EE600 **Homework #1 Solutions** Bell

1. (a) $(A \cap \bar{B}) \cup (B \cap \bar{A}) = A \Delta B$ (this doesn't really simplify things much.)

(b) $(A \cap \bar{B}) \cap (A \cap B) = (A \cap \bar{B}) \cap (B \cap A) = A \cap (\bar{B} \cap B) \cap A$
$= A \cap \phi \cap A = \phi$.

(c) $\overline{A \cap (B \cup C)} = \overline{(A \cap B) \cup (A \cap C)} = (\overline{A \cap B}) \cap (\overline{A \cap C})$
$= (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \bar{C}) = \bar{A} \cup (\bar{B} \cap \bar{C})$

(d) $\overline{A \cap B \cap C} = \overline{A \cap B} \cup \bar{C} = \bar{A} \cup \bar{B} \cup \bar{C}$

2. In order to show that $\{B_1, B_2, \ldots, B_n\}$ is a partition of G, we must show that

(i) $B_i \cap B_j = \phi$, $\forall i,j = 1, \ldots, n$, $i \neq j$ (disjoint)

(ii) $\bigcup_{i=1}^{n} B_i = G$ (collectively exhaustive for G)

Lets show this:

(i): $B_i \cap B_j = (A_i \cap G) \cap (A_j \cap G)$
$= A_i \cap (G \cap G) \cap A_j$
$= A_i \cap G \cap A_j$
$= (A_i \cap A_j) \cap G$
$= \phi \cap G$, $i \neq j$ because the A_i's are a partition
$= \phi \Rightarrow$ The B_i's are disjoint.

(ii): $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A_i \cap G)$
$= G \cap \left(\bigcup_{i=1}^{n} A_i \right)$, because $\cap$ is distributive over $\cup$.
$= G \cap \mathcal{S}$, because the A_i are a partition
$= G$.

3. Let $F_r = [0, 1/r)$, $r \in (0,1]$

$$\bigcup_{r\in(0,1]} F_r = \bigcup_{r\in(0,1]} [0, 1/r) = \{\omega : \omega \in [0, 1/r) \text{ for at least one } r \in (0,1]\}$$
$$= [0, \infty).$$

$$\bigcap_{r\in(0,1]} F_r = \bigcap_{r\in(0,1]} [0, 1/r) = \{\omega : \omega \in [0, 1/r) \text{ for all } r \in (0,1]\}$$
$$= [0, 1).$$

4. There are a number of possible proofs. One simple one is a proof by induction. Clearly it is true for $n = 1$. We must now show that if it is true for $n = k$, then it is also true for $n = k+1$.

Assume there are 2^k subsets of the set $\{a_1, \ldots, a_k\}$. If we now consider subsets of the set $\{a_1, \ldots, a_{k+1}\}$, we note that each of the subsets of $\{a_1, \ldots, a_k\}$ is also a subset of $\{a_1, \ldots, a_{k+1}\}$, and in addition, we can construct 2^k more subsets by adding the element a_{k+1} to each of these original subsets, yielding a total of

$$2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

subsets of $\{a_1, \ldots, a_{k+1}\}$. There are no other subsets, so we are finished.

You can also prove the result using a binary encoding scheme as follows: Suppose $A = \{a_1, \ldots, a_n\}$. Any subset $B \subset A$ either contains a_k or it does not. Define n "binary indicator bits" for $B \subset A$ as

$$b_k(B) = \begin{cases} 1, & a_k \in B \\ 0, & a_k \notin B. \end{cases}$$

(4.-continued)

Then there is an n-bit binary codeword $\underline{b}(B)$ that uniquely describes each $B \subset A$, and to each n-bit codeword there is a unique subset of A.

$\underline{\text{n.b.}}$ $\underline{b}(B) = b_1(B)\, b_2(B) \cdots b_n(B)$

and

$$B = \{ a_k \in A : b_k = 1 \}$$

Clearly there are 2^n n-bit binary words, hence there are 2^n subsets of A.

$\underline{5.}$ (a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\underline{\text{proof}}$: $x \in \overline{A \cup B} \Leftrightarrow x \notin A$ and $x \notin B$, by defn. of union and comp.

$\Leftrightarrow x \in \overline{A}$ and $x \in \overline{B}$, by defn. of comp.

$\Leftrightarrow x \in \overline{A} \cap \overline{B}$, by defn. of intersection.

$\therefore \quad \overline{A \cup B} = \overline{A} \cap \overline{B}$.

(b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\underline{\text{proof}}$: $x \in \overline{A \cap B} \Leftrightarrow x \notin A \cap B$

$\Leftrightarrow x \in \overline{A}$ $\underline{\text{or}}$ $x \in \overline{B}$

$\Leftrightarrow x \in \overline{A} \cup \overline{B}$

$\therefore \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$

6. If $A \cup B = A$, then $B \subset A$;
If $A \cap B = A$, then $A \subset B$.

So we have that $A \subset B$ and $B \subset A$. From the "Fact" stated in class, we have $A = B \Leftrightarrow A \subset B$ and $B \subset A$.

Proof of Fact: Two sets are equal iff they have the same elements.

If $B \subset A$, then $x \in B \Rightarrow x \in A$, and
if $A \subset B$, then $x \in A \Rightarrow x \in B$

If both of these conditions are true, then every element of A is an element of B, and every element of B is an element of A. Furthermore A has no elements that are not in B and B has no elements that are not in A.
$\therefore$ A and B have exactly the same elements, so $A = B$.
That $A \subset B$ and $B \subset A$ if $A = B$ follows from the defn. of a subset.

7. (a) $(A_1 \cup A_2) \cap F$ = The set of female students in their freshman or sophmore years.

(b) $\overline{H} \cap F$ = The set of female students who are not Indiana natives

(c) $A_1 \cap \overline{F} \cap H$ = The set of native Indiana male freshman students.

(d) $A_3 \cap F \cap \overline{H}$ = The set of junior female students who are not Indiana natives

(e) $(A_1 \cup A_2) \cap H \cap F$ = The set of native Indiana female students in the freshman or sophmore years.