1. (a) $$p(\underline{\theta}|\underline{X}) = \frac{f(\underline{\theta},\underline{X})}{f(\underline{X})} = \frac{p(\underline{\theta}) f_{\underline{\theta}}(\underline{X})}{f(\underline{X})}$$

and

$$f(\underline{X}) = p_0 f_{\underline{\theta}_0}(\underline{X}) + p_1 f_{\underline{\theta}_1}(\underline{X})$$

Thus we have

$$p(\underline{\theta}|\underline{X}) = \begin{cases} \dfrac{p_0 f_{\underline{\theta}_0}(\underline{X})}{p_0 f_{\underline{\theta}_0}(\underline{X}) + p_1 f_{\underline{\theta}_1}(\underline{X})}, & \underline{\theta} = \underline{\theta}_0 \\[2ex] \dfrac{p_1 f_{\underline{\theta}_1}(\underline{X})}{p_0 f_{\underline{\theta}_0}(\underline{X}) + p_1 f_{\underline{\theta}_1}(\underline{X})}, & \underline{\theta} = \underline{\theta}_1 \end{cases}$$

(b) Define the <u>posterior likelihood ratio</u> as

$$L_p(\underline{X}) \triangleq \frac{p(\underline{\theta}_1|\underline{X})}{p(\underline{\theta}_0|\underline{X})} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Then we note that

$$L_p(\underline{X}) = \frac{p_1 f_{\underline{\theta}_1}(\underline{X}) / f(\underline{X})}{p_0 f_{\underline{\theta}_0}(\underline{X}) / f(\underline{X})} = \frac{p_1}{p_0} \cdot \frac{f_{\underline{\theta}_1}(\underline{X})}{f_{\underline{\theta}_0}(\underline{X})}$$

$$= \frac{p_1}{p_0} L(\underline{X}), \quad \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

where

$$L(\underline{X}) \triangleq \frac{f_{\underline{\theta}_1}(\underline{X})}{f_{\underline{\theta}_0}(\underline{X})}$$ is the standard likelihood ratio

Thus we have

$$L_p(\underline{X}) \underset{H_0}{\overset{H_1}{\gtrless}} 1 \iff \frac{p_1}{p_0} L(\underline{X}) \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\iff L(\underline{X}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0}{p_1}$$

$\therefore$ The posterior likelihood test is in fact a likelihood ratio test with threshold

$$L_0 = \frac{p_0}{p_1}.$$

2(a) Under hypothesis $H_i$, $N_T$ is a Poisson RV with pmf

$$P_i(n) = \frac{(\lambda_i T)^n e^{-\lambda_i T}}{n!}, \quad n = 0, 1, 2, \ldots, \quad i = 0, 1.$$

In order to find the pmf of $M_T$ under hypothesis $H_i$, $i = 0, 1$, we note that each of the $N_T$ photons has a probability $p$ of being detected. Thus

$$p(m|n) = P(M_T = m \mid N_T = n) = \binom{n}{m} p^m (1-p)^{n-m},$$

for $m = 0, 1, 2, \ldots, n$

Now the pmf of $M_T$ under $H_i$ can be written as

$$P^{(i)}_{M_T}(m) = \sum_{n=0}^{\infty} P_{M_T N_T}(m,n) = \sum_{n=0}^{\infty} p(m|n) P_i(n)$$

$$= \sum_{n=m}^{\infty} p(m|n) P_i(n), \quad \text{since } p(m|n) = 0 \text{ for } n < m.$$

$$= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} \frac{(\lambda_i T)^n e^{-\lambda_i T}}{n!}$$

$$= \sum_{n=m}^{\infty} \frac{\cancel{n!}}{m!(n-m)!} p^m (1-p)^{n-m} \frac{(\lambda_i T)^n e^{-\lambda_i T}}{\cancel{n!}}$$

$$= \frac{p^m (\lambda_i T)^m}{m!} \sum_{n=m}^{\infty} (1-p)^{n-m} \frac{(\lambda_i T)^{n-m} e^{-\lambda_i T}}{(n-m)!}$$

$$= \frac{(\lambda_i T p)^m}{m!} e^{-\lambda_i T} \sum_{k=0}^{\infty} \frac{[(1-p)\lambda_i T]^k}{k!}$$

$$= \frac{(\lambda_i T p)^m}{m!} e^{-\lambda_i T} \cdot e^{(1-p)\lambda_i T} = \frac{(\lambda_i T p)^m e^{-\lambda_i T p}}{m!}, \quad m = 0, 1, 2, \ldots$$

Thus we have

$$P^{(i)}_{M_T} = P_{H_i}(\{M_T = m\}) = \frac{(\lambda_i T p)^m e^{-\lambda_i T p}}{m!}, \quad m = 0, 1, 2, \ldots.$$

(2 - continued)

Thus we have

$$P(\{M_T = m\}) = P(H_0)P(\{M_T = m\}|H_0) + P(H_1)P(\{M_T = m\}|H_1)$$
$$= p_0 P_{H_0}(\{M_T = m\}) + p_1 P_{H_1}(\{M_T = m\})$$
$$= p_0 \frac{(\lambda_0 T\rho)^m e^{-\lambda_0 T\rho}}{m!} + p_1 \frac{(\lambda_1 T\rho)^m e^{-\lambda_1 T\rho}}{m!},$$
$$m = 0, 1, 2, \ldots$$

(b) The decision rule that minimizes the probability of error is the Bayesian decision rule with losses $L_{01} = L_{10} = 1$ and $L_{00} = L_{11} = 0$.

(In this case, the Bayes risk is prob. of error).

This is a likelihood ratio test. The likelihood ratio is

$$L(M_T) = \frac{p(M_T | H_1)}{p(M_T | H_0)} = \frac{(\lambda_1 T\rho)^{M_T} e^{-\lambda_1 T\rho}/M_T!}{(\lambda_0 T\rho)^{M_T} e^{-\lambda_0 T\rho}/M_T!}$$
$$= \left(\frac{\lambda_1}{\lambda_0}\right)^{M_T} e^{-(\lambda_1 - \lambda_0)T\rho} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0 L_{01}}{p_1 L_{10}} = \frac{p_0}{p_1}$$

$$\ell(M_T) = \ln L(M_T) = M_T \ln\left(\frac{\lambda_1}{\lambda_0}\right) - (\lambda_1 - \lambda_0)T\rho$$
$$\underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{p_0}{p_1}$$

$$\Rightarrow \quad M_T \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln\left(\frac{p_0}{p_1}\right) + (\lambda_1 - \lambda_0)\rho T}{\ln(\lambda_1/\lambda_0)} \triangleq M_0$$

Thus we have that the Bayes test yielding the minimum probability of error decision rule is

$$\phi(M_T) = \begin{cases} 1, & M_T > \dfrac{\ln(p_0/p_1) + (\lambda_1 - \lambda_0)\rho T}{\ln(\lambda_1/\lambda_0)} \\ 0, & \text{otherwise}. \end{cases}$$

(2 - continued)

(c) The probability of error, $P(E)$, can be written as

$$P(E) = p_0 \cdot P_{H_0}(\{M_T > M_0\}) + p_1 P_{H_1}(\{M_T < M_0\})$$

$$= p_0 \cdot \left(1 - \sum_{m=0}^{\lfloor M_0 \rfloor} \frac{(\lambda_0 T_p)^m}{m!} e^{-\lambda_0 T_p}\right) + p_1 \sum_{m=0}^{\lfloor M_0 \rfloor} \frac{(\lambda_1 T_p)^m}{m!} e^{-\lambda_1 T_p}$$

3 (a) We know the Bayes decision rule has the form of a likelihood ratio test, so we compute the likelihood ratio:

$$L(X) = \frac{f_1(X)}{f_0(X)} = \frac{\exp\left\{-\frac{(X^2-2XA+A^2)}{2\sigma^2}\right\}}{\exp\left\{\frac{-X^2}{2\sigma^2}\right\}}$$

$$= \exp\left\{\frac{2XA-A^2}{2\sigma^2}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0 L_{01}}{p_1 L_{10}}$$

$$\ell(X) = \ln L(X) = \frac{2XA-A^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{p_0 L_{01}}{p_1 L_{10}}\right)$$

$$\Rightarrow AX \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{p_0 L_{01}}{p_1 L_{10}}\right) + \frac{A^2}{2}$$

We must consider two cases:

(i) $\underline{A > 0}$:
$$X \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{A} \ln\left(\frac{p_0 L_{01}}{p_1 L_{10}}\right) + \frac{A}{2}$$

$$\Rightarrow \quad \phi(X) = \begin{cases} 1, & X > X_0 = \frac{\sigma^2}{A}\ln\left(\frac{p_0 L_{01}}{p_1 L_{10}}\right) + \frac{A}{2} \\ 0, & \text{otherwise} \end{cases}$$

(ii) $\underline{A < 0}$:
$$X \underset{H_0}{\overset{H_1}{\lessgtr}} \frac{\sigma^2}{A} \ln\left(\frac{p_0 L_{01}}{p_1 L_{10}}\right) + \frac{A}{2}$$

$$\Rightarrow \quad \phi(X) = \begin{cases} 1, & X < X_0 = \frac{\sigma^2}{A}\ln\left(\frac{p_0 L_{01}}{p_1 L_{10}}\right) + \frac{A}{2} \\ 0, & \text{otherwise.} \end{cases}$$

(3 - continued)

(b) Here, because $A = 1 > 0$, we are dealing with form (i) of the test. We have that the threshold $X_0$ is

$$X_0 = \frac{\sigma^2}{A} \ln\left(\frac{P_0 L_{01}}{P_1 L_{10}}\right) + \frac{A}{2} \quad \text{with } A = 1,\ \sigma = 1,\ L_{01} = 1, \text{ and } L_{10} = 2.$$

$$\Rightarrow \quad X_0 = \frac{1}{1} \ln\left(\frac{1-P_1}{2P_1}\right) + \frac{1}{2} = \ln\left(\frac{1-P_1}{2P_1}\right) + \frac{1}{2}$$

$$= \ln(1-P_1) - \ln P_1 + \tfrac{1}{2} - \ln 2$$

(c) The probability of type I error for the parameters given in part (b) is

$$\alpha = P_{H_0}(\{X > X_0\}) = \int_{X_0}^{\infty} f_0(x)\,dx$$

$$= \int_{X_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-x^2}{2}\right\} dx = 1 - \Phi(X_0)$$

$$= 1 - \Phi\left(\ln(1-P_1) - \ln P_1 + \tfrac{1}{2} - \ln 2\right)$$

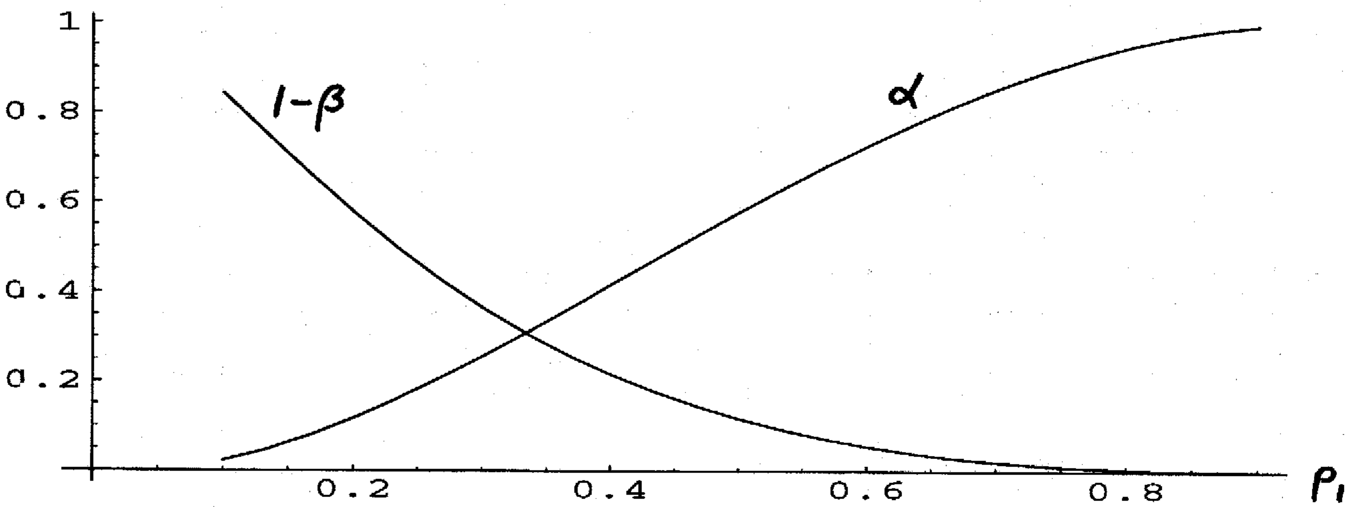
The probability of type II error for the parameters given in (b) is

$$1-\beta = P_{H_1}(\{X \le X_0\}) = \int_{-\infty}^{X_0} f_1(x)\,dx$$

$$= \int_{-\infty}^{X_0} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-1)^2}{2}\right\} dx = \Phi(X_0 - 1)$$

$$= \Phi\left(\ln(1-P_1) - \ln P_1 - \tfrac{1}{2} - \ln 2\right)$$

Plots of $\alpha$ and $1-\beta$ as a function of $P_1$ appear on the next page.

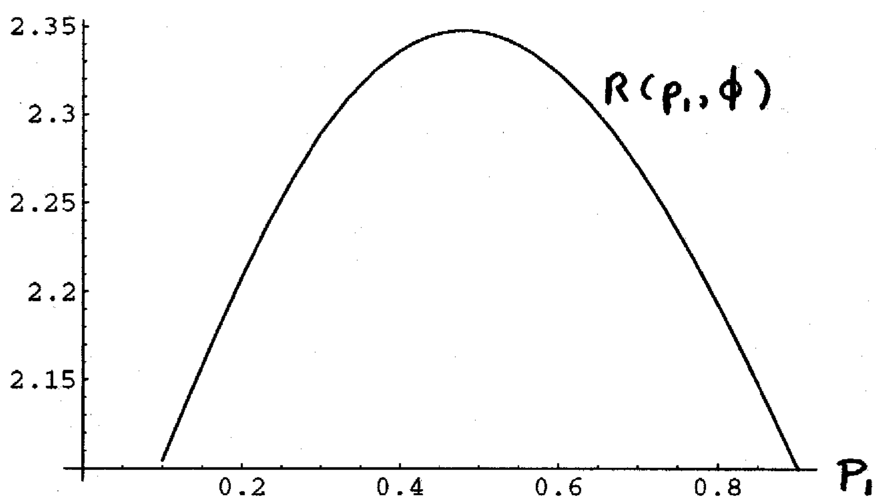



(d) The Bayes Risk for the values given in part (b) is

$$R(\underline{p}, \phi) = R(p_1, \phi) = (1-p_1) L_{01}\, \alpha + p_1 L_{10} (1-\beta)$$

$$\Rightarrow \quad R(p_1, \phi) = (1-p_1)\left[1 - \Phi\left(\ln(1-p_1) - \ln p_1 + \tfrac{1}{2} - \ln 2\right)\right] + p_1\, 2\, \Phi\left(\ln(1-p_1) - \ln p_1 - \tfrac{1}{2} - \ln 2\right)$$

A plot of $R(p_1, \phi)$ as a fcn. of $p_1$ appears as follows:



(e) The minimax test is found by determining the value of $p_1$ that maximizes $R(p_1, \phi)$.

Taking the derivative w.r.t. $p_1$ and setting the resulting expression equal to zero, we get an equation we can solve numerically for the value of $p_1$ that maximizes $R(p_1, \phi)$. The numerical solution yields

$$p_1^* = 0.480206$$

$$R(p_1^*) = 2.34717$$

So using this value of $p_1^*$, we can compute the threshold of the minimax test

$$x_0^* = \ln(1-p_1^*) - \ln p_1^* + \tfrac{1}{2} - \ln 2$$

$$= -0.11393$$

Thus we have the minimax test

$$\phi_{MM}(X) = \begin{cases} 1, & X > -0.11393 \\ 0, & \text{otherwise} \end{cases}.$$

At first glance, the threshold $x_0 = -0.11393$, being negative, looks odd, but this is due to the high cost of deciding $H_0$ when $H_1$ is true ($L_{10} = 2$) relative to that of deciding $H_1$ is true when $H_0$ is true ($L_{01} = 1$).

4 (a) If we detect the signal $s(t)$ in AWGN and sample the output at time $t=T$, we have that the impulse response $h(t)$ of the matched filter is

$$h(t) = \frac{2}{N_0} s^*(T-t)$$

$$= \frac{2}{N_0} \left(A \cdot 1_{[0,T]}(T-t)\right)^*$$

$$= \frac{2A}{N_0} \cdot 1_{[0,T]}(t) .$$

(b) The peak SNR, which occurs at time $t=T$, has value

$$SNR_T = \frac{2E_s}{N_0}$$

For this signal,

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_0^T A^2 dt = A^2 T.$$

Thus

$$SNR_T = \frac{2A^2 T}{N_0}$$

5 (a) If we detect the signal $s(t)$ in AWGN and sample the output at time $t=T$, we have that the impulse response $h(t)$ of the matched filter is

$$h(t) = \frac{2}{N_0} s^*(T-t)$$

$$= \frac{2}{N_0} \left(A e^{i\pi\alpha(T-t)^2}\right)^* \cdot 1_{[0,T]}(T-t)$$

$$= \frac{2A}{N_0} e^{-i\pi\alpha(T-t)^2} \cdot 1_{[0,T]}(t)$$

(b) The peak SNR, which occurs at time $t=T$, has value

$$SNR_T = \frac{2E_s}{N_0}$$

For this signal,

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_0^T A e^{i\pi\alpha t^2} \cdot A^* e^{-i\pi\alpha t^2} dt$$

$$= \int_0^T |A|^2 dt = A^2 T, \text{ assuming } A \text{ is real}$$

Thus

$$SNR_T = \frac{2A^2T}{N_0}$$