ECE 678 **Homework #5 Solutions** Prof. Bell

5.1 Assume $\alpha > 0$ and $v(t) = s(\alpha t)$. Then

$$\beta_v(\tau,\nu) = \int_{-\infty}^{\infty} v(t)\, v^*(t-\tau)\, e^{-i2\pi\nu t}\, dt$$

$$= \int_{-\infty}^{\infty} s(\alpha t)\, s^*(\alpha t - \alpha\tau)\, e^{-i2\pi\nu t}\, dt$$

(let $p = \alpha t \Rightarrow t = p/\alpha \Rightarrow dt = \frac{dp}{\alpha}$)

$$= \int_{-\infty}^{\infty} s(p)\, s^*(p-\alpha\tau)\, e^{-i2\pi\nu \frac{p}{\alpha}}\, \frac{dp}{\alpha}$$

$$= \frac{1}{\alpha}\int_{-\infty}^{\infty} s(p)\, s^*(p-\alpha\tau)\, e^{-i2\pi\left(\frac{\nu}{\alpha}\right)p}\, dp = \frac{1}{\alpha}\beta_s\left(\alpha\tau, \frac{\nu}{\alpha}\right) \quad \text{---}(*)$$

Now assume $\alpha < 0$. Then

$$\beta_v(\tau,\nu) = \int_{-\infty}^{\infty} v(t)\, v^*(t-\tau)\, e^{-i2\pi\nu t}\, dt$$

$$= \int_{-\infty}^{\infty} s(\alpha t)\, s^*(\alpha t - \alpha\tau)\, e^{-i2\pi\nu t}\, dt$$

(let $p = \alpha t \Rightarrow t = \frac{p}{\alpha} \Rightarrow dt = \frac{dp}{\alpha}$)

$$= \int_{+\infty}^{-\infty} s(p)\, s^*(p-\alpha\tau)\, e^{-i2\pi\nu\frac{p}{\alpha}}\, \frac{dp}{\alpha}$$

$$= -\frac{1}{\alpha}\int_{-\infty}^{+\infty} s(p)\, s^*(p-\alpha\tau)\, e^{-i2\pi\left(\frac{\nu}{\alpha}\right)p}\, dp = -\frac{1}{\alpha}\beta_s\left(\alpha\tau, \frac{\nu}{\alpha}\right) \quad \text{---}(**)$$

Combining $(*)$ and $(**)$ in one expression, we have

$$\beta_v(\tau,\nu) = \frac{1}{|\alpha|}\beta_s\left(\alpha\tau, \frac{\nu}{\alpha}\right) \blacksquare$$

5.2 Show that for a unit energy signal $u(t)$,

$$\frac{\partial^2 \beta_u(0,0)}{\partial \upsilon^2} = -4\pi^2 \int_{-\infty}^{\infty} t^2 |u(t)|^2 dt.$$

Proof: By definition, $\beta_u(\tau,\nu) = \int u(t) u^*(t-\tau) e^{-i2\pi\nu t} dt$, from which it follows that

$$\frac{\partial \beta_u(\tau,\nu)}{\partial \nu} = \frac{\partial}{\partial \nu}\left[\int_{-\infty}^{\infty} u(t) u^*(t-\tau) e^{-i2\pi\nu t} dt\right]$$

$$= \int_{-\infty}^{\infty} u(t) u^*(t-\tau) \frac{\partial e^{-i2\pi\nu t}}{\partial \nu} dt = \int_{-\infty}^{\infty} (-i2\pi t) u(t) u^*(t-\tau) e^{-i2\pi\nu t} dt$$

and thus

$$\frac{\partial^2 \beta_u(\tau,\nu)}{\partial \nu^2} = \frac{\partial}{\partial \nu}\left[\int_{-\infty}^{\infty} (-i2\pi t)\, u(t) u^*(t-\tau) e^{-i2\pi\nu t} dt\right]$$

$$= \int_{-\infty}^{\infty} (-i2\pi t)^2 u(t) u^*(t-\tau) e^{-i2\pi\nu t} dt$$

$$= -4\pi^2 \int_{-\infty}^{\infty} t^2 u(t) u^*(t-\tau) e^{-i2\pi\nu t} dt.$$

Now setting $\tau = 0$ and setting $\nu = 0$, we have

$$\left.\frac{\partial^2 \beta_u(\tau,\nu)}{\partial \nu^2}\right|_{\substack{\tau=0\\ \nu=0}} = -4\pi^2 \int_{-\infty}^{\infty} t^2 u(t) u^*(t) e^{-i2\pi\cdot 0\cdot t} dt$$

or

$$\frac{\partial^2 \beta_u(0,0)}{\partial \nu^2} = -4\pi^2 \int_{-\infty}^{\infty} t^2 |u(t)|^2 dt. \quad \blacksquare$$

<u>5.3</u> Prove that the 2-dimensional Fourier transform of $|\beta_s(\tau,\nu)|^2$ is equal to the function $|\beta_s(t,f)|^2$, itself, that is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\beta_s(\tau,\nu)|^2 e^{-i2\pi f\tau} \cdot e^{+i2\pi\nu t} d\tau\, d\nu = |\beta_s(t,f)|^2.$$

<u>Proof:</u>

$$\iint_{\mathbb{R}^2} |\beta_s(\tau,\nu)|^2 e^{-i2\pi f\tau} e^{+i2\pi\nu t} = \iint_{\mathbb{R}^2} \beta_s(\tau,\nu)\beta_s^*(\tau,\nu) e^{-i2\pi f\tau} e^{+i2\pi\nu t} d\tau\, d\nu$$

$$= \iint_{\mathbb{R}^2} \left[\int_{\mathbb{R}} s(t_1) s^*(t_1-\tau) e^{-i2\pi\nu t_1} dt_1\right]\left[\int_{\mathbb{R}} s(t_2) s^*(t_2-\tau) e^{-i2\pi\nu t_2} dt_2\right]^* \cdot e^{-i2\pi f\tau} e^{+i2\pi\nu t} d\tau\, d\nu$$

$$= \iiint_{\mathbb{R}^3} s(t_1) s^*(t_1-\tau) s^*(t_2) s(t_2-\tau) \left[\int_{\mathbb{R}} e^{-i2\pi\nu(t_1-t_2-t)} d\nu\right] e^{-i2\pi f\tau} dt_1\, dt_2\, d\tau$$

$$= \iiint_{\mathbb{R}^3} s(t_1) s^*(t_1-\tau) s^*(t_2) s(t_2-\tau)\, \delta(t_1-t_2-t)\, e^{-i2\pi f\tau} dt_1\, dt_2\, d\tau$$

$$= \iint_{\mathbb{R}^2} s(t_2+t) s^*(t_2+t-\tau) s^*(t_2) s(t_2-\tau) e^{-i2\pi f\tau} dt_2\, d\tau \quad \text{--- } (*)$$

Substituting $z = t_2-\tau \Rightarrow \tau = t_2 - z \Rightarrow d\tau = -dz$ yields

$$(*) = \int_{+\infty}^{-\infty}\int_{-\infty}^{\infty} s(t_2+t) s^*(z+t) s^*(t_2) s(z) e^{-i2\pi f(t_2-z)} dt_2\,(-dz)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(z) s^*(z+t)\, s^*(t_2) s(t_2+t)\, e^{-i2\pi f t_2} \cdot e^{+i2\pi f z} dt_2\, dz$$

$$\left(\begin{array}{l}\text{let } p_1 = z+t \Rightarrow z = p_1 - t \Rightarrow dz = dp_1 \\ \phantom{\text{let }} p_2 = t_2+t \Rightarrow t_2 = p_2 - t \Rightarrow dt_2 = dp_2\end{array}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^*(p_1) s(p_1-t) e^{+i2\pi f(p_1-t)}\, s(p_2) s^*(p_2-t) e^{-i2\pi f(p_2-t)} dp_2\, dp_1$$

$$= e^{-i2\pi f t} \cdot e^{+i2\pi f t}\left[\int_{-\infty}^{\infty} s(p_2) s^*(p_2-t) e^{-i2\pi f p_2} dp\right]\left[\int_{-\infty}^{\infty} s(p_1) s^*(p_1-t) e^{-i2\pi f p_1} dp_1\right]^*$$

$$= e^{i0}\, \beta_s(t,f)\, \beta_s^*(t,f) = |\beta_s(t,f)|^2.$$

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\beta_s(\tau,\nu)|^2 e^{-i2\pi f\tau} e^{+i2\pi\nu t} d\tau\, d\nu = |\beta_s(t,f)|^2 \quad \blacksquare$$

5.4 Calculate the asymmetric ambiguity function $\beta_s(\tau,\nu)$ of the signal

$$s(t) = Be^{-t^2/T^2}.$$

What value of B makes the signal $s(t)$ unit energy?

Solution: Note that the value of B giving a unit energy signal can be found from

$$1 = \int_{-\infty}^{\infty} |s(t)|^2 dt = B^2 \int_{-\infty}^{\infty} \exp\left\{-\frac{2t^2}{T^2}\right\} dt = B^2 \int_{-\infty}^{\infty} e^{-w^2/2}\left(\frac{T}{2}\right) dw$$

(letting $w = \frac{2t}{T}$)

$$= \frac{B^2 T}{2}\sqrt{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw}_{1} = B^2 T \sqrt{\frac{\pi}{2}}$$

$$\therefore \quad B = \sqrt{\frac{1}{T}\sqrt{\frac{2}{\pi}}}$$

Now $\beta_s(\tau,\nu)$ can be calculated as follows:

$$\beta_s(\tau,\nu) = \int_{-\infty}^{\infty} s(t)\, s^*(t-\tau)\, e^{-i2\pi\nu t} dt = B^2 \int_{-\infty}^{\infty} \exp\left[\frac{-t^2}{T}\right] \exp\left[\frac{-(t-\tau)^2}{T^2}\right] e^{-i2\pi\nu t} dt$$

$$= B^2 \int_{-\infty}^{\infty} \exp\left\{\frac{-2t^2 + 2t\tau + \tau^2}{T}\right\} e^{-i2\pi\nu t} dt$$

$$= B^2 e^{-\tau^2/T^2} \int_{-\infty}^{\infty} \exp\left\{\frac{-(t^2 - t\tau)}{T^2/2}\right\} e^{-i2\pi\nu t} dt$$

$$= B^2 e^{-\tau^2/T^2} e^{+\tau^2/2T^2} \int_{-\infty}^{\infty} \exp\left\{-\frac{(t-\tau/2)^2}{T^2/2}\right\} e^{-i2\pi\nu t}$$, completing the square

(let $w = t - \tau/2$) $\Rightarrow$ ($t = w + \tau/2 \Rightarrow dt = dw$)

$$= B^2 \exp\left\{\frac{-\tau^2}{2T^2}\right\} e^{-i\pi\nu\tau} \int_{-\infty}^{\infty} \exp\left\{\frac{-w^2}{T^2/2}\right\} e^{-i2\pi\nu w} dw$$

Now consider the well known Fourier transform pair $e^{-4\pi t^2} \overset{\mathcal{F}}{\rightleftharpoons} e^{-4\pi f^2}$. Using this and the above integral, we can write

$$\int_{-\infty}^{\infty} \exp\left\{\frac{-w^2}{T^2/2}\right\} e^{-i2\pi\nu w} = \int_{-\infty}^{\infty} \exp\left\{-\pi\left(\frac{1}{T}\sqrt{\frac{\pi}{2}}\, w\right)^2\right\} e^{-i2\pi\nu w} dw$$

$$= T\sqrt{\frac{\pi}{2}} \exp\left\{-\pi\left(T\sqrt{\frac{\pi}{2}}\,\nu\right)^2\right\} = T\sqrt{\frac{\pi}{2}} \exp\left\{\frac{-\pi^2 T^2}{2}\nu^2\right\},$$

From which it follows that

$$\beta_s(\tau,\nu) = \boxed{B^2 T\sqrt{\frac{\pi}{2}} \exp\left\{\frac{-\tau^2}{2T^2}\right\} \exp\{-i\pi\nu\tau\} \exp\left\{-\frac{(\pi T)^2}{2}\nu^2\right\}}$$

(5.4 - continued)

Or, we can rewrite this as

$$\beta_s(\tau,\nu) = \boxed{B^2 T\sqrt{\frac{\pi}{2}}\, \exp\{-i\pi\nu\tau\}\, \exp\left\{-\frac{1}{2}\left[\left(\frac{\tau}{T}\right)^2 + (\pi T\nu)^2\right]\right\}}$$

For the unit energy case where $B^2 = \frac{1}{T}\sqrt{\frac{2}{\pi}}$, this becomes

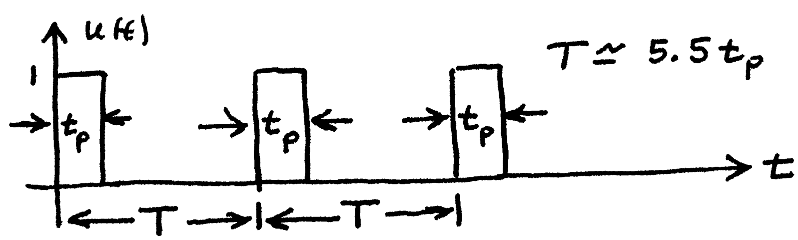
$$\beta_s(\tau,\nu) = \exp\{-i\pi\nu\tau\}\, \exp\left\{-\frac{1}{2}\left[\left(\frac{\tau}{T}\right)^2 + (\pi\tau\nu)^2\right]\right\}.$$

5.5 We can view the signal $s(t)$ in this problem as the "$s(t)$" in the previous problem multiplied by $e^{i\pi\alpha t^2}$. So using the quadratic phase shift property of the asymmetric ambiguity function, we have that if $s(t) = e^{i\pi\alpha t^2} p(t)$, then $\beta_s(\tau,\nu) = e^{-i\pi\alpha\tau^2} \cdot \beta_p(\tau, \nu-\alpha\tau)$. So using this result and the result of problem 5.4, we have

$$\beta_s(\tau,\nu) = e^{-i\pi\alpha\tau^2} \cdot e^{-i\pi(\nu-\alpha\tau)\tau} \cdot B^2 T\sqrt{\frac{\pi}{2}} \exp\left\{-\frac{1}{2}\left[\left(\frac{\tau}{T}\right)^2 + \pi^2 T^2(\nu-\alpha\tau)^2\right]\right\}$$

$$= B^2 T\sqrt{\frac{\pi}{2}}\, e^{-i\pi\nu\tau} \exp\left\{-\frac{1}{2}\left[\left(\frac{\tau}{T}\right)^2 + \pi^2 T^2(\nu-\alpha\tau)^2\right]\right\}$$

5.6 The pulse train appears as follows:



$T \simeq 5.5 t_p$

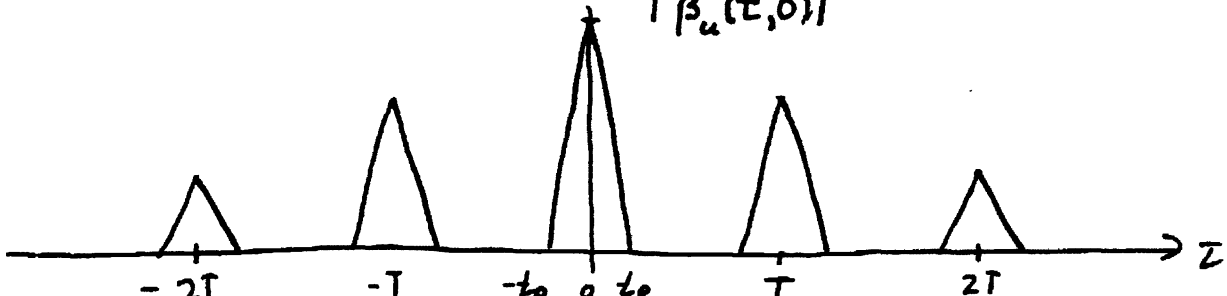
We can write the pulse train $u(t)$ as

$$u(t) = \sum_{m=1}^{3} 1_{[0,t_p]}(t-[m-1]T) = \begin{cases} 1, & t\in[0,t_p]\cup[T,T+t_p]\cup[2T,2T+t_p], \\ 0, & \text{elsewhere} \end{cases}$$

$$\beta_u(\tau,\nu) = \int_{-\infty}^{\infty} u(t)\,u^*(t-\tau)\,e^{-i2\pi\nu t}\,dt$$

While we can write a general expression as we did in class, it is easier to consider the specific cases

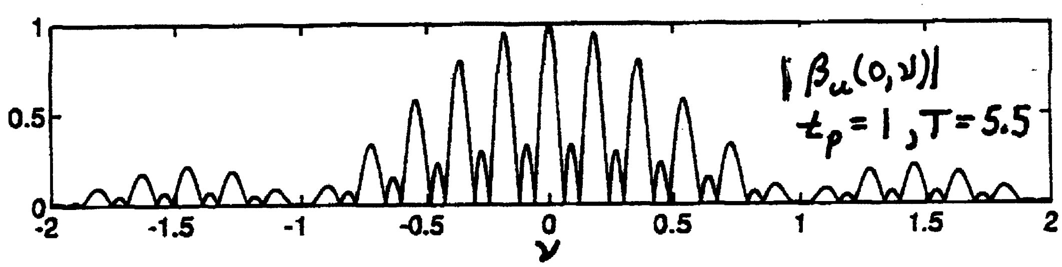
$|\beta_u(\tau,0)|$ is the "time autocorrelation of the signal $u(t)$. graphically it appears as follows:



$|\beta_u(0,\nu)|$ can be written as (simple Fourier transform of the 3-pulse pulse-train)

$$|\beta_u(0,\nu)| = \frac{1}{3}\left|\frac{\sin(\pi t_p \nu)}{\pi t_p \nu}\right| \cdot \left|\frac{\sin(3\pi\nu T)}{\sin(\pi\nu T)}\right|$$

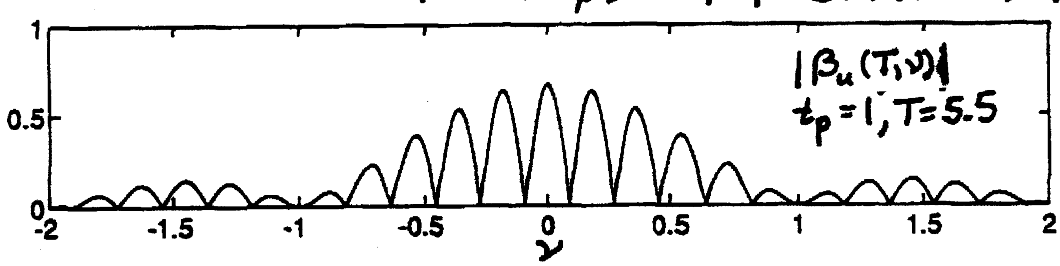
A plot is as follows:



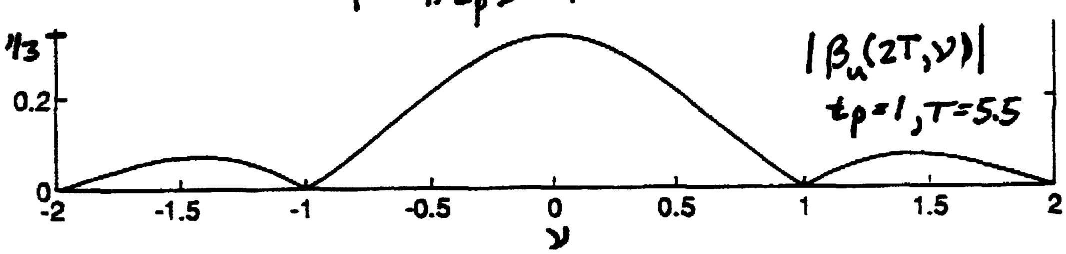
$|\beta_u(T,\nu)|$ can be written as

$$|\beta_u(T,\nu)| = \frac{1}{3}\left|\frac{\sin(\pi t_p \nu)}{\pi t_p \nu}\right| \cdot \left|\frac{\sin(2\pi T\nu)}{\sin(\pi T\nu)}\right|$$

A plot is as follows:



and $|\beta_u(2T,\nu)| = \frac{1}{3}\left|\frac{\sin(\pi t_p \nu)}{\pi t_p \nu}\right|$. A plot appears below.



(5.6 – continued)

$$|\beta_u(0,\tfrac{1}{2T})| = \frac{1}{3}\left|\frac{\sin(\pi t_p \nu)}{\pi t_p \nu}\right| \cdot \left|\frac{\sin 3\pi\nu T}{\sin \pi\nu T}\right|\Bigg|_{\nu=\frac{1}{2T}}$$

$$= \frac{1}{3}\left|\frac{\sin(\frac{\pi}{11})}{\pi/11}\right| \cdot \left|\frac{\sin 3\pi/2}{\sin \pi/2}\right| = \frac{1}{3}\left|\frac{\sin(\frac{\pi}{11})}{\frac{\pi}{11}}\right| \cdot \left|\frac{-1}{1}\right|$$

$$= \frac{1}{3}\left|\frac{\sin(\frac{\pi}{11})}{\pi/11}\right| = \boxed{0.3288}$$

$$|\beta_u(T,\tfrac{1}{2T})| = \frac{1}{3}\left|\frac{\sin(\pi(\frac{t_p}{2T}))}{\pi(t_p/2T)}\right| \cdot \left|\frac{\sin(2\pi T/2T)}{\sin(\pi T/2T)}\right|$$

$$= \frac{1}{3}\left|\frac{\sin(\frac{\pi}{11})}{\pi/11}\right| \cdot \left|\frac{\sin \pi}{\sin \frac{\pi}{2}}\right| = \frac{1}{3}\left|\frac{\sin(\frac{\pi}{11})}{\pi/11}\right| \cdot \left|\frac{0}{1}\right|$$

$$= \boxed{0}$$

$$|\beta_u(2T,\tfrac{1}{2T}) = \frac{1}{3}\left|\frac{\sin(\frac{\pi}{11})}{(\frac{\pi}{11})}\right| \cdot \left|\frac{\sin(\pi 2T/2T)}{\sin(\pi 2T/2T)}\right|$$

$$= \frac{1}{3}\left|\frac{\sin(\pi/11)}{\pi/11}\right| \cdot 1 = \boxed{0.3288}$$

5.7 n.b. Our intuition tells us the bias will be negative (target appears further than it actually is) because a positive Doppler shift means the correlation peak will occur later in a chirp going up in instantaneous frequency with time

$f_c = 10\,GHz$

$t_p = 20\,\mu s$

$\dot{R} = 150\, m/s$

$\Delta f = 1\,MHz$ (Bandwidth of chirp)

Now

$$\Delta f = \alpha t_p \Rightarrow \alpha = \frac{\Delta f}{t_p} = \frac{1\,MHz}{20\,\mu s} = \frac{1\times 10^6\ 1/s}{20\times 10^6\ s} = 5\times 10^{10}\ 1/s^2.$$

The Doppler shift in this case would be

$$\nu = \frac{2v}{\lambda} = \frac{2v f_c}{c} = \frac{2\,(150\,m/s)\cdot 10^9\,Hz}{3\times 10^8\,m/s} = 10^4\,Hz$$

$$= 10{,}000\,Hz$$

Assuming we use the peak of the matched filter output matched to a zero-Doppler version of the signal to determine range (delay), we have that the peak will occur when

$$\nu - \alpha\tau_b = 0 \Rightarrow \tau_b = \frac{\nu}{\alpha} = \frac{10^4\,Hz}{5\times 10^{10}\ 1/s^2} = 2\times 10^{-7}\ sec.$$

The range error will thus be

$$R_b = \frac{c\tau_b}{2} = \frac{(2\times 10^{-7}\,s)(3\times 10^8\,m/s)}{2} = \boxed{30\,m}$$

5.8 Prove that if $SAF\{s(t)\} = \Gamma_s(\tau,\nu)$, then if $u(t) = s(t)e^{i\pi\alpha t^2}$, then $\Gamma_u(\tau,\nu) = SAF\{u(t)\} = \Gamma_s(\tau,\nu-\alpha\tau)$

Proof: $$\Gamma_u(\tau,\nu) = \int_{-\infty}^{\infty} u(t+\tau/2)u^*(t-\tau/2)e^{-i2\pi\nu t}dt$$
$$= \int_{-\infty}^{\infty} s(t+\tau/2)e^{i\pi\alpha(t+\tau/2)^2}\left[s(t-\tau/2)e^{i\pi\alpha(t-\tau/2)^2}\right]^* e^{-i2\pi\nu t}dt$$
$$= \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{i\pi\alpha(t^2+\tau t+\tau^2/4)}e^{-i\pi\alpha(t^2-\tau t+\tau^2/4)}e^{-i2\pi\nu t}dt$$
$$= \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{+i2\pi\alpha\tau t}\cdot e^{-i2\pi\nu t}dt$$
$$= \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{-i2\pi(\nu-\alpha\tau)t}dt = \Gamma_s(\tau,\nu-\alpha\tau)$$

5.9 (a) $$SAF\{1_{[-T/2,T/2]}(t)\} = (T-|\tau|)\frac{\sin\pi\nu(T-|\tau|)}{\pi\nu(T-|\tau|)}\cdot 1_{[-T,T]}(\tau)$$

So using the above result (prob. 5.8) we have

$$SAF\{e^{i\pi\alpha t^2}1_{[-T/2,T/2]}(t)\} = (T-|\tau|)\frac{\sin[\pi(\nu-\alpha\tau)(T-|\tau|)]}{\pi(\nu-\alpha\tau)(T-|\tau|)}\cdot 1_{[-T,T]}(\tau)$$

(b) $$SAF\{e^{-\pi\beta t^2}\} = \sqrt{\frac{1}{2\beta}}\exp\left\{-\pi\left(\frac{\nu^2}{\beta}+\beta\tau^2\right)\right\}$$

Thus from the result of (prob. 5.8) we have

$$SAF\{e^{i\pi\alpha t^2}e^{-\pi\beta t^2}\} = \sqrt{\frac{1}{2\beta}}\exp\left\{-\pi\left[\frac{(\nu-\alpha\tau)^2}{\beta}+\beta\tau^2\right]\right\}$$