_X+b)}_{E[Y]})^2]$$

$$= E[(aX - a\eta_X)^2] = a^2\,E[(X-\eta_X)^2]$$

$$= a^2 \cdot \mathrm{Var}[X] = a^2\sigma_X^2.$$

$$\Rightarrow \sigma_Y = \sqrt{\mathrm{var}(Y)} = \sqrt{a^2\sigma_X^2} = |a|\sigma_X$$

(b) $$Y = \frac{X-\eta_X}{\sigma_X} \Rightarrow E[Y] = E\left[\frac{X-\eta_X}{\sigma_X}\right] = \frac{E[X]-\eta_X}{\sigma_X} = \frac{0}{\sigma_X} = 0.$$

$$\mathrm{Var}(Y) = E[Y^2] - \cancel{\eta_Y^2} \overset{=0}{=} E\left[\left(\frac{X-\eta_X}{\sigma_X}\right)^2\right]$$

$$= \frac{1}{\sigma_X^2}\,E[(X-\eta_X)^2] = \frac{\sigma_X^2}{\sigma_X^2} = 1.$$

2. Papoulis 5-23 (5-19 in 3rd Edition):

X is Rayleigh with parameter $\alpha \Rightarrow f_X(x) = \frac{x}{\alpha^2}e^{-x^2/2\alpha^2}\cdot 1_{[0,\infty)}(x)$

The n-th noncentral moment of X is given by

$$E[X^n] = \frac{1}{\alpha^2}\int_0^\infty x^{n+1}e^{-x^2/2\alpha^2}dx = \frac{1}{2\alpha^2}\int_{-\infty}^{\infty}|x|^{n+1}e^{-x^2/2\alpha^2}dx$$

$$= \frac{1}{\alpha}\sqrt{\frac{\pi}{2}}\underbrace{\int_{-\infty}^{\infty}|x|^{n+1}\cdot\frac{1}{\sqrt{2\pi}\alpha}e^{-x^2/2\alpha^2}dx}_{\text{n-th noncentral moment of zero-mean Gaussian with variance } \alpha^2 \text{ (see p. 148 of Papoulis)}} = \begin{cases} 1\cdot 3\cdots n\,\alpha^n\sqrt{\frac{\pi}{2}}, & n \text{ odd} \\ 2^{n/2}(n/2)!\,\alpha^n, & n \text{ even.} \end{cases}$$

It follows that $E[X^2] = 2\alpha^2$ and $E[X^4] = 8\alpha^4$

Thus $E[Y] = E[b+cX^2] = b + c\,E[X^2] = b + 2c\alpha^2$

and $E[Y^2] = E[b^2 + 2bcX^2 + c^2X^4] = b^2 + 4\alpha^2bc + 8\alpha^4c^2$

$$\Rightarrow \mathrm{var}[Y] = E[Y^2] - (E[Y])^2 = \cancel{b^2} + \cancel{4\alpha^2bc} + 8\alpha^4c^2 - (\cancel{b^2} + \cancel{4bc\alpha^2} + 4c^2\alpha^4)$$

$$= 4\alpha^4c^2$$

3. Papoulis 5-33 (5-21 in 3rd Edition):

$$E[|\mathbb{X}|] = \int_{-\infty}^{\infty} |x| f_{\mathbb{X}}(x)\,dx = \int_0^{\infty} x f_{\mathbb{X}}(x)\,dx - \int_{-\infty}^{0} x f_{\mathbb{X}}(x)\,dx$$

and

$$E[\mathbb{X}] = \eta_x = \int_{-\infty}^{\infty} x\, f_{\mathbb{X}}(x)\,dx = \int_0^{\infty} x\, f_{\mathbb{X}}(x)\,dx + \int_{-\infty}^{0} x\, f_{\mathbb{X}}(x)\,dx$$

So

$$\frac{E[|\mathbb{X}|] + E[\mathbb{X}]}{2} = \int_0^{\infty} x\, f_{\mathbb{X}}(x)\,dx = \int_0^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\eta_x)^2/2\sigma^2}\,dx$$

$$= \int_{-\eta_x}^{\infty} (y+\eta_x) \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2\sigma^2}, \quad \text{letting } y = x - \eta_x \Rightarrow dy = dx$$

$$= \int_{-\eta_x}^{\infty} y \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2\sigma^2}\,dy + \eta_x \int_{-\eta_x}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2\sigma^2}\,dy$$

$$= \frac{-\sigma}{\sqrt{2\pi}} \int_{-\eta_x}^{\infty} \left(\frac{-y}{\sigma^2}\right) e^{-y^2/2\sigma^2}\,dy + \eta_x \left[1 - \Phi\left(\frac{-\eta_x}{\sigma}\right)\right]$$

$$= -\frac{\sigma}{\sqrt{2\pi}} e^{-y^2/2\sigma^2} \Big|_{-\eta_x}^{\infty} + \eta_x \Phi\left(\frac{\eta_x}{\sigma}\right)$$

because $\Phi(\alpha) = 1 - \Phi(-\alpha)$ $\forall \alpha \in \mathbb{R}$.

$$= \frac{\sigma}{\sqrt{2\pi}} e^{-\eta_x^2/2\sigma^2} + \eta_x \Phi\left(\frac{\eta_x}{\sigma}\right)$$

Thus it follows that

$$E[|\mathbb{X}|] = 2\left[\frac{\sigma}{\sqrt{2\pi}} e^{-\eta_x^2/2\sigma^2} + \eta_x \Phi\left(\frac{\eta_x}{\sigma}\right)\right] - E[\mathbb{X}]$$

$$= \sigma\sqrt{\frac{2}{\pi}}\, e^{-\eta_x^2/2\sigma^2} + 2\eta_x \Phi\left(\frac{\eta_x}{\sigma}\right) - \mu_x.$$

4. Papoulis 5-22 (5-18 in 3rd Edition)

Because $f_X(x) = \sum_{i=1}^{n} f_X(x|A_i)\,P(A_i)$, we have

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-\infty}^{\infty} x\left(\sum_{i=1}^{n} f_X(x|A_i)\,P(A_i)\right) dx$$

$$= \sum_{i=1}^{n}\left[P(A_i)\int_{-\infty}^{\infty} x\, f_X(x/A_i)\, dx\right]$$

$$= \sum_{i=1}^{n} P(A_i)\, E[X\,|\,A_i].$$

5. Although we can apply the definitions of $\overline{X}$ and σ_X^2 directly, in this case it is much easier to use the characteristic function of generating function:

$$\phi_X(s) = E[e^{sX}] = \sum_{k=0}^{n} e^{sk}\binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}(e^s p)^k (1-p)^{n-k}$$

$$= [e^s p + (1-p)]^n = [1 + p(e^s - 1)]^n$$

$$\overline{X} = E[X] = \left.\frac{\partial \phi_X(s)}{\partial s}\right|_{s=0} = \left. e^s np\left(1+p(e^s-1)\right)^{n-1}\right|_{s=0}$$

$$= \boxed{np}$$

$$E[X^2] = \left.\frac{\partial^2 \phi_X(s)}{\partial s^2}\right|_{s=0} = \left. e^s np(1+p(e^s-1))^{n-2}(1+p(ne^s-1))\right|_{s=0}$$

$$= np + n(n-1)p^2$$

Thus

$$\sigma_X^2 = Var(X) = E[X^2] - (E[X])^2 = np + n(n-1)p^2 - (np)^2$$

$$= np - np^2 = \boxed{np(1-p)}$$

Alternatively, one can use the definitions of expectation directly, although this requires a bit more cleverness to direct the sums:

$$E\{\#\} = \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1}(1-p)^{n-k}$$

$$= np \underbrace{\sum_{s=0}^{m} \frac{m!}{s!(m-s)!} p^s (1-p)^{m-s}}_{1} = \boxed{np}, \text{ where } \begin{array}{l} s=k-1 \\ m=n-1. \end{array}$$

Similarly

$$E\{\#^2\} = \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} k p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{s=0}^{m} \frac{m!}{s!(m-s)!} (s+1) p^s (1-p)^{m-s}, \quad \text{where } \begin{array}{l} s=k-1 \\ m=n-1 \end{array}$$

$$= np \left[\sum_{s=0}^{m} \frac{m!}{s!(m-s)!} s p^s (1-p)^{m-s} + \sum_{s=0}^{m} \frac{m!}{s!(m-s)!} p^s (1-p)^{m-s} \right]$$

$$= np \left[\underbrace{(n-1)}_{m} p + 1 \right] = n^2p^2 - np^2 + np$$

$$= n^2p^2 + np(1-p)$$

$$\therefore \quad \sigma_\#^2 = \mathrm{Var}\{\#\} = E\{\#^2\} - (E\{\#\})^2 = n^2p^2 + np(1-p) - (np)^2$$

$$= \boxed{np(1-p)}$$

Yet another way of simplifying the sums involves using derivatives:

$$E\{\#\} = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = (1-p)^n \sum_{k=0}^{n} \binom{n}{k} k \left(\frac{p}{1-p}\right)^k$$

$$= p(1-p)^{n-1} \sum_{k=0}^{n} \binom{n}{k} k \left(\frac{p}{1-p}\right)^{k-1} = p(1-p)^{n-1}(1-p)^2 \sum_{k=0}^{n} \binom{n}{k} \left[k \left(\frac{p}{1-p}\right)^{k-1} \cdot \frac{1}{(1-p)^2} \right]$$

$$= p(1-p)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{d}{dp} \left\{ \left(\frac{p}{1-p}\right)^k \right\} = p(1-p)^{n+1} \frac{d}{dp} \left\{ \sum_{k=0}^{n} \binom{n}{k} \left(\frac{p}{1-p}\right)^k \cdot 1^{n-k} \right\}$$

$$= p(1-p)^{n+1} \cdot \frac{d}{dp} \left\{ \left(\frac{p}{1-p} + 1 \right)^n \right\} = p(1-p)^{n+1} \cdot \frac{d}{dp} \left\{ (1-p)^{-n} \right\}$$

$$= p(1-p)^{n+1} (-n)(1-p)^{-(n+1)}(-1) = \boxed{np}.$$ A similar computation can be done for $E\{\#^2\}$.

6. let $y = \beta x \Rightarrow dy = \beta\, dx$

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{0}^{\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\beta x}\, dx = \frac{1}{\beta\Gamma(\alpha)} \underbrace{\int_0^\infty y^\alpha e^{-y}\, dy}_{\Gamma(\alpha+1)}$$

$$= \frac{1}{\beta} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \boxed{\frac{\alpha}{\beta}}, \text{ since } \Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

Similarly

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha+1} e^{-\beta x}\, dx = \frac{1}{\beta^2\Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-y}\, dy$$

$$= \frac{1}{\beta^2} \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\beta^2}.$$

Now $\sigma_X^2 = E\{X^2\} - (E\{X\})^2 = \dfrac{\alpha(\alpha+1)}{\beta^2} - \dfrac{\alpha^2}{\beta^2} = \boxed{\dfrac{\alpha}{\beta^2}}$

since $\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1)$
$= (\alpha+1)\alpha\Gamma(\alpha)$.

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\alpha!}{(\alpha-1)!} = \alpha$$

7.
$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{1}{b-a}\int_a^b x\, dx = \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \boxed{\frac{b+a}{2}}$$

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \frac{1}{b-a}\int_a^b x^2\, dx = \frac{x^3}{3(b-a)}\Big|_a^b = \cdots = \frac{a^2+ab+b^2}{3}$$

The variance σ_x^2 is given by

$$\sigma_x^2 = E\{X^2\} - (E\{X\})^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \boxed{\frac{(b-a)^2}{12}}$$

8. Papoulis 5-30: The Markov Inequality (Papoulis Eq. (5-58)) states that if X is a random variable with $f_X(x) = 0$ for $x < 0$, then

$$P\{X \geq \alpha\} \leq \frac{E\{X\}}{\alpha}, \quad \alpha > 0. \qquad (*)$$

Now note that for any real RV X, when s is a real number $Y = e^{sX}$ is a real random variable that is always non-negative. Thus applying $(*)$, we have

$$P(\{e^{sX} \geq \alpha\}) \leq \frac{E\{e^{sX}\}}{\alpha} = \frac{\Phi_X(s)}{\alpha}$$

Now letting $\alpha = e^{sA}$, we get

$$P(\{e^{sX} \geq e^{sA}\}) = P(\{sX \geq sA\}) = P\{X \geq A\} \leq \frac{\Phi_X(s)}{e^{sA}} = e^{-sA}\Phi_X(s), \quad s > 0$$

and $P(\{e^{sX} \geq e^{sA}\}) = P(\{sX \geq sA\}) = P\{X \leq A\} \leq e^{-sA}\Phi_X(s), \quad s < 0$.

9. Papoulis 5-31:

(a) If $\Phi_x(\omega) = e^{-\alpha|\omega|}$, then

$$f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha|\omega|} e^{-i\omega x}\, d\omega = \frac{1}{2\pi}\int_{0}^{\infty} e^{-(\alpha+ix)\omega}\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{0} e^{(\alpha-ix)\omega}\, d\omega$$

$$= \frac{1}{2\pi}\left\{ \left.\frac{e^{-(\alpha+ix)\omega}}{-(\alpha+ix)}\right|_0^{\infty} + \left.\frac{e^{(\alpha-ix)\omega}}{\alpha-ix}\right|_{-\infty}^{0} \right\} = \frac{1}{2\pi(\alpha+ix)} + \frac{1}{2\pi(\alpha-ix)}$$

$$= \frac{1}{2\pi}\,\frac{2\alpha}{\alpha^2+x^2} = \frac{\alpha}{\pi(\alpha^2+x^2)}$$

(b) If $f_x(x) = \frac{\alpha}{2} e^{-\alpha|x|}$

$$\Phi_x(\omega) = \int_{-\infty}^{\infty} \frac{\alpha}{2} e^{-\alpha|\omega|} e^{i\omega x}\, dx = \frac{\alpha}{2}\int_0^{\infty} e^{-(\alpha-i\omega)x}\, dx + \frac{\alpha}{2}\int_{-\infty}^{0} e^{+(\alpha+i\omega)x}\, dx$$

$$= \cdots \underset{\text{similar to part (a)}}{=} \frac{\alpha^2}{\alpha^2+\omega^2}$$

10. (a) If $Y = aX+b$, then

$$\Phi_Y(\omega) = E\{e^{i\omega Y}\} = E\{e^{i\omega[aX+b]}\} = E\{e^{i\omega b} e^{i\omega a X}\}$$

$$= e^{i\omega b} E\{e^{i\omega a X}\} = e^{i\omega b}\,\Phi_X(a\omega)$$

(b)
$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - i2\omega x)}\, dx$$

$$\underset{\text{completing the square}}{=} e^{-\omega^2/2}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-i\omega)^2}\, dx$$

<u>Formally</u>, we can evaluate this integral by the change of variable by letting $y = x - i\omega \Rightarrow dy = dx$

$$\Rightarrow \quad e^{-\omega^2/2}\,\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy}_{1} = \boxed{e^{-\omega^2/2}}$$

Contour integration can be used to rigorously justify this formal computation.

(c) If X is an $N(0,1)$ R.V, $Y = \sigma X + \mu$ has mean μ and variance σ^2

$$\therefore\ \Phi_Y(\omega) = \Phi_X(\sigma\omega)e^{i\omega\mu} = e^{i\omega\mu}e^{-\sigma^2\omega^2/2} = e^{i\omega\mu - \omega^2\sigma^2/2} \quad (\text{from part (a)})$$

Homework # 6 Solutions
Addendum

Papoulis 4 ed., Problem 5-30:

(a) $\mathbb{X}$ is a RV with pdf $f_{\mathbb{X}}(x) = \frac{1}{2} \cdot \mathbb{1}_{[10,12]}(x)$.

A new RV $\mathbb{Y}$ is defined by $\mathbb{Y} = g(\mathbb{X}) = \mathbb{X}^3$.
(n.b. $y = g(x) = x^3$ is one-to-one)
Using the direct pdf method, we find $f_{\mathbb{Y}}(y)$ as follows:

$$f_{\mathbb{Y}}(y) = f_{\mathbb{X}}[x(y)] \left| \frac{dx(y)}{dy} \right|$$

Here we have $x(y) = y^{1/3}$

$$\Rightarrow \quad \frac{dx(y)}{dy} = \frac{1}{3} y^{-2/3} = \frac{1}{3y^{2/3}}$$

Thus it follows that

$$f_{\mathbb{Y}}(y) = \frac{1}{2} \cdot \mathbb{1}_{[10,12]}(y^{1/3}) \cdot \left| \frac{1}{3y^{2/3}} \right|$$

$$= \frac{1}{6\,y^{2/3}} \cdot \mathbb{1}_{[1000,1728]}(y)$$

(b) $$E[\mathbb{Y}] = E[\mathbb{X}^3] = \int_{10}^{12} x^3 \cdot \frac{1}{2}\, dx = \frac{x^4}{8}\Big|_{10}^{12}$$

$$= 1342$$

n.b. Eq. (5-86) states $\eta_y \simeq g(\eta_x) + g''(\eta_x)\frac{\sigma_x^2}{2}$
$g''(x) = 6x$, $\sigma_x^2 = \frac{1}{3}$. Thus by (5-86)
$E[\mathbb{Y}] \simeq 11^3 + 6(11) \cdot \frac{1}{6} = 11^3 + 11 = 1342$.

Papoulis 4ed., Problem 5-31

Using Eq. (5-86), we have

$$E[Y] \cong g(\eta_x) + g''(\eta_x) \cdot \frac{\sigma_x^2}{2}$$

Here X has mean 100 and $\sigma_x^2 = 9^2 = 81$

Thus we have

$$g(x) = \frac{1}{x} \Rightarrow g''(x) = \frac{2}{x^3}$$

Thus

$$E[Y] \cong \frac{1}{100} + \frac{2}{(100)^3} \cdot \frac{81}{2} = 0.010081$$

A numerical computation of the integral giving the expectation yielded 0.010083