ECE 600 — **Homework # 10 Rev1 Solutions** — Bell

1. Papoulis 7-20: The event $\{X \le x\}$ occurs if at least one point falls in the interval $(0, x]$. The event $\{Y \le y\}$ occurs if all of the points fall in the interval $(0, y]$.

Let $A_x \triangleq \{X \le x\}$ and $B_y = \{Y \le y\} = \{\text{no points fall in } (y, 1)\}$

Hence for $x \in [0,1]$ and $y \in [0,1]$, we have

$$F_X(x) = P(A_x) = 1 - P(\bar{A}_x) = 1 - (1-x)^n$$

$$F_Y(y) = P(B_y) = y^n$$

$F_{XY}(x,y) = P(A_x \cap B_y)$, and $B_y = (A_x \cap B_y) \cup (\bar{A}_x \cap B_y)$

Now if $x \le y$, then $\bar{A} \cap B = \{\text{all points in interval } (x, y]\}$

$$P(\bar{A} \cap B) = (y-x)^n$$

If $x > y$, then $\bar{A} \cap B = \phi$ and $P(\bar{A} \cap B) = 0$

Thus

$$F_{XY}(x,y) = P(A \cap B) = P(B) - P(\bar{A} \cap B) = \begin{cases} y^n - (y-x)^n, & x \le y \\ y^n, & x > y. \end{cases}$$

2. Papoulis 7-25 Note that

$$E\{(X_n - a)^2\} = E\{[(X_n - a_n) + (a_n - a)]^2\}$$

$$= E\{(X_n - a_n)^2\} + 2(a_n - a)E\{(X_n - a_n)\} + (a_n - a)^2$$

Now as $n \to \infty$, we are given 1) $a_n \to a$
2) $E\{(X_n - a_n)^2\} \to 0$

Thus we have $\lim_{n\to\infty} E\{(X_n - a)^2\} = \lim_{n\to\infty} \{E\{(X_n - a_n)^2\} + 2(a_n - a)E\{X_n - a_n\} + (a_n - a)^2\}$

$$= 0 + 0 \cdot 0 + 0 = 0$$

$\therefore E\{(X_n - a)^2\} \to 0$ as $n \to \infty$.

3. Papoulis (7-27): We have $E[X_k]=0$ and $E[X_k^2]=\sigma_k^2$.

$\Leftarrow$: Given $\sum_{k=1}^{\infty}\sigma_k^2<\infty$, we wish to show that $\sum_{k=1}^{\infty}X_k$ converges in mean square.

$\sum_{k=\phi}^{\infty}\sigma_k^2<\infty \Rightarrow$ For any $\varepsilon>0$, there exists an $n_0\in\mathbb{N}$ such that for any $n>n_0$,

$$\sum_{k=n}^{\infty}\sigma_k^2<\varepsilon.$$

For such an n_0, it follows that if $Y_n=\sum_{k=1}^{n}X_k$,

$$E[(Y_{n+m}-Y_n)^2]\overset{(*)}{=}\sum_{k=n+1}^{n+m}\sigma_k^2\le\sum_{k=n}^{\infty}\sigma_k^2<\varepsilon,\quad \forall n>n_0,\ \forall m\ge 1$$

Aside (*)

(*) Note: $E[(Y_{n+m}-Y_n)^2]=E[(\sum_{k=n+1}^{n+m}X_k^2)]$

$$=E\left[\sum_{j=n+1}^{n+m}\sum_{k=n+1}^{n+m}X_jX_k\right]=\sum_{k=n+1}^{n+m}E[X_k^2]+\sum_{k=n+1}^{n+m}\sum_{\substack{j=n+1\\ j\neq k}}\cancelto{0}{E[X_j]}\cdot\cancelto{0}{E[X_k]}$$

$$=\sum_{k=n+1}^{n+m}\sigma_k^2$$

Thus $E[(Y_{n+m}-Y_n)^2]\to 0$, $\forall m\ge 1$ as $n\to\infty$

$\Rightarrow \sum_{k=1}^{\infty}X_k$ converges (m.s.)

$\Rightarrow$: Show that given convergence (m.s.), $\sum_{k=1}^{\infty}\sigma_k^2<\infty$.

If the sequence converges (m.s.), then the sequence of partial sums $\{Y_n\}$ converges (m.s).
This means that by the Cauchy criterion

$$E[(Y_{n+m}-Y_n)^2]\to 0 \text{ as } n\to\infty \text{ for all } m\ge 1.$$

$\Rightarrow$ For any $\varepsilon>0$, there exists an $n_0\in\mathbb{N}$, such that for $n>n_0$ and $m\ge 1$, $E[(Y_{n+m}-Y_n)^2]=\sum_{k=n+1}^{n+m}\sigma_k^2<\varepsilon$,
From which it follows that

$$\sum_{k=1}^{\infty}\sigma_k^2=\lim_{n\to\infty}\sum_{k=1}^{n}\sigma_k^2<\infty.$$

4. We have from the specification of the RVs X_k that

$$P(\{X_n = k\}) = \begin{cases} \frac{\alpha}{n^2}, & k = n \\ 1 - \frac{\alpha}{n^2}, & k = 0 \\ 0, & \text{elsewhere} \end{cases}$$

(recall that the RVs take on only non-negative integer values.)

In the limit, as $n \to \infty$, we have

$$\lim_{n\to\infty} X_n = \begin{cases} \lim_{n\to\infty} n \text{ (does not exist), with prob } 0 \\ 0, \text{ with probability one} \end{cases}$$

$\therefore$ $X_n \to 0$ as $n \to \infty$ <u>almost everywhere</u> or <u>with probability one.</u>

However

$$E[(X_n - 0)^2] = E[X_n^2] = n^2 \cdot \frac{\alpha}{n^2} + 0\left(1 - \frac{\alpha}{n^2}\right) = \alpha.$$

Thus

$$\lim_{n\to\infty} E[(X_n - 0)^2] = \lim_{n\to\infty} \alpha = \alpha$$

$\therefore$ $\{X_n\}$ does <u>not</u> converge in the mean-square sense.

5. <u>Papoulis 7-13</u>

$$\phi_Y(s) = E\{e^{sY}\} = E\{e^{s(X_1+\cdots+X_N)}\}$$
$$= E\{E\{e^{s(X_1+\cdots+X_N)} \mid N=k\}\}$$

Now because N and the $\{X_j\}$ are stat. indep., it follows that

$$E\{e^{sY} \mid N=k\} = E\{e^{s(X_1+\cdots+X_K)}\}$$
$$= E\{e^{sX_1}\}\cdots E\{e^{sX_K}\}$$
$$= \Phi_{X_1}(s)\cdots\Phi_{X_K}(s)$$

Because the $\{X_j\}$ are i.i.d, this becomes

$$E\{e^{sY} / N=k\} = [\Phi_X(s)]^K$$

Hence we have

$$\phi_Y(s) = E_N\{E\{e^{sY}/N\}\} = E_N\{[\Phi_X(s)]^N\}$$
$$= \Gamma_N(\phi_X(s)), \text{ since } E\{z^N\} = \Gamma_N(z)$$

<u>Special case:</u> Now if N is Poisson with parameter a, this becomes

$$\phi_Y(s) = \sum_{k=0}^{\infty} \frac{a^k e^{-a}}{k!}[\Phi_X(s)]^k = e^{-a}\sum_{k=0}^{\infty}\frac{[a\Phi_X(s)]^k}{k!}$$
$$= e^{-a} e^{a\Phi_X(s)} = e^{a\Phi_X(s)-a}$$
$$= e^{a[\Phi_X(s)-1]}$$

6. Papoulis 9-1

(a) $E\{X(t)\} = \frac{1}{2} \cdot \sin \pi t + \frac{1}{2} \cdot 2t = \frac{1}{2} \sin(\pi t) + t$

(b) $F_{X(t)}(x) = F(x,t) = P\{X(t) \leq x\}$

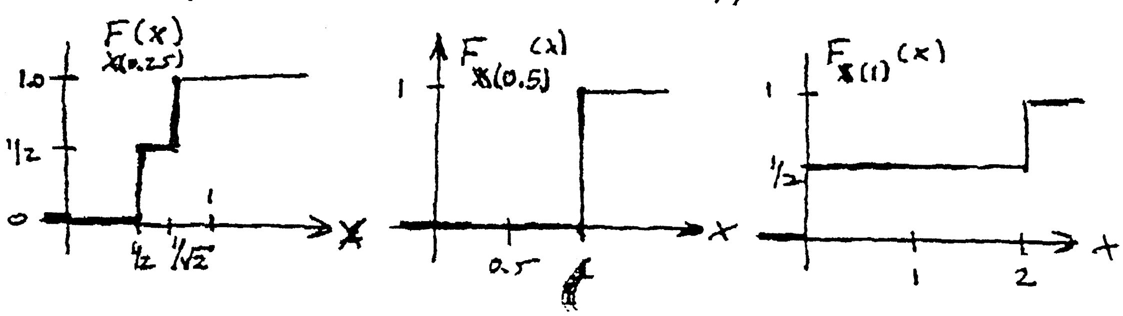
For any particular t, $X(t)$ is a RV:

$$X(0.25) = \begin{cases} \sin \pi/4 = \frac{1}{\sqrt{2}}, & \text{prob } 1/2 \\ 2 \cdot \frac{1}{4} = 1/2, & \text{prob } 1/2 \end{cases}$$

$$X(0.5) = \begin{cases} \sin \frac{\pi}{2} = 1, & \text{prob } \frac{1}{2} \\ 2 \cdot \frac{1}{2} = 1, & \text{prob } 1/2 \end{cases} = 1 \text{ with prob } 1$$

$$X(1) = \begin{cases} \sin \pi = 0, & \text{prob } 1/2 \\ 2 \cdot 1 = 2, & \text{prob } 1/2 \end{cases}$$

Thus $F_{X(t)}(x)$ for these 3 cases appear as follows:



7. Papoulis 9-2:

$X(t) = e^{At} \Rightarrow A = \frac{1}{t} \ln X(t)$

$$\eta_x(t) = \int_{-\infty}^{\infty} e^{at} f_A(a)\, da = \phi_A(t) \quad \text{(generating ftn.)}$$

$$R_{xx}(t_1,t_2) = \int_{-\infty}^{\infty} e^{at_1} e^{at_2} f_A(a)\, da = \int_{-\infty}^{\infty} e^{a(t_1+t_2)} f_A(a)\, da = \phi_A(t_1+t_2)$$

$$f_{X(t)}(x) = f_A(a(X,t)) \left| \frac{da}{dx} \right|, \quad \frac{da}{dx} = \frac{d}{dx}\left[\frac{1}{t} \ln x\right] = \frac{1}{tx} = \frac{1}{|t|x}$$

$$\therefore \quad f_{X(t)}(x) = \frac{1}{x|t|} f_A\left(\frac{1}{t} \ln x\right) 1_{[0,\infty)}(x)$$

8. Papoulis 9-3: From the discussion of Poisson processes on pages 290-291 of Papoulis, we know that

$$E\{\mathbf{x}(t)\} = \lambda t \ , \quad E\{\mathbf{x}^2(t)\} = \lambda t + \lambda^2 t^2, \text{ and thus } var(\mathbf{x}(t)) = \lambda t.$$

(a) If $E\{\mathbf{x}(9)\} = 6 = \lambda \cdot 9 \Rightarrow \lambda = \frac{6}{9} = \frac{2}{3}$

Thus $E\{\mathbf{x}(8)\} = \lambda t\big|_{t=8} = \frac{2}{3}\cdot 8 = \frac{16}{3}$

$var\{\mathbf{x}(8)\} = \lambda t\big|_{t=8} = \frac{2}{3}\cdot 8 = \frac{16}{3}$

(b) $\mathbf{x}(2)$ is a Poisson RV. with mean $\lambda t\big|_{t=2} = \frac{2}{3}\cdot 2 = 4/3$

$$P(\{\mathbf{x}(2) \le 3\}) = \sum_{k=0}^{3} \frac{e^{-4/3}(4/3)^k}{k!} = \cdots =$$

(c) $U = \mathbf{x}(2)$ is a Poisson RV with mean $\lambda \cdot 2 = 4/3$. $V = \mathbf{x}(4) - \mathbf{x}(2)$ is also a Poisson RV with mean $4/3$.

Hence $P(\{U=k\}) = \frac{e^{-4/3}(4/3)^k}{k!}$, $P(\{V=m\}) = \frac{e^{-4/3}(4/3)^m}{m!}$

and by independent increments, U and V are stat. indep $\Rightarrow P(\{U=k\}\cap\{V=m\}) = \frac{e^{-8/3}(4/3)^k(4/3)^m}{k!\,m!}$

Hence $P(\{\mathbf{x}(4) \le 5\} \mid \{\mathbf{x}(2) \le 3\}) = \frac{P(\{U \le 3\}\cap\{V \le 5-U\})}{P(\{U \le 3\})}$

where $P(\{U \le 3\}) = \sum_{k=0}^{3} \frac{e^{-4/3}(4/3)^k}{k!} =$

$$P(\{U\le 3\}\cap\{V \le 5-U\}) = \sum_{k=0}^{3}\sum_{m=0}^{5-k} \frac{e^{-8/3}(4/3)^k(4/3)^m}{k!\,m!}$$

$$= e^{-8/3}\sum_{k=0}^{3}\sum_{m=0}^{5-k}\frac{(4/3)^{k+m}}{k!\,m!} = \frac{48373}{3645}e^{-8/3}$$

$$= 0.9221$$

9. Papoulis 9-8:

(a) $X(t)$ is Gaussian WSS with zero mean and variance $\sigma^2_{X(t)} = E\{X^2(t)\} = R_{XX}(0) = 4$

$$\Rightarrow f_{X(t)}(x) = \frac{1}{2\sqrt{2\pi}} \exp\left\{\frac{-x^2}{8}\right\} \sim N[0,2] \Rightarrow P(\{X(t) \le 3\}) = F_{X(t)}(3) = \Phi\left(\tfrac{3}{2}\right) = 0.933$$

(b) $E\{[X(t+1) - X(t-1)]^2\} = E\{X^2(t+1)\} - 2E\{X(t+1)X(t-1)\} + E\{X^2(t-1)\}$

$= R_{XX}(0) - 2R_{XX}(2) + R_{XX}(0) = 2[R_{XX}(0) - R_{XX}(2)]$

$= 8[1 - e^{-2|2|}] = 8(1 - e^{-4}) = 8(0.98168) = 7.853$

10. Papoulis 9-10 We wish to show that if $X(t)$ is a ~~zero-mean~~ W.S.S. R.P. with autocovariance $C_{XX}(\tau)$, then $Z(t) = X^2(t)$ has autocovariance $C_{ZZ}(\tau) = 2C^2_{XX}(\tau)$.

If X_1, X_2, X_3, X_4 are jointly distributed zero-mean Gaussian RVs, then

$$E\{X_1X_2X_3X_4\} = E\{X_1X_2\}E\{X_3X_4\} + E\{X_1X_3\}E\{X_2X_4\} + E\{X_1X_4\}E\{X_2X_3\}$$

[See Example 8-9 of Papoulis, pp. 197-8 3rd. Edition]

Let $X_1 = X_2 = X(t+\tau)$, $X_3 = X_4 = X(t)$.
Then it follows that

$$R_{ZZ}(\tau) = E\{Z(t+\tau)Z(t)\} = E\{X^2(t+\tau)X^2(t)\}$$

$$= E\{X(t+\tau)X(t+\tau)\}E\{X(t)X(t)\} + E\{X(t+\tau)X(t)\}E\{X(t+\tau)X(t)\} + E\{X(t+\tau)X(t)\}E\{X(t+\tau)X(t)\}$$

$$= R_{XX}(0) \cdot R_{XX}(0) + R_{XX}(\tau) \cdot R_{XX}(\tau) + R_{XX}(\tau) \cdot R_{XX}(\tau)$$

$$= [C_{XX}(0)]^2 + 2[C_{XX}(\tau)]^2, \quad R_{XX}(\tau) = C_{XX}(\tau) \text{ because } E\{X(t)\} = 0.$$

But $C_{ZZ}(\tau) = R_{ZZ}(\tau) - [E\{Z(t)\}]^2 = R_{ZZ}(\tau) - [R_{XX}(0)]^2$

$= R_{ZZ}(\tau) - [C_{XX}(0)]^2 = 2C^2_{XX}(\tau)$.

11. Papoulis 9-37:

Just as in problem 9 of this assignment

$$E\{x^2(t+\tau)x^2(t)\} = E\{x^2(t+\tau)\}E\{x^2(t)\} + 2E\{x(t+\tau)x(t)\}E\{x(t+\tau)x(t)\}$$

$$= [R_{xx}(0)]^2 + 2[R_{xx}(\tau)]^2 = I^2 + 2I^2 e^{-2\alpha|\tau|}\cos^2\beta\tau$$

$$= I^2[1 + e^{-2\alpha|\tau|} + e^{-2\alpha|\tau|}\cos 2\beta\tau] = R_{YY}(\tau)$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i\omega\tau}d\tau = I^2\int_{-\infty}^{\infty}[1+e^{-2\alpha|\tau|} + e^{-2\alpha|\tau|}\cos 2\beta\tau]e^{-i\omega\tau}d\tau$$

$$= I^2\left[2\pi\delta(\omega) + \frac{4\alpha}{4\alpha^2+\omega^2} + \frac{2\alpha}{4\alpha^2+(\omega-2\beta)^2} + \frac{2\alpha}{4\alpha^2+(\omega+2\beta)^2}\right]$$

Furthermore

$$C_{YY}(\tau) = R_{YY}(\tau) - [E\{y(t)\}]^2, \quad E\{y(t)\} = E\{x^2(t)\} = R_{xx}(0) = I$$

$$\therefore C_{YY}(\tau) = I^2[1+e^{-2\alpha|\tau|}(1+\cos 2\beta\tau)] - I^2 = I^2 e^{-2\alpha|\tau|}(1+\cos 2\beta\tau)$$

12. Papoulis 10-39

(a) $S_{xx}(\omega) = \dfrac{1}{1+\omega^4} \Rightarrow S_x(s) = \dfrac{1}{1+s^4} = \dfrac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)}$

This is a special case of example 10-27b with $b=\sqrt{2}$ and $c=1$.
Hence

$$R_{xx}(\tau) = \frac{1}{2\sqrt{2}}e^{-|\tau|/\sqrt{2}}\left[\cos\frac{\tau}{\sqrt{2}} + \sin\frac{|\tau|}{\sqrt{2}}\right]$$

(b) We know that $\mathcal{F}\{e^{-2|\tau|}\} = \dfrac{4}{4+\omega^2}$, Thus by the convolution theorem

$$\mathcal{F}\{e^{-2|\tau|} * e^{-2|\tau|}\} = \frac{16}{(4+\omega^2)^2}$$

Now $S_{xx}(\omega) = \dfrac{1}{(4+\omega^2)^2}$, thus

$$R_{xx}(\tau) = \frac{e^{-2|\tau|} * e^{-2|\tau|}}{16} = \dots = \frac{1+2|\tau|}{32}e^{-2|\tau|}$$

13. Papoulis 9-42: $y(t) = 2x(t) + 3x'(t)$

The system function corresponding to this diff. eq. is
$H(\omega) = 2 + i\omega 3$.

The mean and covariance of $x(t)$ are $\eta_x = 5$ and $C_{xx}(\tau) = 4e^{-2|\tau|}$
$R_{xx}(\tau) = 25 + 4e^{-2|\tau|}$

Hence
$$\eta_y = \eta_x \int_{-\infty}^{\infty} h(t)\,dt = \eta_x H(0) = 5 \cdot 2 = 10$$
and
$$S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2 = \left[25 \cdot 2\pi\delta(\omega) + \frac{16}{4+\omega^2}\right]\left[4+9\omega^2\right]$$
$$= \frac{16\left[4+9\omega^2\right]}{4+\omega^2} + 50\pi\,\delta(\omega)\left[4+9\omega^2\right] \checkmark$$
$$= 144 - \frac{512}{4+\omega^2} + 200\pi\,\delta(\omega)$$

Thus
$$R_{yy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{i\omega\tau}\,d\omega = 144\,\delta(\tau) - 128e^{-2|\tau|} + 100$$
$$= 144\,\delta(\tau) + 100 - 128e^{-2|\tau|}$$

14. If $z(t)$ is WSS, then $E\{z(t)\} = \text{constant}$ and $R_{zz}(t_1,t_2) = R_{zz}(t_1-t_2)$.
Lets check this

$$E\{z(t)\} = E\{x(t)y(t)\} = E\{x(t)\}E\{y(t)\} = \eta_x \cdot \eta_y = \text{constant}$$
$$E\{z(t_1)z(t_2)\} = E\{x(t_1)y(t_1)x(t_2)y(t_2)\} = E\{x(t_1)x(t_2)\}E\{y(t_1)y(t_2)\}$$
$$= R_{xx}(t_1-t_2)R_{yy}(t_1-t_2)$$

Thus $z(t)$ is always WSS in the above scenario.

15. $X(t)$ is W.S.S. with $R_{XX}(\tau) = \frac{1}{2}\cos\omega_0\tau$

$Y(t)$ is W.S.S. with $R_{YY}(\tau) = e^{-\alpha|\tau|}$

Given that Θ is stat. indep of $Y(t)$, we have $X(t) \perp\!\!\!\perp Y(t)$. It follows that

$$R_{ZZ}(t_1,t_2) = E\{X(t_1)Y(t_1)X(t_2)Y(t_2)\}$$
$$= E\{X(t_1)X(t_2)\}E\{Y(t_1)Y(t_2)\}$$
$$= R_{XX}(t_1-t_2)\cdot R_{YY}(t_1-t_2)$$
$$= R_{ZZ}(t_1-t_2),$$

just as in problem 13 above.

Now by the convolution theorem (and duality) of Fourier transforms, we have

$$S_{ZZ}(\omega) = \frac{1}{2\pi}\left[S_{XX}(\omega) * S_{YY}(\omega)\right].$$

Now $S_{XX}(\omega) = \frac{\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

$$S_{YY}(\omega) = \frac{2\alpha}{\alpha^2+\omega^2}$$

$$\therefore\ S_{ZZ}(\omega) = \frac{1}{2\pi}\left[\frac{\pi\alpha}{\alpha^2+(\omega-\omega_0)^2} + \frac{\pi\alpha}{\alpha^2+(\omega+\omega_0)^2}\right]$$

$$= \frac{\alpha/2}{\alpha^2+(\omega-\omega_0)^2} + \frac{\alpha/2}{\alpha^2+(\omega+\omega_0)^2}$$