EE600 Homework # 3 Solutions Bell

1. Papoulis 2-10: Proof by induction.

- True for $n=2$, since $P(A_1 \cap A_2) = P(A_2|A_1)P(A_1)$.
- Now suppose its true for general n. We must show its true for $n+1$ as well (induction step)

$$P(A_{n+1} \cap A_n \cap \cdots \cap A_1) = P(A_{n+1}|A_n \cap \cdots \cap A_1)P(A_n \cap \cdots \cap A_1)$$

$$= P(A_{n+1}|A_n \cap \cdots \cap A_1)P(A_n|A_{n-1} \cap \cdots \cap A_1)P(A_{n-1} \cap \cdots \cap A_1)$$

$$\vdots$$

$$= P(A_{n+1}|A_n \cap \cdots \cap A_1)P(A_n|A_{n-1} \cap \cdots A_1) \cdots P(A_2|A_1)P(A_1)$$

So we conclude it is true for $n+1$. This completes the proof by induction.

2. Papoulis 2-11: The total number of subsets with m elements is $\binom{n}{m}$. The total number of m element subsets containing $\mathcal{S}_0$ is $\binom{n-1}{m-1}$; this is because knowing $\mathcal{S}_0$ is in the subset, you can selecting the remaining $m-1$ elements of the subset from the remaining $n-1$ elements in $\binom{n-1}{m-1}$ ways.

If each m element subset is equally likely, then the probability p of selecting one with $\mathcal{S}_0$ in it is

$$p = \frac{\begin{pmatrix}\text{no. ways to pick} \\ \text{subset with } \mathcal{S}_0\end{pmatrix}}{\begin{pmatrix}\text{total no. ways to} \\ \text{pick subset}\end{pmatrix}} = \frac{\binom{n-1}{m-1}}{\binom{n}{m}} = \frac{\frac{(n-1)!}{(m-1)!(n-1-m+1)!}}{\frac{n!}{m!(n-m)!}}$$

$$= \frac{(n-1)!\, m!}{n!\,(m-1)!} = \frac{m}{n}.$$

3. Papoulis 2-12: "At random" $\Rightarrow$ All times equally likely.
$\Rightarrow$ prob. of call in any time interval proportional to interval length.

(a) $P(\{6 \le t \le 8\}) = \dfrac{|8-6|}{|10-0|} = \dfrac{2}{10} = \boxed{1/5}$

(b) $P(\{6 \le t \le 8\} | \{t > 5\}) = \dfrac{P(\{6 \le t \le 8\} \cap \{t > 5\})}{P(\{t>5\})}$

$$= \frac{P(\{6 \le t \le 8\})}{P(\{t>5\})} = \frac{|8-6|}{|10-5|} = \boxed{\frac{2}{5}}$$

4. Papoulis 2-13: Here $\mathcal{S} = (0,\infty)$ and $\mathcal{F}(\mathcal{S}) = \mathcal{B}((0,\infty))$. We will assume the probability of events is specified by a pdf $\alpha(t)$. Thus we have

$$P(\{t_0 < t \le t_0 + t_1\} | \{t \ge t_0\}) = \frac{P(\{t_0 < t \le t_0+t_1\} \cap \{t \ge t_0\})}{P(\{t \ge t_0\})}$$

$$= \frac{\int_{t_0}^{t_0+t_1} \alpha(t)\,dt}{\int_{t_0}^{\infty} \alpha(t)\,dt}, \text{ and } P(\{t \le t_1\}) = \int_0^{t_1} \alpha(t)\,dt$$

Equating these two probabilities (by hypothesis), we have

$$\frac{\int_{t_0}^{t_0+t_1} \alpha(t)\,dt}{\int_{t_0}^{\infty} \alpha(t)\,dt} = \int_0^{t_1} \alpha(t)\,dt \quad ---- \quad (*)$$

We must solve $(*)$ for $\alpha(t)$. We do this by setting $t_1 = \Delta t$, dividing both sides by Δt, and letting $\Delta t \to 0$:

<u>LHS(*)</u>: $\displaystyle \lim_{\Delta t \to 0} \frac{\frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} \alpha(t)\,dt}{\int_{t_0}^{\infty} \alpha(t)\,dt} = \lim_{\Delta t \to 0} \frac{\frac{1}{\Delta t}\left[A(t_0+\Delta t) - A(t_0)\right]}{A(\infty) - A(t_0)}$

$$= \frac{\alpha(t_0)}{A(\infty) - A(t_0)}. \text{ Here } A(t) \text{ is the antiderivative of } \alpha(t): \ \frac{dA(t)}{dt} = \alpha(t).$$

<u>RHS(*)</u>: $\displaystyle \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_0^{\Delta t} \alpha(t)\,dt = \cdots = \alpha(0).$

Equating LHS(*) and RHS(*), we get $\dfrac{\alpha(t_0)}{A(\infty) - A(t_0)} = \alpha(0) \quad --- \quad (**)$

(4-continued)

To find $\alpha(t)$ from (**), we integrate both sides w.r.t. t_0 over the interval $(0,t)$.

<u>LHS(**)</u>: $$\int_0^t \frac{\alpha(t_0)}{A(\infty)-A(t_0)}\,dt_0 = -\int_{A(\infty)-A(0)}^{A(\infty)-A(t)} \frac{du}{u} = -\ln[A(\infty)-A(t)] + \ln[A(\infty)-A(0)]$$

<u>RHS(**)</u>: $$\int_0^t \alpha(0)\,dt_0 = \alpha(0)t$$

Equating LHS(**) and RHS(**), we get

$$-\ln[A(\infty)-A(t)] + \ln[A(\infty)-A(0)] = \alpha(0)t.$$

Solving for $A(t)$ yields

$$A(t) = \left(A(\infty) - \underbrace{[A(\infty)-A(0)]}_{K} e^{-\alpha(0)t}\right) 1_{(0,\infty)}(t)$$

So the pdf $\alpha(t)$ is

$$\alpha(t) = \frac{dA(t)}{dt} = \alpha(0) K e^{-\alpha(0)t} 1_{(0,\infty)}(t)$$

<u>n.b</u> Checking pdf $\int_0^\infty \alpha(t)\,dt = 1 \Rightarrow K = 1$

$$\therefore \quad \alpha(t) = \alpha(0) e^{-\alpha(0)t} 1_{(0,\infty)}(t) \quad \text{(Exponential pdf)}$$

and

$$P(\{t \le t_1\}) = \int_0^{t_1} \alpha(t)\,dt = -e^{-\alpha(0)t}\Big|_0^{t_1} = \boxed{1 - e^{-\alpha(0)t_1}}$$

for $t_1 > 0$.

5. Papoulis 4ed., Problem 2-16

$\mathcal{S}$ = set of all outcomes of k balls selected from n.

$\mathcal{F} = \mathcal{P}(\mathcal{S})$ (power set). $|\mathcal{S}| = \binom{n}{k}$, $|\mathcal{F}| = 2^{\binom{n}{k}}$.

P = classical prob. measure $\Rightarrow$ $P(A) = \dfrac{|A|}{|\mathcal{S}|}, \ \forall A \in \mathcal{F}$,

(a) $$P\left(\left\{\begin{matrix}\text{m is the largest} \\ \text{ball drawn}\end{matrix}\right\}\right) = \frac{\left(\begin{matrix}\text{no. of ways to draw k} \\ \text{balls such that m is largest}\end{matrix}\right)}{\left(\begin{matrix}\text{Total number of ways} \\ \text{to draw k balls from n}\end{matrix}\right)}$$

$$= \frac{\binom{m-1}{k-1}}{\binom{n}{k}} = \frac{\dfrac{(m-1)!}{(k-1)!\,(m-k)!}}{\dfrac{n!}{k!\,(n-k)!}} = \frac{(m-1)!\,k\,(n-k)!}{n!\,(m-k)!}$$

$$= \begin{cases} \dfrac{\binom{m-1}{k-1}}{\binom{n}{k}}, & 0 \le k \le m \le n \\ 0, & \text{elsewhere.} \end{cases}$$

(b) $$P\left(\left\{\begin{matrix}\text{largest ball} \\ \text{drawn} \le m\end{matrix}\right\}\right) = \frac{\left(\begin{matrix}\text{no. of ways to select k balls} \\ \text{such that largest is} \le m\end{matrix}\right)}{\left(\begin{matrix}\text{Total no. of ways to} \\ \text{draw k balls}\end{matrix}\right)}$$

$$= \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{\dfrac{m!}{k!\,(m-k)!}}{\dfrac{n!}{k!\,(n-k)!}} = \begin{cases} \dfrac{m!\,(n-k)!}{n!\,(m-k)!}, & 0 \le k \le m \\ 0, & \text{elsewhere.} \end{cases}$$

6. Papoulis 4th Ed., Problem 2-19

Box contains $\left\{\begin{matrix} m \text{ white} \\ n \text{ black}\end{matrix}\right\}$ balls. k balls are drawn from a total of $m+n$ balls. We want the probability there is at least one white ball

Let $B \triangleq$ all k balls selected are black
$W \triangleq$ at least one ball is white.

Then $P(W) = 1 - P(\overline{W}) = 1 - P(\overline{B})$

$$P(B) = \frac{\left(\begin{matrix}\text{no. of ways to choose } k \text{ balls} \\ \text{such that all are black}\end{matrix}\right)}{\left(\begin{matrix}\text{Total no. of ways to} \\ \text{select } k \text{ balls}\end{matrix}\right)}$$

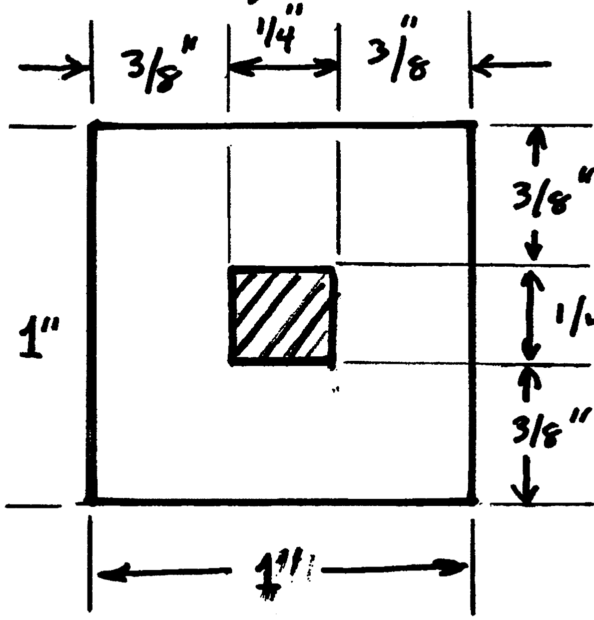
$$= \frac{\binom{n}{k}}{\binom{n+m}{k}} = \frac{\dfrac{n!}{k!\,(n-k)!}}{\dfrac{(n+m)!}{k!\,(n+m-k)!}}$$

$$= \frac{n!\,(n+m-k)!}{(n-k)!\,(n+m)!}$$

$$\therefore\ P(W) = 1 - \frac{\binom{n}{k}}{\binom{n+m}{k}} = 1 - \frac{n!\,(n+m-k)!}{(n-k)!\,(n+m)!}$$

7. Papoulis 4ed., Problem 2-20

Consider the ruled square in which the center of the coin is sitting. For the penny to lie entirely in the square, its center must lie in the shaded region



Given that the center of the penny has landed in a 1 in² square, we assume all points in the square are equally likely as the position of the coins center. If we define

$$A \triangleq \text{penny is entirely in square}$$
$$= \text{center of penny is in shaded region}$$

Then

$$P(A) = \frac{\text{Area of shaded region}}{\text{Area of } 1'' \text{ square}} = \frac{\frac{1}{4}'' \cdot \frac{1}{4}''}{1'' \cdot 1''}$$

$$= \frac{1/16 \text{ in}^2}{1 \text{ in}^2} = \frac{1}{16}$$

8. <u>Papoulis 4ed., Problem 3-1</u>

Let E_k = A occurs at least k times in n repetitions

A_k = A occurs exactly k times in n repetitions.

(a) $P(E_2) = 1 - P(\overline{E_2}) = 1 - P(A_0 \cup A_1)$

$= 1 - [P(A_0) + P(A_1)]$

$= 1 - \left[\binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} \right]$

$= 1 - (1-p)^n - np(1-p)^{n-1}$

(b) $P(E_3) = 1 - P(\overline{E_3}) = 1 - P(A_0 \cup A_1 \cup A_2)$

$= 1 - \left[\binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2} \right]$

$= 1 - (1-p)^n + np(1-p)^{n-1} - \frac{n(n-1)}{2} p^2 (1-p)^{n-2}$

9. <u>Papoulis 4ed., Problem 3-2</u> There are 36 equiprobable outcomes in the rolling of a pair of dice. They have the form (j,k), $j = 1, \ldots, 6$; $k = 1, \ldots, 6$.

Thus we have $|S| = 36$. Define the event A = double six = $\{(6,6)\}$. Clearly $P(A) = |A|/|S| = \frac{1}{36}$ for the role of a pair of dice. Now consider 50 rolls of a pair of dice (Bernoulli trials). Define

E_k = double six occurs at least k-times

B_k = " " " exactly k-times.

$P(E_3) = 1 - P(\overline{E_3}) = 1 - P(B_0 \cup B_1 \cup B_2) = 1 - [P(B_0) + P(B_1) + P(B_2)]$

$= 1 - \left[\binom{50}{0} \left(\frac{1}{36}\right)^0 \left(\frac{35}{36}\right)^{50} + \binom{50}{1} \left(\frac{1}{36}\right)^1 \left(\frac{35}{36}\right)^{49} + \binom{50}{2} \left(\frac{1}{36}\right)^2 \left(\frac{35}{36}\right)^{48} \right]$

$= 1 - \left(\frac{35}{36}\right)^{50} - 50\left(\frac{1}{36}\right)\left(\frac{35}{36}\right)^{49} - 1225 \left(\frac{1}{36}\right)^2 \left(\frac{35}{36}\right)^{48} = 0.161718$

10. Papoulis 4ed., Problem 3-3

When a pair of fair dice are rolled, the sample space is $\mathcal{S} = \{(j,k) : j=1,\ldots,6;\ k=1,\ldots,6\}$ $\Rightarrow |\mathcal{S}| = 36$. The event

$A_7 =$ outcome sums to seven

$= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

has six outcomes, i.e. $|A_7| = 6$

$$\therefore\ p = P(A_7) = \frac{|A_7|}{|\mathcal{S}|} = \frac{6}{36} = \frac{1}{6}.$$

If we now form the combined experiment of 10 tosses of the pair of fair dice, we can define the event

$E_1 =$ A seven occurs at least once in 10 throws

and the event

$B_k =$ A seven occurs exactly k-times in 10 throws. Then

$$P(E_1) = 1 - P(\overline{E}_1) = 1 - P(B_0) = 1 - \binom{10}{0} p^0 (1-p)^{10}$$

$$= 1 - (1-p)^{10} = 1 - \left(\frac{5}{6}\right)^{10}$$

$$= 1 - \frac{9765625}{60466176} = 0.838494$$

11. Papoulis 4ed., Problem 3-8

Let A_r = r successes in n trials

B_i = Success on i-th trial

$$P(B_i|A_r) = \frac{P(B_i \cap A_r)}{P(A_r)} = \frac{P(A_r|B_i)P(B_i)}{P(A_r)}$$

$$= \frac{\binom{n-1}{r-1} p^{r-1}(1-p)^{n-r} \cdot p}{\binom{n}{r} p^r (1-p)^{n-r}}$$

$$= \frac{\binom{n-1}{r-1}}{\binom{n}{r}} = \frac{\frac{(n-1)!}{(r-1)!\,(n-r)!}}{\frac{n!}{r!\,(n-r)!}}$$

$$= \frac{r}{n}$$