EE678 Radar   **Homework #3 Solutions**   Mark R. Ball

3.1 We are testing two simple hypotheses, $H_0: \Theta_0 = \{\mu_0\}$ and $H_1: \Theta_1 = \{\mu_1\}$, where the sample distribution is

$$f_\mu(x) = \frac{1}{\mu} \exp\left\{\frac{-x}{\mu}\right\} \cdot 1_{[0,\infty)}(x), \quad \mu > 0,$$

and $\mu_1 > \mu_0 > 0$.

(a) By the Neyman-Pearson lemma, we know that the most powerful test of size α is the likelihood ratio test:

$$L(X) = \frac{\mu_0 \exp\{-X/\mu_1\}}{\mu_1 \exp\{-X/\mu_0\}} \underset{H_0}{\overset{H_1}{\gtrless}} L_0$$

$$\Rightarrow \ell(X) = \ln L(X) = \ln\left(\frac{\mu_0}{\mu_1}\right) + X\left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ell_0 = \ln L_0$$

$$\Rightarrow X \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\mu_0 \mu_1}{\mu_1 - \mu_0}\left[\ell_0 + \ln\left(\frac{\mu_1}{\mu_0}\right)\right] = \lambda_0 .$$

So the most powerful test of size α is

$$\boxed{\phi(X) = \begin{cases} 1, & X > \lambda_0 \\ 0, & X < \lambda_0 \end{cases}}$$

<u>n.b.</u> $P(\{X = \lambda_0\}) = 0$ under H_0, so randomization is not necessary.

We can calculate λ_0 from $\alpha = P_{\mu_0}(\{X > \lambda_0\}) = \int_{\lambda_0}^{\infty} \frac{1}{\mu_0} \exp\left(\frac{-x}{\mu_0}\right) dx = \exp\left(\frac{-\lambda_0}{\mu_0}\right)$

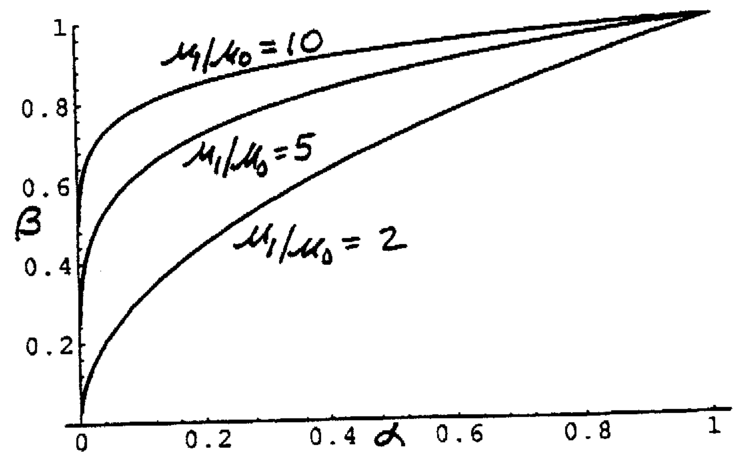
$$\Rightarrow \boxed{\lambda_0 = -\mu_0 \ln \alpha}$$

(b) $$\beta = \int_{\lambda_0}^{\infty} \frac{1}{\mu_1} \exp\left(\frac{-x}{\mu_1}\right) dx = \exp\left(-\frac{\lambda_0}{\mu_1}\right) = \exp\left(\frac{\mu_0}{\mu_1} \ln \alpha\right)$$

$$= \exp\left\{\ln \alpha^{\mu_0/\mu_1}\right\} = \alpha^{\mu_0/\mu_1}$$

$$\therefore \boxed{\beta(\alpha, \mu_1/\mu_0) = \alpha^{1/(\mu_1/\mu_0)}}$$

(c) A plot of the ROC appears as follows:



3.2 (a) $$f_{\mu_1}(\underline{X}) = \frac{1}{\mu_1^N} \exp\left\{\frac{-1}{\mu_1}\sum_{k=1}^{N} X_k\right\}\cdot 1_{[0,\infty)}(\min\{X_k\})$$

$$f_{\mu_0}(\underline{X}) = \frac{1}{\mu_0^N} \exp\left\{\frac{-1}{\mu_0}\sum_{k=1}^{N} X_k\right\}\cdot 1_{[0,\infty)}(\min\{X_k\})$$

$$L(\underline{X}) = \frac{f_{\mu_1}(\underline{X})}{f_{\mu_0}(\underline{X})} = \left(\frac{\mu_0}{\mu_1}\right)^N \exp\left\{\left(\frac{1}{\mu_0}-\frac{1}{\mu_1}\right)\sum_{k=1}^{N} X_k\right\} \underset{H_0}{\overset{H_1}{\gtrless}} L_0$$

$$\ell(\underline{X}) = \ln L(\underline{X}) = N \log\left(\frac{\mu_0}{\mu_1}\right) + \frac{\mu_1-\mu_0}{\mu_0\mu_1}\sum_{k=1}^{N} X_k \underset{H_0}{\overset{H_1}{\gtrless}} \ln L_0 = \ell_0$$

$$\Rightarrow t(\underline{X}) \triangleq \sum_{k=1}^{N} X_k \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\mu_0\mu_1}{\mu_1-\mu_0}\left[N \ln\left(\frac{\mu_1}{\mu_0}\right) + \ell_0\right] = \lambda_0$$

Thus $t(\underline{X})$ is a sufficient statistic for the most powerful test of size α, and we can write the test as

$$\boxed{\phi(\underline{X}) = \begin{cases} 1, & t(\underline{X}) > \lambda_0 \\ 0, & t(\underline{X}) < \lambda_0 \end{cases}}$$

Again, randomization is unnecessary because for any λ_0, $P_{\mu_0}(\{t(\underline{X}) = \lambda_0\}) = 0$.

The sufficient statistic $t(\underline{X}) = \sum_{k=1}^{N} X_k$ is, under both H_0 and H_1 a sum of N i.i.d. exponentially distributed RVs (each having mean μ_0 under H_0 and μ_1 under H_1). Hence properly normalized, $t(\underline{X})$ is a chi-square RV with $2N$ degrees of freedom (or a gamma RV with $\alpha = N$ and $\beta = \frac{1}{\mu_i}$). So under hypothesis H_0, the pdf of $t(\underline{X})$ is

$$f_{t;\mu_0}^{(N)}(t) = \frac{t^{N-1}\exp\{-t/\mu_0\}}{\mu_0^N\,\Gamma(N)}\cdot 1_{[0,\infty)}(t)$$

The threshold λ_0 yielding a size α test can be found by solving

$$\alpha = \int_{\lambda_0}^{\infty} f_{t;\mu_0}^{(N)}(t)\,dt = 1 - \int_0^{\lambda_0} f_{t;\mu_0}^{(N)}(t)\,dt = 1 - F_{t,\mu_0}^{(N)}(\lambda_0), \quad (*)$$

where $F_{t,\mu_0}^{(N)}(t)$ is the cdf corresponding to pdf $f_{t;\mu_0}^{(N)}(t)$.

It is probably best to solve for λ_0 yielding a size α test — call this threshold $\lambda_0(\mu_0, N, \alpha)$ — by solving $(*)$ numerically, but a "closed form" solution can be found as follows:

From $(*)$, we have

$$\alpha = \int_{\lambda_0}^{\infty} f_{t;\mu_0}^{(N)}(t)\,dt = \int_{\lambda_0}^{\infty} \frac{t^{N-1}\exp(-t/\mu_0)}{\mu_0^N\,\Gamma(N)}\,dt$$

$$= \frac{1}{\Gamma(N)} \int_{\frac{\lambda_0}{\mu_0}}^{\infty} z^{N-1}\exp(-z)\,dz = \frac{\Gamma\left(N, \frac{\lambda_0}{\mu_0}\right)}{\Gamma(N)}$$

where

$$\Gamma(\rho, \gamma) \triangleq \int_{\gamma}^{\infty} z^{\rho-1}\exp(-z)\,dz$$

is the incomplete gamma function. It then follows that

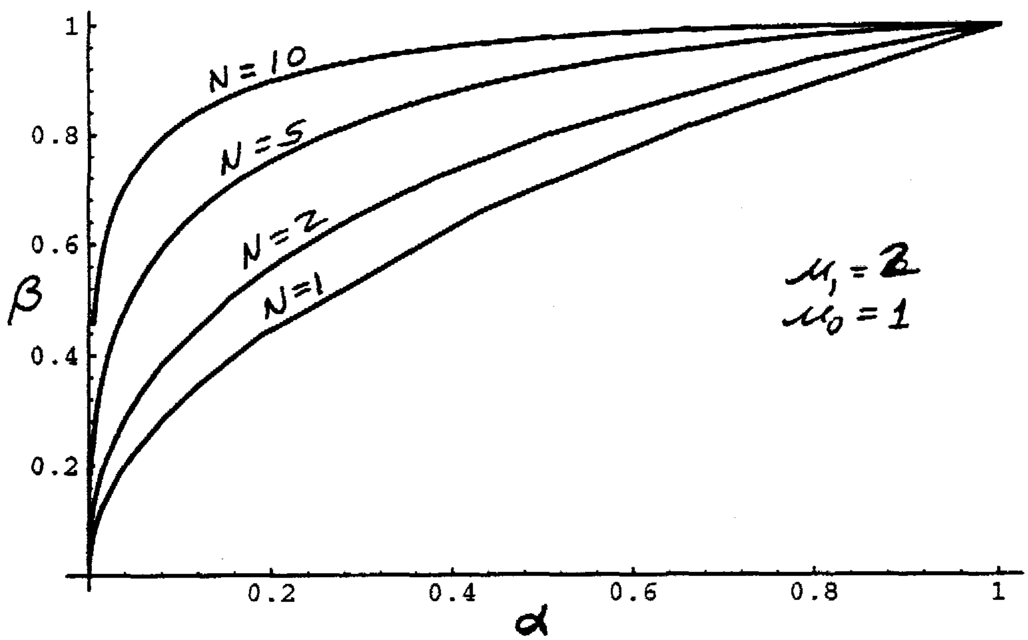
$$\lambda_0 = \mu_0\,\Gamma^{-1}(N, \alpha\Gamma(N)) \triangleq \lambda_0(\mu_0, N, \alpha)$$

where $\Gamma^{-1}(N,\cdot)$ is the inverse incomplete gamma function (where the inverse is w.r.t. the second argument).

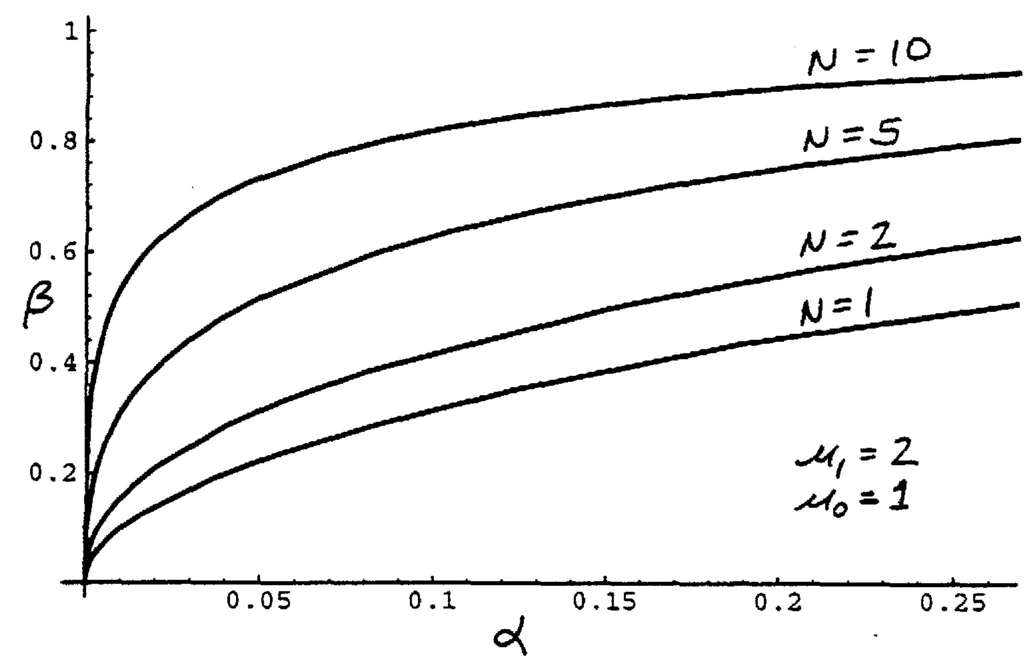
(b) The power of the test is given by

$$\beta(\alpha, \mu_0, \mu_1, N) = \int_{\lambda_0(\mu_0, N, \alpha)}^{\infty} \frac{t^{N-1}\exp(-t/\mu_1)}{\mu_1^N\,\Gamma(N)}\,dt = 1 - F_{t;\mu_1}^{(N)}(\lambda_0(\mu_0, N, \alpha)).$$

(C) The R.O.C can be generated by parametrically plotting $(\alpha(\lambda_0), \beta(\lambda_0))$, where $\alpha(\lambda_0) = 1 - F_{t,\mu_0}^{(N)}(\lambda_0)$ See plot on next page.
varying λ_0 from 0 to $+\infty$. $\beta(\lambda_0)$ $1 - F_{t,\mu_1}^{(N)}(\lambda_0)$



Complete ROC



Blown-Up View for $0 < \alpha < 0.25$

3.3 N, the number of photons observed, has a Poisson distribution with mean $\lambda_0 = 1$ under H_0 and $\lambda_1 = 5$ under H_1. The likelihood ratio is

$$L(N) = P_{\lambda_1}(N) / P_{\lambda_0}(N) = \frac{\left(\frac{\lambda_1^N e^{-\lambda_1}}{N!}\right)}{\left(\frac{\lambda_0^N e^{-\lambda_0}}{N!}\right)} = \left(\frac{\lambda_1}{\lambda_0}\right)^N e^{-(\lambda_1-\lambda_0)} \underset{H_0}{\overset{H_1}{\gtrless}} L_0$$

The log-likelihood ratio is $\ell(N) = \ln L(N) = N \log\left(\frac{\lambda_1}{\lambda_0}\right) - (\lambda_1 - \lambda_0) \underset{H_0}{\overset{H_1}{\gtrless}} \mu$

$$\Rightarrow N \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln L_0 + (\lambda_1 - \lambda_0)}{\ln(\lambda_1/\lambda_0)} = V_0 \Rightarrow \phi(N) = \begin{cases} 1, & N > V_0 \\ \gamma, & N = V_0 \\ 0, & N < V_0 \end{cases}$$

where V_0 and γ are selected to give a size α test under H_0. For integer value V_0 under hypothesis H_0 we have

$$\therefore P_{H_0}(\{N > V_0\}) = 1 - P(\{N \le V_0\}) = 1 - \sum_{k=0}^{V_0} \frac{e^{-1}}{k!}$$

Evaluating for various integer values of V_0, we have

V_0	0	1	2	3	4
$P_{H_0}(\{N > V_0\})$	0.632121	0.264241	0.080301	0.0189882	0.00365985

Thus in order to get a size $\alpha = 0.01$ test, we take $V_0 = 4$ and

$$\gamma = \frac{P_{H_0}(\{N \le V_0\}) - (1-\alpha)}{P_{H_0}(\{N = V_0\})} = \frac{\alpha - P_{H_0}(\{N > 4\})}{P_{H_0}(\{N = 4\})}$$

$$= \frac{0.01 - 0.00365985}{0.0153283} = 0.413624$$

$$\therefore \phi(N) = \begin{cases} 1, & N > 4 \\ 0.413624, & N = 4 \\ 0, & N < 4 \end{cases}$$ is the most powerful test of size $\alpha = 0.01$.

The resulting power β – or prob. of detection P_D, is

$$P_D = P_{H_1}(\{N > 4\}) + \gamma P(\{N = 4\}) = 0.6321$$

3.4 The pdfs of $\underline{X}$ are given by: (assuming $\underline{X}=(X_1,\dots,X_N)^T$)

Under H_0: $$f_0(\underline{X})=\frac{1}{(2\pi)^{N/2}|C_0|^{1/2}}\exp\left\{-\frac{1}{2}(\underline{X}-\underline{m}_0)^T C_0^{-1}(\underline{X}-\underline{m}_0)\right\};$$

Under H_1: $$f_1(\underline{X})=\frac{1}{(2\pi)^{N/2}|C_1|^{1/2}}\exp\left\{-\frac{1}{2}(\underline{X}-\underline{m}_1)^T C_1^{-1}(\underline{X}-\underline{m}_1)\right\}.$$

$$L(\underline{X})=\frac{f_1(\underline{X})}{f_0(\underline{X})}=\frac{\frac{1}{(2\pi)^{N/2}|C_1|^{1/2}}\exp\left\{-\frac{1}{2}(\underline{X}-\underline{m}_1)^T C_1^{-1}(\underline{X}-\underline{m}_1)\right\}}{\frac{1}{(2\pi)^{N/2}|C_1|^{1/2}}\exp\left\{-\frac{1}{2}(\underline{X}-\underline{m}_1)^T C_1^{-1}(\underline{X}-\underline{m}_1)\right\}}\ \underset{H_0}{\overset{H_1}{\gtrless}}\ L_0$$

$$\ell(\underline{X})=\ln L(\underline{X})=\frac{1}{2}\ln\left(\frac{|C_0|}{|C_1|}\right)-\frac{1}{2}(\underline{X}-\underline{m}_1)^T C_1^{-1}(\underline{X}-\underline{m}_1)+\frac{1}{2}(\underline{X}-\underline{m}_0)^T C_0^{-1}(\underline{X}-\underline{m}_0)\underset{H_1}{\overset{H_1}{\gtrless}}\ln L$$

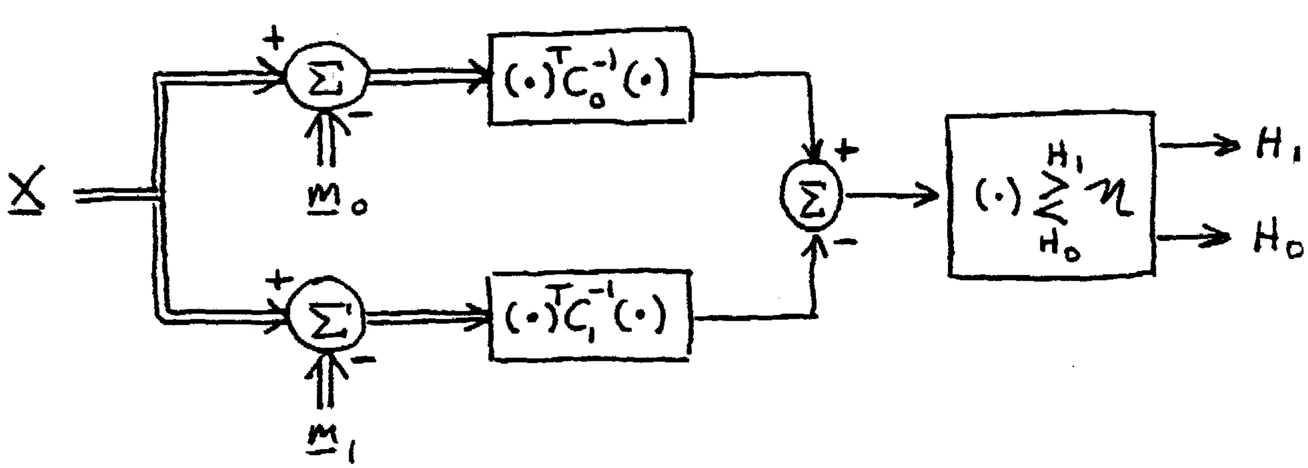
$$\Rightarrow\ (\underline{X}-\underline{m}_0)^T C_0^{-1}(\underline{X}-\underline{m}_0)-(\underline{X}-\underline{m}_1)C_1^{-1}(\underline{X}-\underline{m}_1)\underset{H_0}{\overset{H_1}{\gtrless}}\ell_0'=2\ln L_0+\ln\left(\frac{|C_1|}{|C_0|}\right)$$

Thus we have a test of the form

$$\phi(\underline{X})=\begin{cases}1, & Q_0(\underline{X})-Q_1(\underline{X})>\eta\ ; \quad Q_0(\underline{X})=(\underline{X}-\underline{m}_0)^T C_0^{-1}(\underline{X}-\underline{m}_0)\\ 0, & Q_0(\underline{X})-Q_1(\underline{X})\le\eta\ ; \quad Q_1(\underline{X})=(\underline{X}-\underline{m}_1)C_1^{-1}(\underline{X}-\underline{m}_1)\end{cases}$$

where $\eta=\ln\left[L_0^2\,\frac{|C_1|}{|C_0|}\right]$

A block diagram of the implementation of this test appears as follows



3.5 Consider the likelihood ratio

$$L_{\mu_1}(X) = \frac{\mu_0 \exp\{-X/\mu_1\}}{\mu_1 \exp\{-X/\mu_0\}} = \frac{\mu_0}{\mu_1} \exp\left\{X\left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right)\right\}$$

For every pair (μ_0, μ_1) such that $\mu_1 > \mu_0$, $L_{\mu_1}(X)$ is a non-decreasing function of μ_1. Hence it follows that the conditions of the Karlin-Rubin theorem are met. Hence, the UMP test of $H_0: \Theta_0 = [0,1)$ versus $H_1: \Theta_1 = [1, \infty)$ is of the form

$$\phi(X) = \begin{cases} 1, & X > X_0 \\ \gamma, & X = X_0 \\ 0, & X < X_0 \end{cases}$$

We must find X_0 and γ for a size α test. Note that under both H_0 and H_1, we have $P(\{X = X_0\} | H_0) = P(\{X = X_0\} | H_1) = 0 \Rightarrow \gamma = 0$
So we don't have to worry about randomization

The X_0 yielding a size α set can be found from

$$\alpha = \sup_{\mu_0 \in [0,1)} P_{\mu_0}(\{X > X_0\}) = \exp\left(-\frac{X_0}{1}\right) = e^{-X_0} \Rightarrow X_0 = -\ln \alpha$$

The power $\beta(\mu_1)$ of the size α test is thus given by

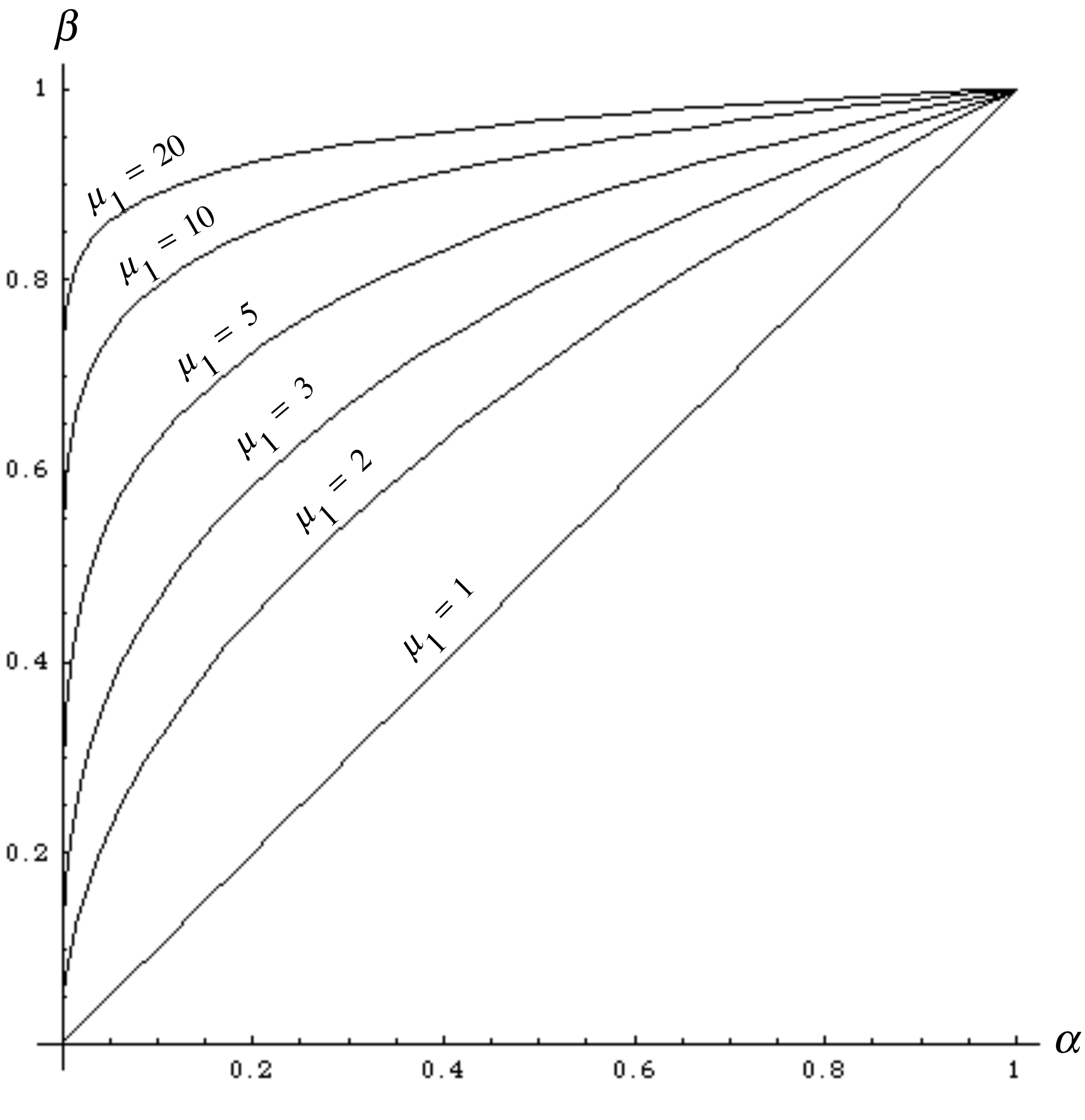
$$\beta(\mu_1) = P_{\mu_1}(\{X > X_0\}) = e^{+\frac{\ln \alpha}{\mu_1}} = e^{+\ln \alpha^{1/\mu_1}}$$

$$= \alpha^{(1/\mu_1)}$$

$$\therefore \quad \phi(X) = \begin{cases} 1, & X > -\ln \alpha \\ 0, & X \le -\ln \alpha \end{cases}$$

and $\beta(\mu_1) = \alpha^{(1/\mu_1)}$

Typical plots of the power $\beta(\mu)$ for various sizes α are shown on the next page.



β
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
α
μ1 = 20
μ1 = 10
μ1 = 5
μ1 = 3
μ1 = 2
μ1 = 1

3.6 (a) The likelihood ratio in this case is given by

$$L(r) = \frac{f_1(r)}{f_0(r)} = \exp\left\{\frac{-A^2}{2\sigma^2}\right\} I_0\left(\frac{rA}{\sigma}\right) \underset{H_0}{\overset{H_1}{\gtrless}} L_0$$

$I_0(z)$ is the modified Bessel function of order zero, and is a monotonically increasing function of its argument.

Thus it follows that the most powerful test of size α for testing H_1 versus H_0 will be of the form

$$\phi(r) = \begin{cases} 1, & r > r_0 \\ 0, & r \le r_0 \end{cases}.$$

for some threshold $r_0 \in [0, \infty)$.

(b) To find the threshold r_0 yielding a size α test, we solve

$$\alpha = \int_{r_0}^{\infty} \frac{r}{\sigma^2} \exp\left(\frac{-r^2}{2\sigma^2}\right) dr = 1 - \int_0^{r_0} \frac{r}{\sigma^2} \exp\left(\frac{-r^2}{2\sigma^2}\right) dr$$

$$= 1 - \left[-e^{-r^2/2\sigma^2}\right]_0^{r_0} = e^{-r_0^2/2\sigma^2}$$

$$\Rightarrow \frac{-r_0^2}{2\sigma^2} = \ln\alpha \Rightarrow r_0^2 = -2\sigma^2 \ln\alpha$$

$$\Rightarrow r_0 = \sqrt{-2\sigma^2 \ln\alpha}$$

(c) It follows that the power of the test is

$$\beta = \int_{r_0}^{\infty} f_1(r)\, dr = \int_{r_0}^{\infty} \frac{r}{\sigma^2} \exp\left\{-\frac{(r^2+A^2)}{2\sigma^2}\right\} I_0\left(\frac{rA}{\sigma^2}\right) dr$$

$$= \int_{\sqrt{-2\sigma^2\ln\alpha}}^{\infty} \frac{r}{\sigma^2} \exp\left\{-\frac{(r^2+A^2)}{2\sigma^2}\right\} I_0\left(\frac{rA}{\sigma^2}\right) dr \quad \text{---} (*)$$

(d) Let $z = \frac{r}{\sigma} \Rightarrow dr = \sigma dz$. Then we have

$$\beta = \int_{r_0/\sigma}^{\infty} \frac{z}{\sigma} \exp\left\{-\frac{(z^2 + A^2/\sigma^2)}{2}\right\} \cdot I_0\left(\frac{zA}{\sigma}\right) \sigma\, dz$$

$$= \int_{r_0/\sigma}^{\infty} z\, e^{-(z^2 + A^2/\sigma^2)/2} \cdot I_0\left(\frac{zA}{\sigma}\right) dz = Q\left(\frac{A}{\sigma}, \frac{r_0}{\sigma}\right)$$

$$= Q\left(\frac{A}{\sigma}, \frac{\sqrt{-2\sigma^2\ln\alpha}}{\sigma}\right) = Q\left(\frac{A}{\sigma}, \sqrt{-2\ln\alpha}\right)$$