ECE 678 — **Homework #6 Solutions** — Mark R. Bell

6.1. Levanon 8.1: The frequency coding sequence is of the form

$$\{a_j\} = 2, 4, 5, a_3, a_4, a_5,$$

where $\{a_3, a_4, a_5\}$ is a sequence (permutation) of the set $\{1, 3, 6\}$.

There are $3! = 6$ such permutations, and we will check each one of them using the distance matrix D described in Ch. 8 of Levanon (pp. 147-152), with

$$D_{ij} = a_{i+j} - a_j \quad , \quad i+j \le N.$$

i	2	4	5	1	6	3
1	2	1	-4	5	3	
2	3	-3	1	2		
3	-1	2	-2			
4	4	-1				
5	1					

$\Rightarrow$ Costas sequence

	2	4	5	3	6	1
1	2	1	-2	3	-5	
2	3	-1	1	-2		
3	1	2	-4			
4	4	-3				
5	-1					

$\Rightarrow$ Costas sequence

	2	4	5	1	3	6
1	2	-1	-4	2	3	
2						
3						
4						
5						

Not Costas

i	2	4	5	6	1	3
1	2	1	1			
2						
3						
4						
5						
6						

Not Costas

	2	4	5	6	3	1
1	2	1	1			
2						
3						
4						
5						

Not Costas

	2	4	5	3	1	6
1	2	1	-2	-2		
2						
3						
4						
5						

Not Costas

6.2 Levanon 8.2: The difference matrix of the Costas sequence $\{a_n\} = 5, 3, 2, 7, 1, 8, 4, 6, 9$ is

i	5	3	2	7	1	8	4	6	9
1	-2	-1	5	-6	7	-4	2	3	
2	-3	4	-1	1	3	-2	5		
3	2	-2	6	-3	5	1			
4	-4	5	2	-1	8				
5	3	1	4	2					
6	-1	3	7						
7	1	6							
8	4								

From this difference matrix D, we can construct the full sidelobe matrix. (We use the fact that $|\chi_s(\tau,\nu)| = |\chi_s(-\tau,-\nu)|$).

The full sidelobe matrix is shown on the next page.

Doppler \ Delay	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
+8													1				
+7										1					1		
+6								1				1				1	
+5										1	1	1	1				
+4					1			1			1			1			1
+3						1	1			1	1			1	1		
+2						1	1	1		1		1	1	1			
+1			1		1		1	1			1	1		1		1	
0									9								
-1		1		1		1	1			1	1		1		1		
-2				1	1	1		1		1	1	1					
-3			1	1			1	1			1	1					
-4	1			1			1			1			1				
-5					1	1	1	1									
-6		1				1				1							
-7			1					1									
-8					1												

6.3. In order to find a Costas sequence $\{a_n\}$ with length 16, we can use the Welch procedure.

Here $N = p-1 = 16 \Rightarrow p = 17$

We must find a primitive element π that generates the whole set $\{1, \ldots, 16\}$ using the relation:

$$\pi^j \bmod p \;;\quad j = 1, \ldots 16$$

We note that there are a number of primitive elements that generate the set $\{1, \ldots, 16\}$. Using a simple program, they can easily be found. The complete list of primitive elements is

$$\pi = 3, 5, 7, 10, 11, 12, 14$$

Hence there are 7 Costas sequences of length 16 that can be generated using the Welch method. They are:

$$\{a_n\}^{[3]} = 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$$

$$\{a_n\}^{[5]} = 5, 8, 6, 13, 14, 2, 10, 16, 12, 9, 11, 4, 3, 15, 7, 1$$

$$\{a_n\}^{[7]} = 7, 15, 3, 4, 11, 9, 12, 16, 10, 2, 14, 13, 6, 8, 5, 1$$

$$\{a_n\}^{[10]} = 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3, 13, 11, 8, 12, 1$$

$$\{a_n\}^{[11]} = 11, 2, 5, 4, 10, 8, 3, 16, 6, 15, 12, 13, 7, 9, 14, 1$$

$$\{a_n\}^{[12]} = 12, 8, 11, 13, 3, 2, 7, 16, 5, 9, 6, 4, 14, 15, 10, 1$$

$$\{a_n\}^{[14]} = 14, 9, 7, 13, 12, 15, 6, 16, 3, 8, 10, 4, 5, 2, 11, 1$$

There are many more Costas signals of length 16 than this, however, we have a direct constructive approach for finding these 7. There are in fact thousands (or more) Costas signals of length 16, but an exhaustive search would require checking all

$$16! = 20{,}922{,}789{,}888{,}000.$$

The fraction of such signals that are Costas is small.

6.4 (a) Because $N = M^2$ and $N = 9$, we have $M = 3$. Thus the 3×3 ($M\times M$) matrix of phase

$$\phi_{pq} = \frac{2\pi}{M}(p-1)(q-1), \quad p = 1,\dots,3, \; q = 1,\dots,3$$

is of the form

$$\Phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2\pi}{3} & \frac{4\pi}{3} \\ 0 & \frac{4\pi}{3} & \frac{8\pi}{3} \end{pmatrix}$$

and the matrix of signals $u_{pq} = \exp(i\,\phi_{pq})$ is given by

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix}$$

Thus the sequence $\{u_1, u_2, \dots, u_9\}$ is

$$\{1, 1, 1, 1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, 1, e^{i\frac{4\pi}{3}}, e^{i\frac{8\pi}{3}}\}$$

(b) The autocorrelation sequence is given by

$$R(k) = \sum_{n=1}^{9-k} u_{n+k} \cdot u_n^* \quad , \; k = 0, \dots, 8$$

and for $k = -8, -7, \dots, -1$, we can make use of the fact that

$$R(k) = R^*(-k) \quad , \; k = -1, -2, \dots, -8 .$$

Thus we have

$$R(0) = \sum_{n=1}^{9} u_n u_n^* = 1 \cdot 1 + \cdots + e^{-i 8\pi/3} \cdot e^{+i\frac{8\pi}{3}} = 9$$

$$R(1) = \sum_{n=1}^{8} u_{n+1} u_n^* = \cdots = 3 + 3e^{-i\frac{2\pi}{3}} + 2e^{i\frac{2\pi}{3}}$$

$$|R(1)| = 1$$

$$R(2) = \sum_{n=1}^{7} u_{n+2} u_n^* = \cdots = 3 + 2e^{-i\frac{2\pi}{3}} + 2e^{i2\pi/3}$$

$$|R(2)| = 1$$

$$R(3) = \sum_{n=1}^{6} u_{n+3} u_n^* = \cdots = 2 + 2e^{-i\frac{2\pi}{3}} + 2e^{i\frac{2\pi}{3}}$$

$$|R(3)| = 0$$

$$R(4) = \sum_{n=1}^{5} u_{n+4} u_n^* = \cdots = 2 + e^{-i\frac{2\pi}{3}} + 2e^{i2\pi/3}$$

$$|R(4)| = 1$$

$$R(5) = \sum_{n=1}^{4} u_{n+5} u_n^* = \cdots = 1 + e^{-i\frac{2\pi}{3}} + 2e^{i\frac{2\pi}{3}}$$

$$|R(5)| = 1$$

$$R(6) = \sum_{n=1}^{3} u_{n+6} u_n^* = \cdots = 1 + e^{-i\frac{2\pi}{3}} + e^{i\frac{2\pi}{3}}$$

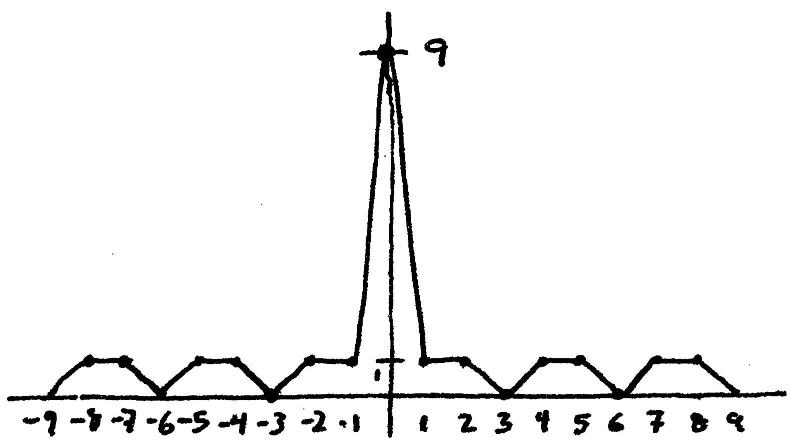
$$|R(6)| = 0$$

$$R(7) = \sum_{n=1}^{2} u_{n+7} u_n^* = \cdots = e^{-i\frac{2\pi}{3}} + e^{+i\frac{2\pi}{3}}$$

$$|R(7)| = 1$$

$$R(8) = u_9 u_1^* = e^{-i\frac{2\pi}{3}} \quad , \; |R(8)| = 1$$

If we plot $|R(k)|$ for $k = -8, -7, \ldots, 0, \ldots, 7, 8$, we get the following



6.5 Levanon 8.16: using the algorithm of Eq. (8.17) of Levanon, we construct our pair as follows:

$$S_1 = (++) \qquad \hat{S}_1 = (+-)$$
$$S_2 = (+++-) \qquad \hat{S}_2 = (++-+)$$
$$S_3 = (+++-++-+) \qquad \hat{S}_3 = (+++---+-)$$

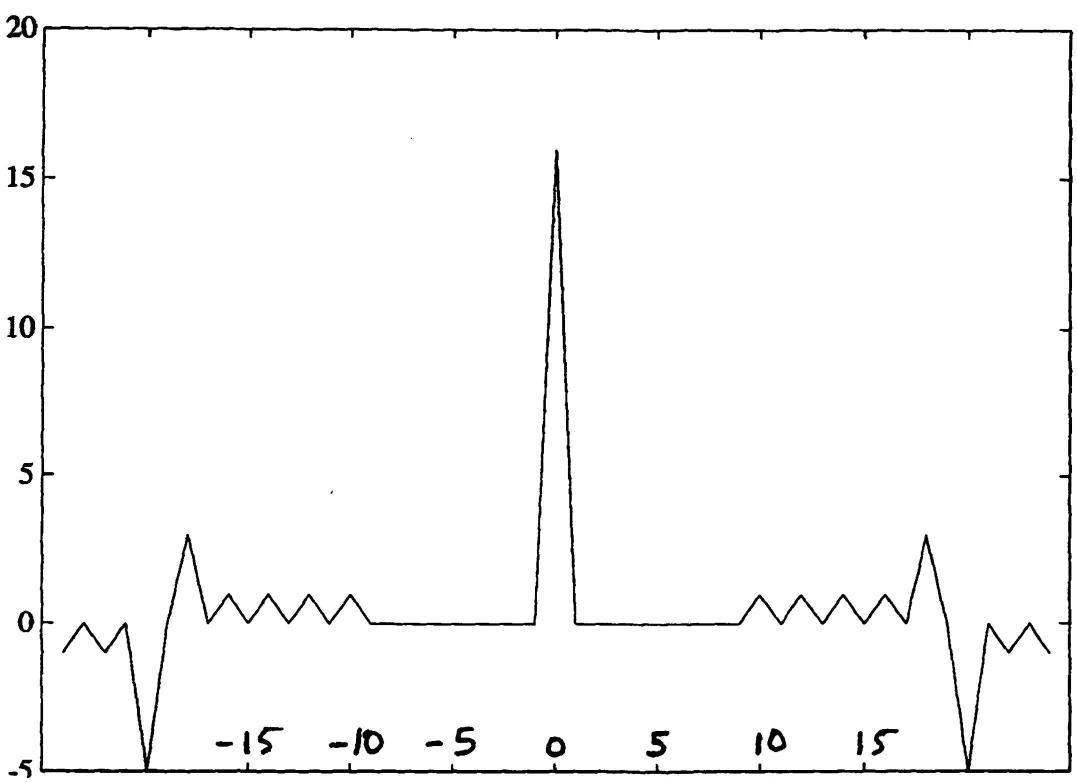
Hence our complementary pair is

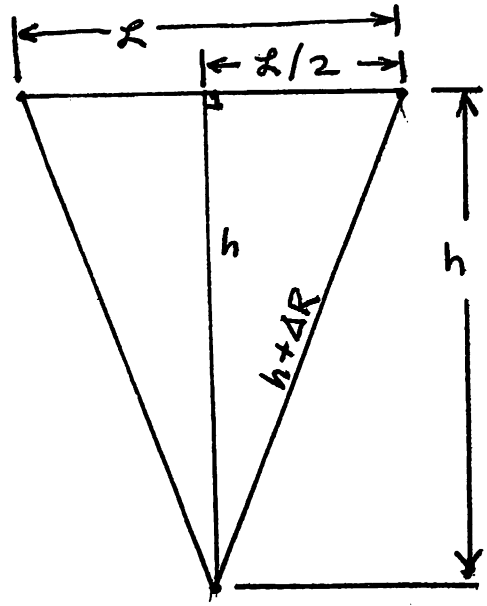
(+++-++-+) and (+++---+-), where "+" = +1, "-" = -1

Assuming a spacing of length 9 between the complementary signals, we can construct a length 8+8+9 = 25 sequence u^c as

$\underline{u}$ = (+++-++-+000000000+++---+-)

If we compute the autocorrelation sequence of u, we get the following plot (with zero index as peak):





6.6. (a) If the endpoint of the synthetic aperture has a (two-way) phase difference of $\frac{\pi}{2}$, then we have

$$2\left(\frac{2\pi}{\lambda}\right)\Delta R = \frac{\pi}{2}$$

$$\Rightarrow \Delta R = \frac{\pi}{2}\left(\frac{\lambda}{4\pi}\right) = \frac{\lambda}{8}$$

Now by the Pythagorean Theorem

$$\cancel{h^2} + \frac{\mathcal{L}^2}{4} = \cancel{h^2} + 2h\frac{\lambda}{8} + \frac{\lambda^2}{64}$$

$$\Rightarrow \frac{\mathcal{L}^2}{4} = 2h\frac{\lambda}{8} + \frac{\lambda^2}{64} \simeq \frac{h\lambda}{4}, \text{ since } \lambda \ll h$$

$$\Rightarrow \frac{h\lambda}{4} \gg \frac{\lambda^2}{64}$$

$$\Rightarrow \mathcal{L}^2 = h\lambda \Rightarrow \boxed{\mathcal{L} = \sqrt{h\lambda}}$$

(b) The along track resolution of the synthetic aperture is

$$\Delta X = h\theta_s,$$

where θ_s, the beamwidth of the synthetic aperture, is given by

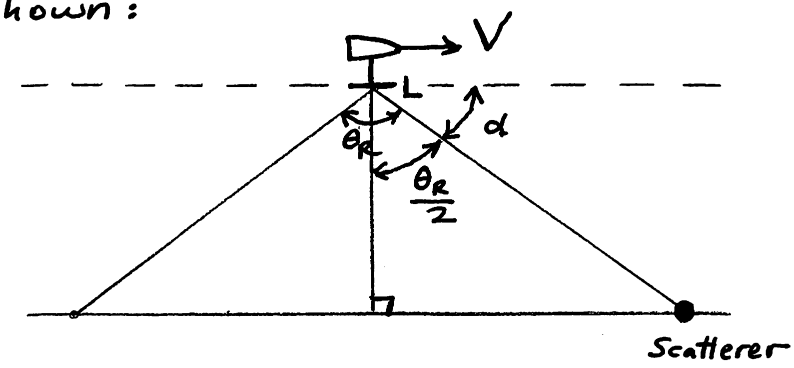
$$\theta_s = \frac{\lambda}{2\mathcal{L}}.$$

Thus

$$\Delta X = h\theta_s = h\left(\frac{\lambda}{2\mathcal{L}}\right) = h\left(\frac{\lambda}{2\sqrt{h\lambda}}\right) = \frac{1}{2}\sqrt{h\lambda}$$

$$= \boxed{\frac{(h\lambda)^{1/2}}{2}}$$

6.7. Consider a SAR flying with velocity V and having an antenna of physical dimension L along track as shown:



The beamwidth of the real antenna is

$$\theta_R \cong \frac{\lambda}{L}, \quad \text{assuming } L >> \lambda.$$

As the SAR flies along, the maximum Doppler shift from a stationary scatterer on the ground occurs from a scatterer just entering the beam. Such a scatterer will have Doppler shift

$$f_{D,max} = \frac{2V}{\lambda}\cos\alpha = \frac{2V}{\lambda}\cos\left(\frac{\pi}{2} - \frac{\theta_R}{2}\right)$$

$$= \frac{2V}{\lambda}\sin\left(\frac{\theta_R}{2}\right) = \frac{2V}{\lambda}\sin\left(\frac{\lambda}{2L}\right)$$

Since for $\theta << 1$, we have $\sin\theta \cong \theta$, and here we assume $\frac{\lambda}{2L} << 1$, this becomes

$$f_{D,max} = \frac{2V}{\lambda}\sin\left(\frac{\lambda}{2L}\right) \cong \frac{2V}{\lambda}\cdot\frac{\lambda}{2L} = \frac{V}{L}$$

Similarly, the Doppler shift of a scatterer just leaving the beam is $-V/L$.

In order to sample the signal without aliasing, we must sample at twice the maximum Doppler rate. But in a pulsed radar, we get one sample per pulse, so the PRF is the sampling rate, and thus

$$PRF \geq \frac{2V}{L}.$$