ECE 678 Radar Engr. Homework #1 Solutions Mark R. Bell

1.1 The exact expression for the received Doppler shifted frequency (incorporating relativistic effects) is

$$f_R = f_T \frac{1-\dot{R}/c}{1+\dot{R}/c}. \text{ Thus } f_{D,R} = f_R - f_T = \frac{-2\dot{R}/c}{1+\dot{R}/c} f_T .$$

The common approximation for the Doppler shift in the signal is

$$f_D = -\frac{2\dot{R}}{\lambda} = -2(\dot{R}/c) f_T .$$

Thus the error in the approximation is

$$\Delta f_D = f_D - f_{D,R} = \frac{-2(\dot{R}/c) f_T (1+\dot{R}/c) + 2(\dot{R}/c) f_T}{1+\dot{R}/c}$$

$$= \frac{-2(\dot{R}/c)^2}{1+\dot{R}/c} f_T$$

Thus the relative error is $\varepsilon_R = \dfrac{\Delta f_D}{f_{D,R}} = \dfrac{\left(\frac{-2(\dot{R}/c)^2 f_T}{1+\dot{R}/c}\right)}{\left(\frac{2(\dot{R}/c) f_T}{1+\dot{R}/c}\right)} = \dot{R}/c$

For $\dot{R} = 300\text{m/s}$, $\dot{R}/c = 10^{-6}$

$$\therefore \varepsilon_R = \boxed{10^{-6}} = \boxed{0.0001\ \%}$$

1.2 (a) The total power received by antenna G_2 is

$$P_{TOT,2} = \frac{P_T \lambda^2 G_1 G_2}{(4\pi)^2 R^2} \quad \text{(using the Friis equation written with gains)}$$

Half of this power goes to the receiver attached to G_2 and half is retransmitted by the antenna with gain G_2. Thus the receiver sees

$$P_R = \frac{P_T \lambda^2 G_1 G_2}{2(4\pi)^2 R^2} .$$

Thus the scattered power received by antenna G_1 is

$$P_S = \frac{P_R \lambda^2 G_1 G_2}{(4\pi)^2 R^2} = \frac{P_T \lambda^4 G_1^2 G_2^2}{2(4\pi)^4 R^4} \quad \left(\text{recall the half of the power retransmitted equals } P_R\right)$$

and thus the ratio of P_R/P_S is

$$\frac{P_R}{P_S} = \boxed{\frac{(4\pi)^2}{G_1 G_2}\left(\frac{R}{\lambda}\right)^2}$$

(b) Taking $G_1 = G_2 = 100$ and $\frac{R}{\lambda} = 10^4$, we have

$$\frac{P_R}{P_S} = \frac{(4\pi)^2}{(100)(100)}\left(10^4\right)^2 = \boxed{1.57914 \times 10^6 = 61.98\,\text{dB}}$$

1.3 The police radar receives signal power

$$P_R = \frac{P_T G^2 \lambda^2 \sigma}{(4\pi)^3 R^4} = \frac{P_T (100)^2 (0.02m)^2 \sigma}{(4\pi)^3 (50m)^4} = \frac{P_T \sigma \cdot 10^{-8}}{\pi^3}\left(\frac{1}{W\text{-}m^2}\right)$$

The radar detector receives power

$$P_D = \frac{A_T A_D P_T}{\lambda^2 R_D^2} = \frac{\left(\frac{\lambda^2 G_T}{4\pi}\right) k \sigma P_T}{4\pi \lambda^2 R_D^2} = \frac{0.025 P_T \sigma}{\pi R_D^2}.$$

If the two receivers have identical sensitivities, they must have equal received power for signal detection. Thus $P_D = P_R$. Setting these two expressions equal and solving for R_D, we get

$$\frac{P_T \sigma \cdot 10^{-8}}{\pi} = \frac{0.025 P_T \sigma}{\pi R_D^2} m^2 \Rightarrow R_D^2 = 2.5\pi^2 \times 10^6 m^2$$

$$\Rightarrow R_D = 1581\pi m = 4967 m$$

$$= \boxed{4.967 km \cong 5 km}$$

1.4

$$A_T = \pi r_T^2 = \pi (1m)^2 = \pi \ (m^2)$$

$$A_R = \eta_R A_{RG} = 0.6\pi (32m)^2 = (614.4)\pi \ (m^2)$$

$$= 1930.19 \ (m^2)$$

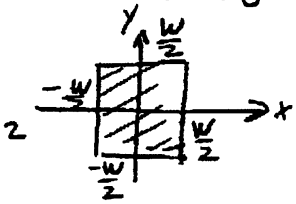
$$P_R = \frac{P_T A_T A_R}{\lambda^2 R^2} = \frac{(50 W)(\pi m^2)(1930.2 m^2)}{(0.1 m)^2 (10^9 m)^2}$$

$$= \boxed{3.032 \times 10^{-11} W}$$

1.5 $G(\theta,\phi) = \dfrac{4\pi A_{TG}}{\lambda^2} f(\theta,\phi)$

where $f(\theta,\phi) = \dfrac{1}{A_{TG}} \cdot \dfrac{\left| \iint\limits_{A_{TG}} \exp\left\{ i \frac{2\pi}{\lambda}\left[x \sin\theta + y \sin\phi\right]\right\} E(x,y)\, dx\, dy \right|^2}{\iint\limits_{A_{TG}} |E(x,y)|^2 dx\, dy}$

For small angles θ and ϕ such that $\sin\theta \sim \theta$ and $\sin\phi \sim \phi$, and a uniform aperture illumination of E_0 over the square aperture with sides of length W as specified we have that



$$f(\theta,\phi) = \frac{1}{W^2} \frac{\left| \int_{-W/2}^{W/2}\int_{-W/2}^{W/2} E_0 \exp\left\{ i \frac{2\pi}{\lambda}\left[x\theta + y\phi\right]\right\} dx\, dy \right|^2}{\int_{-W/2}^{W/2}\int_{-W/2}^{W/2} |E_0|^2 dx\, dy}$$

Clearly $\int_{-\frac{W}{2}}^{\frac{W}{2}}\int_{-\frac{W}{2}}^{\frac{W}{2}} |E_0|^2 dx\, dy = W^2 |E_0|^2$, and

$$\int_{-\frac{W}{2}}^{\frac{W}{2}}\int_{-\frac{W}{2}}^{\frac{W}{2}} E_0 \exp\left\{ i \frac{2\pi}{\lambda}\left[x\theta + y\phi\right]\right\} dx\, dy = E_0 \int_{-\frac{W}{2}}^{\frac{W}{2}} \exp\left\{ i \frac{2\pi}{\lambda} x\theta \right\} dx \int_{-\frac{W}{2}}^{\frac{W}{2}} \exp\left\{ \frac{i2\pi}{\lambda} y\phi \right\} dy$$

$$= E_0 \left[W \frac{\sin(\pi W\theta/\lambda)}{(\pi W\theta/\lambda)} \right] \left[W \frac{\sin(\pi W\phi/\lambda)}{(\pi W\phi/\lambda)} \right]$$

Thus $f(\theta,\phi)$ becomes

$$f(\theta,\phi) = \frac{1}{W^2} \frac{|E_0|^2 W^4 \left[\frac{\sin(\pi W\theta/\lambda)}{(\pi W\theta/\lambda} \right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda} \right]^2}{|E_0|^2 W^2}$$

$$= \left[\frac{\sin(\pi W\theta/\lambda)}{(\pi W\theta/\lambda)} \right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{(\pi W\phi/\lambda)} \right]^2$$

Thus we have

$$G(\theta,\phi) = \frac{4\pi W^2}{\lambda^2} \left[\frac{\sin(\pi W\theta/\lambda)}{(\pi W\theta/\lambda)} \right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{(\pi W\phi/\lambda)} \right]^2$$

1.6. $\mathcal{G}(\theta,\phi) = \dfrac{4\pi A_{TG}}{\lambda^2} f(\theta,\phi)$

where

$$f(\theta,\phi) = \frac{1}{A_{TG}} \cdot \frac{\left| \iint_{A_{TG}} \exp\left\{ i\frac{2\pi}{\lambda}\left[x\sin\theta + y\sin\phi\right]\right\} E(x,y)\,dx\,dy \right|^2}{\iint_{A_{TG}} |E(x,y)|^2\,dx\,dy} \quad --- (*)$$

In this case, $A_{TG} = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}$, and

$$\iint_{A_{TG}} |E(x,y)|^2\,dx\,dy = \frac{\pi D^2}{4}|E_0|^2$$

The numerator is given by

$$\iint_{A_{TG}} \exp\left\{ i\frac{2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\} E_0\,dx\,dy$$

$$= E_0 \iint_{A_{TG}} \exp\left\{ -i2\pi\left[x\left(-\frac{\sin\theta}{\lambda}\right) + y\left(-\frac{\sin\phi}{\lambda}\right)\right]\right\} dx\,dy$$

$$= E_0\, G\left(-\frac{\sin\theta}{\lambda}, -\frac{\sin\phi}{\lambda}\right) \quad --- (**)$$

where

$$G(u,v) = \iint_{\mathbb{R}^2} g(x,y)\exp\left\{-i2\pi\left[ux + yv\right]\right\} dx\,dy$$

is the 2-dimensional Fourier transform of the unit-magnitude disk of radius $a = D/2$ given by

$$g(x,y) = \mathbf{1}_{[0,a]}\left(\sqrt{x^2+y^2}\right) = \begin{cases} 1, & \sqrt{x^2+y^2} \le a \\ 0, & \sqrt{x^2+y^2} > a. \end{cases}$$

So our next task is to compute the two-dimensional Fourier transform $G(u,v)$ of $g(x,y)$.

Skip to "*" at bottom of page 8 for a more straightforward determination of G(u,v) us results from a table of integrals.

Let $g(x,y) = 1_{[0,a]}(\sqrt{x^2+y^2}) = \begin{cases} 1, & \sqrt{x^2+y^2} \le a \\ 0, & \sqrt{x^2+y^2} > a \end{cases}$

We wish to find the two-dimensional Fourier transform

$$G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \exp\{-i2\pi(ux+vy)\}\, dx\, dy$$

We note that if we write $g(x,y)$ in polar coordinates with

$$x = r\cos\phi$$
$$y = r\sin\phi$$

we have

$$g_p(r,\phi) = g(r\cos\phi, r\sin\phi) = \hat{g}(r) = 1_{[0,a]}(r)$$

because $g(x,y)$ is circularly symmetric.

Because $g(x,y)$ is circularly, $G(u,v)$ is also circularly symmetric in u and v. Thus it follows that we can write

$$u = \rho\cos\beta$$
$$v = \rho\sin\beta.$$

We can then write

$$G(u,v) = \mathcal{G}(\rho,\beta)$$

$$= \int_0^{2\pi}\int_0^{\infty} g(r\cos\phi, r\sin\phi)\exp\{-i2\pi(ur\cos\phi + vr\sin\phi)\}\left|\frac{\partial(x,y)}{\partial(r,\phi)}\right| dr\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\infty} r\hat{g}(r)\exp\{-i2\pi(\rho\cos\beta\, r\cos\phi + \rho\sin\beta\, r\sin\phi)\}\, dr\, d\phi$$

$$= \int_0^{\infty} r\hat{g}(r)\int_0^{2\pi} \exp\{-i2\pi r\rho\cos(\phi-\beta)\, d\phi\, dr$$

Now the Bessel function of the first kind, order zero, can be written as

$$J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} \exp\{-ia\cos(\psi-\beta)\}\, d\psi$$

(G&R 8.411.1)

Thus it follows that

$$\hat{\mathcal{G}}(\rho,\beta) = \hat{\mathcal{G}}(\rho) = 2\pi\int_0^{\infty} r\,\hat{g}(r)\, J_0(2\pi\rho r)\, dr$$

This is true for any circularly symmetric $g(x,y) = \hat{g}(\sqrt{x^2+y^2}) = \hat{g}(r)$. For our particular $g(x,y) = \hat{g}(r) = 1_{[0,a]}(r)$ this becomes

$$\mathcal{G}(\rho,\beta) = \hat{\mathcal{G}}(\rho) = 2\pi\int_0^{a} r\, J_0(2\pi\rho r)\, dr.$$

Now using the identity

$$\int_0^{x} z\, J_0(z)\, dz = x\, J_1(x),$$

where $J_1(\cdot)$ is the Bessel function of the first kind, order one, we have

$$\hat{\mathcal{G}}(\rho) = 2\pi\int_0^{a} r\, J_0(2\pi\rho r)\, dr$$

$$= 2\pi\int_0^{2\pi\rho a} \frac{z}{2\pi\rho}\, J_0(z)\, \frac{dz}{2\pi\rho} = \frac{2\pi}{(2\pi\rho)^2}\int_0^{2\pi\rho a} z\, J_0(z)\, dz$$

$$= \frac{2\pi}{(2\pi\rho)^2}\cdot 2\pi\rho a \cdot J_1(2\pi\rho a)$$

$$= \frac{a}{\rho}\cdot J_1(2\pi a\rho)$$

Now noting that $\rho = \sqrt{u^2+v^2}$, we have

$$G(u,v) = \frac{a}{\sqrt{u^2+v^2}}\, J_1\left(2\pi a\sqrt{u^2+v^2}\right)$$

Now going back to equation (**) in our aperture problem, we have

$$E_0\, G\left(-\frac{\sin\theta}{\lambda}, -\frac{\sin\phi}{\lambda}\right)$$

$$= E_0 \frac{a\lambda}{\sqrt{\sin^2\theta + \sin^2\phi}} \cdot J_1\left(\frac{2\pi}{\lambda} a \sqrt{\sin^2\theta + \sin^2\phi}\right)$$

$$= \frac{E_0 \lambda D}{2\sqrt{\sin^2\theta + \sin^2\phi}} \cdot J_1\left(\frac{\pi D}{\lambda}\sqrt{\sin^2\theta + \sin^2\phi}\right)$$

Now substituting into equation (*), we get

$$G(\theta,\phi) = \cancel{\frac{4\pi}{\lambda^2}} \cdot \frac{\dfrac{\cancel{|E_0|^2 \lambda^2 D^2}}{\cancel{4}(\sin^2\theta + \sin^2\phi)} \cdot J_1^2\left(\frac{\pi D}{\lambda}\sqrt{\sin^2\theta + \sin^2\phi}\right)}{\cancel{\dfrac{\pi D^2}{4}} \cancel{|E_0|^2}}$$

$$= \frac{4}{(\sin^2\theta + \sin^2\phi)} \cdot J_1^2\left(\frac{\pi D}{\lambda}\sqrt{\sin^2\theta + \sin^2\phi}\right)$$

$$= \frac{4 \cdot J_1^2\left(\frac{\pi D}{\lambda}\sqrt{\sin^2\theta + \sin^2\phi}\right)}{(\sin^2\theta + \sin^2\phi)}$$

$$= \frac{4\pi^2 D^2}{\lambda^2}\left[\frac{J_1\left(\frac{\pi D}{\lambda}\sqrt{\sin^2\theta + \sin^2\phi}\right)}{\frac{\pi D}{\lambda}\sqrt{\sin^2\theta + \sin^2\phi}}\right]^2$$

$$\therefore \quad G(\theta,\phi) = \frac{4\pi^2 D^2}{\lambda^2}\left[\frac{J_1\left(\frac{\pi D}{\lambda}\sqrt{\sin^2\theta + \sin^2\phi}\right)}{\frac{\pi D}{\lambda}\sqrt{\sin^2\theta + \sin^2\phi}}\right]^2$$

For $\theta, \phi \ll 1$, we can approximate this by

$$G(\theta,\phi) \cong \frac{4\pi D^2}{\lambda^2}\left[\frac{J_1\left(\frac{\pi D}{\lambda}\sqrt{\theta^2 + \phi^2}\right)}{\frac{\pi D}{\lambda}\sqrt{\theta^2 + \phi^2}}\right]^2$$

n.b. $\lim_{z \to 0} \frac{J_1(z)}{z} = \frac{1}{2}$

$$\Rightarrow \quad G(0,0) = \frac{4\pi^2 D^2}{\lambda^2} \cdot \left(\frac{1}{2}\right)^2 = \frac{\pi^2 D^2}{\lambda^2}$$

and the gain of an aperture of area A_{TG} is

$$G = \frac{4\pi A_{TG}}{\lambda^2} = \frac{4\pi}{\lambda^2} \pi \left(\frac{D}{2}\right)^2 = \frac{4\pi^2 D^2}{\lambda^2 4} = \frac{\pi^2 D^2}{\lambda^2}$$

☑ check.

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* An alternative method for computing the 2-dimensional Fourier transform of a function that is constant on a disk of radius a:

$$F(u,v) = \iint_{\{(x,y):\, \sqrt{x^2+y^2} \le a\}} K e^{-i2\pi(ux+vy)}\, dx\, dy$$

$$F(u,v) = K \int_{-a}^{a} e^{-i2\pi v y} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} e^{-i2\pi u x}\, dx\, dy$$

$$= K \int_{-a}^{a} e^{-i2\pi v y} \left[\frac{e^{-i2\pi u x}}{-i2\pi u} \right]_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy$$

$$= K \int_{-a}^{a} e^{-i2\pi v y} \left[\frac{e^{-i2\pi u\sqrt{a^2-y^2}} - e^{i2\pi u\sqrt{a^2-y^2}}}{-i2\pi u} \right] dy$$

$$= \frac{K}{\pi u} \int_{-a}^{a} e^{-i2\pi v y} \sin\left[2\pi u \sqrt{a^2-y^2}\right] dy$$

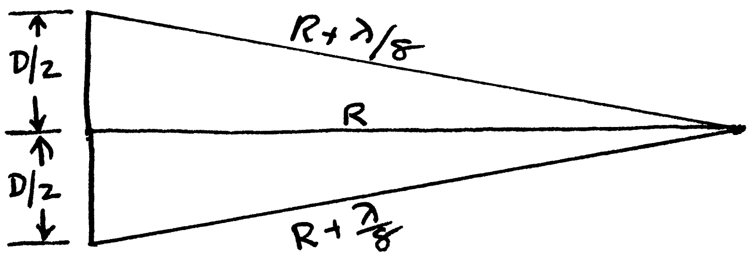
$$= \frac{2K}{\pi u} \int_{0}^{a} \cos(2\pi v y) \cdot \sin\left[2\pi u \sqrt{a^2-y^2}\right] dy$$

using Table of integrals (e.g. G&R, p.399, entry 3.711)

$$\frac{Ka}{\sqrt{u^2+v^2}} \cdot J_1\left[2\pi a \sqrt{u^2+v^2}\right] = Ka \frac{J_1(2\pi a q)}{q}$$

where $q = \sqrt{u^2+v^2}$

1.7 A diagram of the situation under consideration appears as follows:



So if we solve for R is this diagram, we get the range at which the far-field starts.

By the Pythagorian Theorem, we have

$$(R+\lambda/8)^2 = R^2 + \frac{D^2}{4} \Rightarrow R^2 + \frac{R\lambda}{4} + \frac{\lambda^2}{64} = R^2 + \frac{D^2}{4}$$

$$\Rightarrow \frac{R\lambda}{4} + \frac{\lambda^2}{64} = \frac{D^2}{4} \Rightarrow \frac{R\lambda}{4} = \frac{D^2}{4} - \frac{\lambda^2}{64} \cong \frac{D^2}{4}, \text{ since } D^2 \gg \lambda$$

$$\therefore \boxed{R_{FF} \cong \frac{D^2}{\lambda}}$$

(a) $R_{FF} = \frac{(3\,m)^2}{0.075\,m} = \frac{9\,m^2}{0.075\,m} = \boxed{120\,m}$

(b) $R_{FF} = \frac{(100\,m)^2}{0.075\,m} = \boxed{133333\,m \cong 133\,km}$

(c) $R_{FF} = \frac{(30\,m)^2}{0.01\,m} = \frac{900\,m^2}{0.01\,m} = 90\,000\,m = \boxed{90\,km}$

(d) $R_{FF} = \frac{(3\,m)^2}{6000\,\text{Å}} = \frac{9\,m^2}{6\times10^{-9}\,m} = \boxed{1{,}500{,}000\,m = 15{,}000\,km}$

1.8 (a) The field strength on the aperture is $E_0 \exp\{-r^2/a^2\}$. The total power falling on the aperture of diameter D is

$$P_D = \iint_{\{(x,y):\, x^2+y^2 \le D^2/4\}} |E_0|^2 \exp\left(\frac{-2r^2}{a^2}\right) dx\,dy = \int_0^{2\pi}\int_0^{D/2} |E_0|^2 \exp\left\{\frac{-2r^2}{a^2}\right\} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= 2\pi|E_0|^2 \int_0^{D/2} \exp\left\{-\frac{2r^2}{a^2}\right\} |r|\,dr = 2\pi|E_0|^2 \left[\frac{-a^2 e^{-2r^2/a^2}}{4}\right]_0^{D/2}$$

$$= 2\pi|E_0|^2 \cdot \left(\frac{a^2}{4}\right)\left[1 - e^{-D^2/2a^2}\right] = \frac{\pi a^2 |E_0|^2}{2}\left[1 - e^{-D^2/2a^2}\right]$$

The total power emitted from the feed is

$$P_{TOT} = P_D\Big|_{D=\infty} = \frac{\pi a^2 |E_0|^2}{2}$$

Thus the fraction of the power falling on the aperture is

$$\alpha(D) = \frac{P_D}{P_{TOT}} = 1 - e^{-D^2/2a^2}$$

(b) The aperture efficiency for the power falling <u>on the aperture</u> is

$$\eta_T = \frac{|\overline{E}|^2}{\overline{|E^*|^2}} \qquad \overline{E} = \frac{1}{A_{TG}} \iint_{A_{TG}} E(x,y)\,dx\,dy$$

$$A_{TG} = \frac{\pi D^2}{4} \qquad \overline{|E|^2} = \frac{1}{A_{TG}} \iint |E(x,y)|^2\,dx\,dy$$

$$\overline{E} = \frac{4}{\pi D^2} \iint_{A_{TG}} E_0 e^{-r^2/a^2}\,dx\,dy = \frac{4E_0}{\pi D^2} \int_0^{2\pi}\int_0^{D/2} \exp\left(\frac{-r^2}{a^2}\right) |r|\,dr$$

$$= \frac{8E_0}{D^2} \int_0^{D/2} r \cdot \exp\left[\frac{-r^2}{a^2}\right] dr = \frac{8E_0}{D^2} \cdot \left(\frac{a^2}{2}\right)\left[1 - e^{-D^2/4a^2}\right]$$

$$\Rightarrow |\overline{E}|^2 = \frac{16|E_0|^2 a^4}{D^4}\left[1 - e^{-D^2/4a^2}\right]^2 = \frac{16|E_0|^2}{(D/a)^4}\left[1 - e^{-\frac{(D/a)^2}{4}}\right]^2$$

$$\overline{|E|^2} = \frac{1}{A_{TG}}\iint_{A_{TG}} |E(x,y)|^2 dx\,dy = \frac{4|E_0|^2}{\pi D^2}\int_0^{2\pi}\int_0^{D/2} \left|\exp\left(-\frac{r^2}{a^2}\right)\right|^2 \cdot |r|\,dr\,d\theta$$

$$= \frac{8|E_0|^2}{D^2}\cdot\int_0^{D/2} r\cdot\exp\left(\frac{-2r^2}{a^2}\right)dr = \frac{2|E_0|^2}{(D/a)^2}\left[1 - e^{-D^2/2a^2}\right]$$

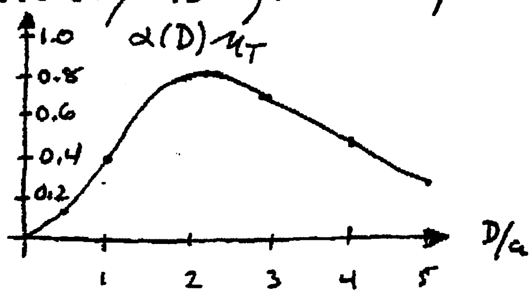
Hence the aperture efficiency of the aperture is

$$\eta_T = \frac{|\overline{E}|^2}{\overline{|E|^2}} = \frac{8}{(D/a)^2}\cdot\frac{\left[1-e^{-\frac{(D/a)^2}{4}}\right]^2}{\left[1-e^{-\frac{(D/a)^2}{2}}\right]}$$

(c) The overall aperture efficiency is given by

$$\alpha(D)\eta_T = \frac{8}{(D/a)^2}\left[1 - e^{-\frac{(D/a)^2}{4}}\right]^2$$

This is the ratio of the effective area of the antenna to the geometric area.



(d) To find the value of D/a that maximizes $\alpha(D)\eta_T$, we need to find the value of the inflection point near 2. Taking the derivative of $\alpha(D)\eta_T$ w.r.t. $u = D/a$ and setting it equal to 0, we get

$$\frac{d}{du}(\alpha(D)\eta_T) = \frac{8e^{-u^2/4}(1-e^{-u^2/4})}{u} - \frac{16(1-e^{-u^2/4})^2}{u^3}$$

$$= \frac{8(1-e^{+u^2/4})(2e^{+u^2/4}-u^2-2)}{e^{+u^2/2}u^3} = 0. \quad \text{Either } (1-e^{+u^2/4}) = 0 \text{ or } (2e^{+u^2/4}-u^2-2) = 0$$

The first gives $u = D/a = 0$ (not interesting). The second is an even ftn of u (via u^2) so it has roots $\pm(2+\varepsilon)$. Solving numerically for the positive (meaningful) root, we get

$$u = \boxed{D/a = 2.24181}$$

The fraction of the energy falling on the disk in this case is

$$\alpha(D)\Big|_{D/a=2.24181} = 1 - \exp\left(-\frac{1}{2}\left(\frac{D}{a}\right)^2\right)\Big|_{D/a=2.24181} = \boxed{0.919}$$

<u>n.b.</u> The total overall aperture efficiency (in order to use Friis Eq.) is

$$\alpha(D)\eta_T\Big|_{D/a=2.24181} = \frac{8}{(D/a)^2}\left[1-e^{-\frac{(D/a)^2}{4}}\right]^2\Bigg|_{D/a=2.24181} = 0.815$$