EE600 Homework #5 Solutions Bell

1. Papoulis 5-2: $\quad Y = -4X + 3$ and $f_X(x) = 2e^{-2x} \cdot 1_{[0,\infty)}(x)$

$$F_Y(y) = P(\{Y \le y\}) = P(\{-4X + 3 \le y\})$$

$$= P(\{X \ge \tfrac{3-y}{4}\}) = 1 - F_X\left(\tfrac{3-y}{4}\right)$$

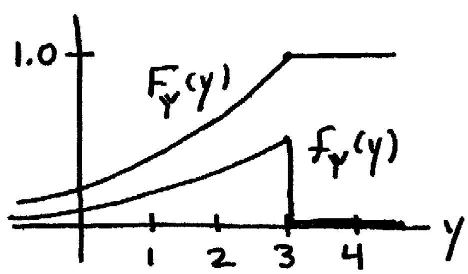
$$F_X(\alpha) = \int_{-\infty}^{\alpha} f_X(x)\,dx = (1 - e^{-2\alpha})\,1_{[0,\infty)}(\alpha)$$

$$\therefore\ F_Y(y) = 1 - F_X\left(\tfrac{3-y}{4}\right) = \exp\left(\tfrac{y-3}{2}\right) \cdot 1_{[0,\infty)}\left(\tfrac{3-y}{4}\right)$$

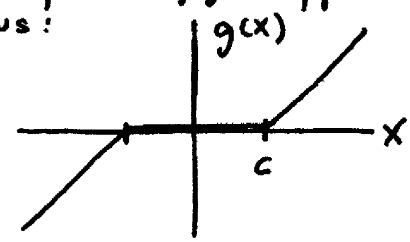
$$= \exp\left(\tfrac{y-3}{2}\right) 1_{(-\infty,3]}(y) + 1_{(3,\infty)}(y)$$

and

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2}\exp\left(\tfrac{y-3}{2}\right) 1_{(-\infty,3]}(y)$$



In this problem, $g(x)$ appears as follows:



2. Papoulis 5-3:

$$g(x) = \begin{cases} x - c, & x > c \\ x + c, & x < c \\ 0, & \text{elsewhere} \end{cases}$$

Here we have $F_X(x) = \Phi\left(\frac{x}{c}\right)$

We want $F_Y(y)$, where $Y = g(X)$.

For $y < 0$: $\quad F_Y(y) = P(\{Y \le y\}) = P(\{X \le y - c\}) = F_X(y - c)$

$$= \Phi(y/c - 1)$$

For $y \ge 0$: $\quad F_Y(y) = P(\{Y \le y\}) = P(\{X \le y + c\}) = F_X(y + c)$

$$= \Phi\left(\tfrac{y+c}{c}\right) = \Phi(y/c + 1)$$

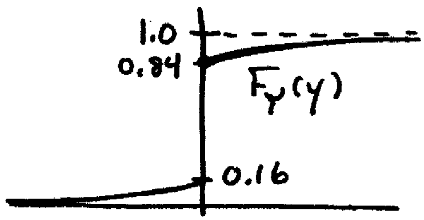
$$\therefore\ F_Y(y) = \begin{cases} \Phi(y/c - 1), & y < 0 \\ \Phi(y/c + 1), & y \ge 0. \end{cases}$$

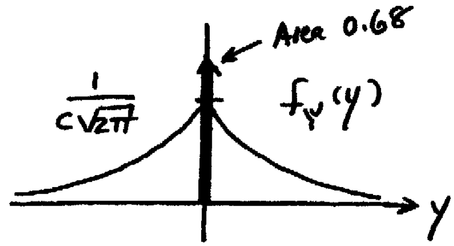
(Problem 2 — Papoulis 5-3 — continued)

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \left(\Phi(1) - \Phi(-1)\right)\delta(y) + \frac{1}{c\sqrt{2\pi}}\left[e^{-\frac{(y+c)^2}{2c^2}} 1_{[0,\infty)}(y) + e^{-\frac{(y-c)^2}{2c^2}} 1_{(-\infty,0]}(y)\right]$$

$$= 0.68\,\delta(y) + \frac{1}{c\sqrt{2\pi}}\left[e^{-\frac{(y+c)^2}{2c^2}} 1_{[0,\infty)}(y) + e^{-\frac{(y-c)^2}{2c^2}} 1_{(-\infty,0]}(y)\right]$$

Plots of $F_Y(y)$ and $f_Y(y)$ appear as follows:





3. Papoulis 5-4: If $Y = X^2$ and

$$F_X(x) = \frac{x+2c}{4c}\cdot 1_{[-2c,2c]}(x) + 1_{(2c,\infty)}(x)$$

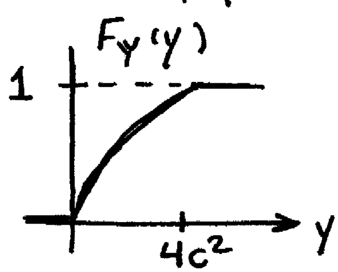
then

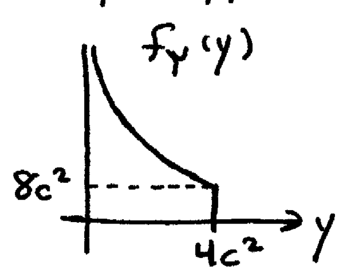
$$F_Y(y) = P(\{Y \le y\}) = P(\{-\sqrt{y} \le X \le +\sqrt{y}\})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$
$$= \frac{\sqrt{y}}{2c} 1_{[0,4c^2]}(y) + 1_{(4c^2,\infty)}(y)$$

and

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{4c\sqrt{y}} 1_{[0,4c^2]}(y)$$

Plots of $F_Y(y)$ and $f_Y(y)$ appear as follows:





4. Papoulis 5-7(a): If $X_1, \ldots, X_{200}$ are the 200 independently placed points in the interval $(0,100)$ with each point in the interval equally likely, we are interested in $Z = \min\{X_1, \ldots, X_{200}\}$.

Let $n(0,z) \triangleq$ no. points in interval $(0,z]$, $z \in (0,100)$. Then

$$F_Z(z) = P(\{Z \le z\}) = P(\{n(0,z) > 0\})$$
$$= 1 - P(\{Z > z\}) = 1 - P(\{n(0,z) = 0\}).$$

The event $E_{k,z}$ that any given point X_k is in the interval $(0,z]$ can be viewed as a Bernouli trial with prob. p_z of success, where p_z is given by

$$p_z = \frac{\text{length } (0,z]}{\text{length } (0,100)} = \frac{z}{100}$$

Hence we have

$$F_Z(z) = 1 - P(\{n(0,z) = 0\}) = 1 - \binom{200}{0} p_z^0 (1-p_z)^{200}$$
$$= 1 - \left(1 - \frac{z}{100}\right)^{200}, \quad z \in (0,100)$$

$$\therefore \quad F(z) = \begin{cases} 0, & z \le 0 \\ 1 - \left(1 - \frac{z}{100}\right)^{200}, & 0 < z < 100 \\ 1, & z \ge 100 \end{cases}$$

<u>n.b.</u> From calculus, you may recall that $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$ noting above that we have the term

$$\left(1 - \frac{z}{100}\right)^{200} = \left(1 + \frac{(-2z)}{200}\right)^{200} = \left(1 + \frac{(-2z)}{n}\right)^n \Bigg|_{n=200}$$

We may be tempted to make the approximation

$$\left(1 - \frac{z}{100}\right)^{200} \simeq e^{-2z}.$$

This yields the approximation $F_Z(z) \cong (1 - e^{-2z}) 1_{(0,100)}(z)$
$\simeq (1 - e^{-2z}) 1_{(0,\infty)}(z)$
which is the Poisson approx. mentioned in Papoulis 5-7(b).

5. Papoulis 5-9: Note that for both (a) and (b), $f_Y(y) = 0,\ y < 0$.

(a) If $y \geq 0$ and $Y = |X|$, then

$$F_Y(y) = P(\{Y \leq y\}) = P(\{-y \leq X \leq y\})$$
$$= [F_X(y) - F_X(-y)]\, 1_{[0,\infty)}(y)$$
$$f_Y(y) = \frac{dF_Y(y)}{dy} = [f_X(y) + f_X(-y)]\, 1_{[0,\infty)}(y)$$

(b) For $y \geq 0$ and $Y = e^{-X} u(X) = e^{-X} 1_{[0,\infty)}(X)$,

we have

$$F_Y(y) = P(\{Y \leq y\}) = P(\{e^{-X} 1_{[0,\infty)}(X) \leq y\})$$
$$= P(\{X \leq 0\} \cup \{X > -\ln y\})$$
$$= F_X(0)\, 1_{[0,\infty)}(y) + 1 - F_X(-\ln y)$$

Thus

$$f_Y(y) = \frac{dF_Y(y)}{dy} = F_X(0)\,\delta(y) + \frac{1}{y} f_X(-\ln y)\, 1_{(0,\infty)}(y).$$

6. Papoulis 5-11: If $y = \tan^{-1} x$, then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

So

$$f_Y(y) = \frac{f_X[x(y)]}{\left|\frac{dy}{dx}\right|} = \frac{1+x^2}{\pi(1+x^2)} = \frac{1}{\pi},\quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

$$\therefore\ f_Y(y) = \frac{1}{\pi}\, 1_{(-\pi/2,\,\pi/2)}(y)$$

7. The cdf will be discontinuous in this case. We note the following:

(i) $P(\{Y = -c\}) = F_X(-b) = \Phi\left(\frac{-b-1}{1}\right) = \Phi(-(b+1))$

(ii) For $-b < X < -a$, we have $y(x) = \left(\frac{c}{b-a}\right)x + \frac{ca}{b-a}$

$$\Rightarrow \quad x = \left(\frac{b-a}{c}\right)y - a$$

Hence

$$F_Y(y) = P(\{Y \le y\}) = P\left(\left\{X \le \left(\frac{b-a}{c}\right)y - a\right\}\right)$$

$$= \Phi\left(\frac{\left(\frac{b-a}{c}\right)y - a - 1}{1}\right) = \Phi\left(\left(\frac{b-a}{c}\right)y - (a+1)\right)$$

(iii) $P(\{Y = 0\}) = P(\{-a < X \le a\}) = F_X(a) - F_X(-a)$

$$= \Phi(a-1) - \Phi(-(a+1))$$

(iv) For $a < X \le b$, we have $y(x) = \frac{c}{b-a} - \frac{ca}{b-a}$

$$\Rightarrow x = \left(\frac{b-a}{c}\right)y + a$$

Hence

$$F_Y(y) = P(\{Y \le y\}) = P\left(\left\{X \le \left(\frac{b-a}{c}\right)y + a\right\}\right)$$

$$= F_X\left(\left(\frac{b-a}{c}\right)y + a\right) = \Phi\left(\left(\frac{b-a}{c}\right)y + a - 1\right)$$

(v) $P(\{Y = c\}) = P(\{X > b\}) = 1 - P(\{X \le b\})$

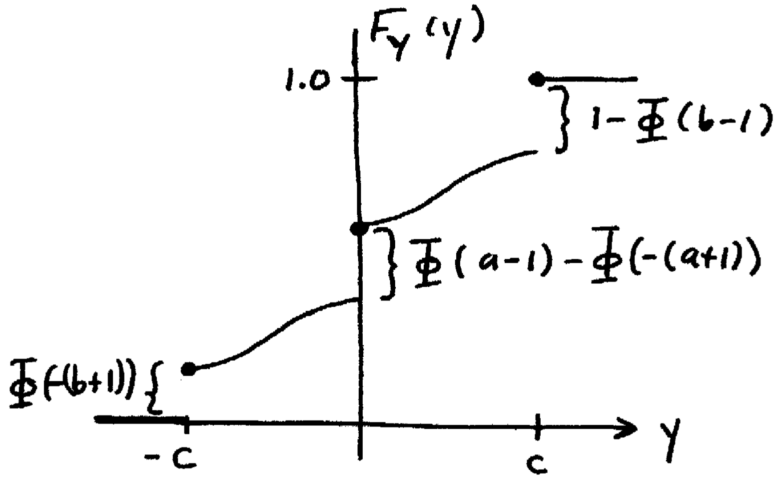
$$= 1 - \Phi(b-1)$$

$\therefore$

$$F_Y(y) = \begin{cases} 0, & y < -c \\ \Phi\left(\left(\frac{b-a}{c}\right)y - (a+1)\right), & -c \le y < 0 \\ \Phi\left(\left(\frac{b-a}{c}\right)y + a - 1\right), & 0 \le y < c \\ 1, & y \ge c \end{cases}$$

(7 - continued)

A plot of $F_Y(y)$ appears as follows:



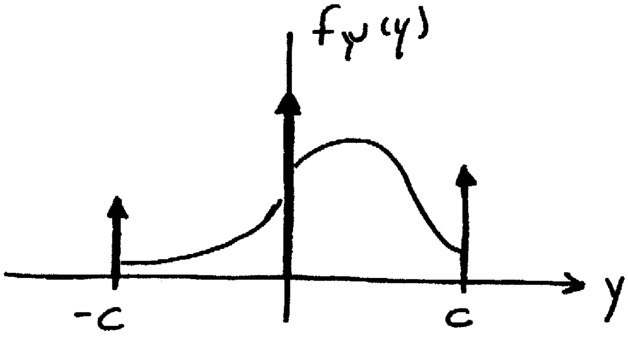
$$f_Y(y) = \frac{dF_Y(y)}{dy} = \Phi(-(b+1))\,\delta(y+c) + \left[\Phi(a-1)-\Phi(-(a+1))\right]\delta(y)$$

$$+ \frac{b-a}{c\sqrt{2\pi}} \exp\left\{ -\frac{\left[\left(\frac{b-a}{c}\right)y-(a+1)\right]^2}{2} \right\} 1_{(-c,0)}(y)$$

$$+ \frac{b-a}{c\sqrt{2\pi}} \exp\left\{ -\frac{\left[\left(\frac{b-a}{c}\right)y+a-1\right]^2}{2} \right\} 1_{(0,c)}(y)$$

$$+ \left[1-\Phi(b-1)\right]\delta(y-c)$$

A plot of $f_Y(y)$ appears as follows:



8. We first note that $F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\,d\alpha = (1-e^{-x})\,1_{[0,\infty)}(x)$

We also note that Y takes on values in the set $\{0, 0.5, 1.0, 1.5, 2.0\}$. Thus

$P(\{Y=0\}) = P(\{X<0\}) = 0$

$P(\{Y=0.5\}) = e^{-0.5} - e^{-1} = 0.239$ (from $F_X(x)$ above).

$P(\{Y=1.0\}) = 1-e^{-1/2} = 0.393$

$P(\{Y=1.5\}) = e^{-3/2} - 0 = 0.223$

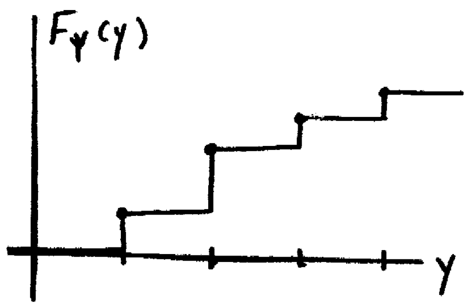
$P(\{Y=2.0\}) = e^{-1} - e^{-3/2} = 0.145$

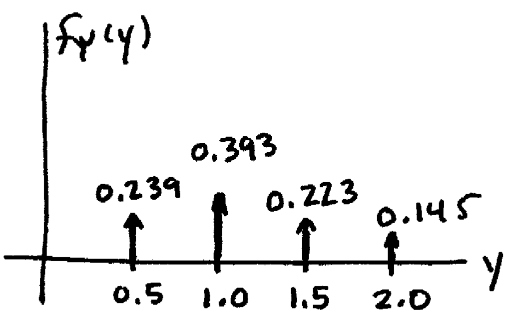
Thus we have

$$F_Y(y) = \begin{cases} 0, & y < 0.5 \\ 0.239, & 0.5 \le y < 1.0 \\ 0.632, & 1.0 \le y < 1.5 \\ 0.855, & 1.5 \le y < 2.0 \\ 1.0, & y \ge 2.0 \end{cases}$$

and $f_Y(y) = \dfrac{dF_Y(y)}{dy} = 0.239\,\delta(y-0.5) + 0.393\,\delta(y-1) + 0.223\,\delta(y-1.5) + 0.145\,\delta(y-2)$.

Plots of $F_Y(y)$ and $f_Y(y)$ appear as follows:





9. $\mathcal{S} = \mathbb{R}$ and $A = [1,3] \subset \mathcal{S}$.

(a) We wish to show that all complements and finite and countable unions of sets in $\mathcal{F} = \{\phi, A, \bar{A}, \mathcal{S}\}$ are also in $\mathcal{S} \Rightarrow \mathcal{F}$ is a σ-algebra.

<u>Checking complements</u>:
$$\overline{(A)} = \bar{A} \in \mathcal{F}$$
$$\overline{(\bar{A})} = \bar{\bar{A}} = A \in \mathcal{F}$$
$$\overline{(\phi)} = \bar{\phi} = \mathcal{S} \in \mathcal{F}$$
$$\overline{(\mathcal{S})} = \bar{\mathcal{S}} = \phi \in \mathcal{F}$$

To check countable and finite unions, we note that with a finite number of elements in $\mathcal{F}$, we need only check a relatively small number of unions, because the union of any element with $\mathcal{S}$ is $\mathcal{S}$ and so once a union produces $\mathcal{S}$, taking the union with additional elements of $\mathcal{S}$ still produces $\mathcal{S}$. In particular, note that

$$B \cup \mathcal{S} = \mathcal{S} \in \mathcal{F}, \quad \forall B \in \mathcal{F}$$
$$B \cup \phi = B \in \mathcal{F}, \quad \forall B \in \mathcal{F}$$
$$A \cup \bar{A} = \mathcal{S}.$$

So all finite and countable unions of the elements of $\mathcal{F} = \{\phi, A, \bar{A}, \mathcal{S}\}$ will yield an element in $\mathcal{F}$.

(b) Consider $X: \mathbb{R} \to \mathbb{R}$ given by $X(\omega) = \omega^2$, and the event space $\mathcal{F} = \{\phi, A, \bar{A}, \mathcal{S}\}$, where $A = [1,3]$. If $X(\omega)$ is a R.V., then it must be measurable on $(\mathbb{R}, \mathcal{F})$ (n.b. $\mathcal{S} = \mathbb{R}$). This means that

$$X^{-1}(F) \in \mathcal{F}, \quad \forall F \in \mathcal{B}(\mathbb{R}).$$

But consider $(0,1) \in \mathcal{B}(\mathbb{R})$. (you could use many other intervals.)

$$X^{-1}((0,1)) = \{\omega : X(\omega) = \omega^2 \in (0,1)\} = (-1,0) \cup (0,1) \notin \mathcal{F}$$

$\Rightarrow X$ is not measurable $\Rightarrow X$ is not a RV.

(9. - continued)

(c) Here we wish to show that if

$$\Psi(\omega) = 2 \cdot 1_A(\omega) + 3 \cdot 1_{\bar{A}}(\omega) = \begin{cases} 2, & \omega \in A \\ 3, & \omega \notin A \end{cases}$$

then $\Psi(\omega)$ is measurable w.r.t. $(\mathbb{R}, \mathcal{F})$, i.e.

$$\Psi^{-1}(F) \in \mathcal{F} = \{\phi, A, \bar{A}, \mathcal{S}\}, \ \forall F \in \mathcal{B}(\mathbb{R}).$$

It is sufficient to show that

$$\Psi^{-1}((a,b)) \in \mathcal{F}$$

for any open interval (a,b), because $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by the open intervals.

For any open interval (a,b), $a, b \in \mathbb{R}$ and $a < b$,

$$\Psi^{-1}((a,b)) = \{\omega : \Psi(\omega) \in (a,b)\} = \begin{cases} \phi, & 2 \notin (a,b) \text{ and } 3 \notin (a,b) \\ A, & 2 \in (a,b),\ 3 \notin (a,b) \\ \bar{A}, & 3 \in (a,b),\ 2 \notin (a,b) \\ \mathcal{S}, & 2 \in (a,b),\ 3 \in (a,b) \end{cases}$$

$\therefore \{\Psi(\omega) \in (a,b)\} = \{\omega : \Psi(\omega) \in (a,b)\} \in \mathcal{F}, \ \forall (a,b).$

$\therefore$ $\Psi(\omega)$ is measurable w.r.t. $(\mathbb{R}, \mathcal{F})$

$\Rightarrow$ $\Psi(\omega)$ is a random variable.