Session 33

Recall...

Constant False Alarm Rate (CFAR) Detection

Recall...

Assume that we must decide between two simple hypotheses

$$H_0: \quad X \sim \exp(\mu_0),$$

$$H_1: \quad X \sim \exp(\mu_1).$$

(assume $\mu_1 > \mu_0$.)

The resulting test will be a threshold test of the form

$$\phi(X) = \begin{cases} 1, & \text{for } X \geq x_0; \\ 0, & \text{for } X < x_0. \end{cases}$$

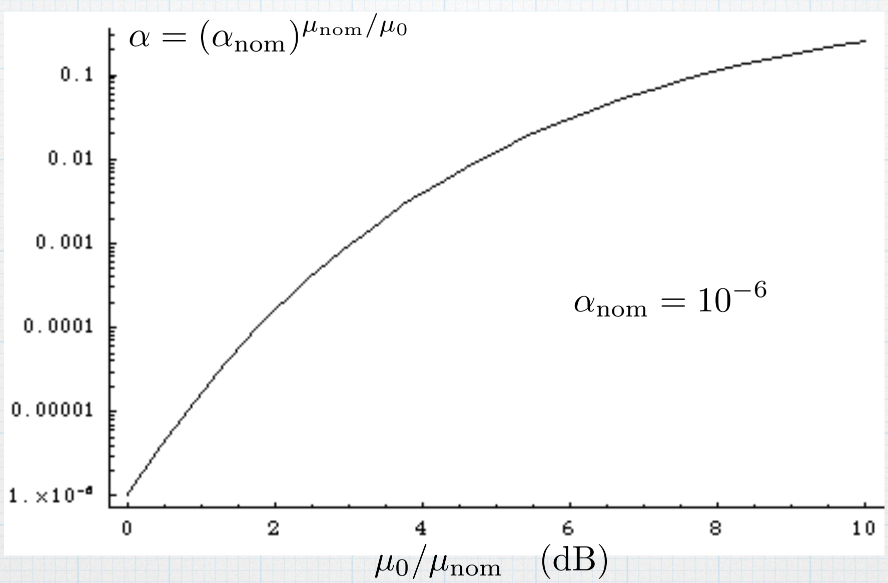
The threshold x_0 that yields a probability of false alarm α is

$$x_0 = -\mu_0 \ln \alpha.$$

If we have an error in μ_0, we will have a significantly different false alarm probability:

$$\alpha = (\alpha_{\text{nom}})^{\mu_{\text{nom}}/\mu_0}.$$

The effects of inaccurate noise estimates



Recall...

The exponential Detection Problem Revisited

Assume that we must decide between two simple hypotheses

$$H_0: \quad Y \sim \exp(\mu_0),$$

$$H_1: \quad Y \sim \exp(\mu_1).$$

$\mu_1 > \mu_0$

Now if we think of

$$\mu_1 = \mu_0 + \mu_s,$$

where

$$\mu_s = \text{signal component of } \mu_1,$$

then if we define the *signal-to-noise ratio* as

$$S = \mu_s/\mu_0,$$

we can rewrite μ_1 as

$$\mu_1 = \mu_0(1+S),$$

and our simple hypotheses can be rewritten as

$$H_0: \quad Y \sim \exp(\mu_0) \qquad \text{versus} \qquad H_1: \quad Y \sim \exp(\mu_0(1+S)).$$

Recall...

The most powerful test of size α is given by

$$\phi(Y) = \begin{cases} 1, & \text{for } Y > Y_0, \\ 0, & \text{for } Y \le Y_0, \end{cases}$$

where the threshold Y_0 is given by

$$Y_0 = -\mu_0 \ln \alpha.$$

The power of the test is given by

$$\beta = P(Y > Y_0 | H_1) = \cdots = \alpha^{1/(1+S)}.$$

n.b. The threshold Y_0 is a function of μ_0 and the probability of false alarm α.

Recall...

If we don't know the value of μ_0, we cannot set the threshold Y_0 that will yield our size α test. How should we proceed?

In principle, μ_0 could take on a broad range of positive values.

We could view H_0 as a the composite hypothesis that $\mu_0 \in (0, \infty)$.

We could then use a generalized likelihood ratio test to solve the problem.

Under hypothesis H_0, this would correspond to finding the maximum likelihood estimate $\hat{\mu}_0$ and using it in place of μ_0. But for one sample measurement, this does not yield a good estimate.

However, if we had N i.i.d. measurements $X_1, \ldots, X_N$ of the noise, we could use the maximum likelihood (and minimum variance unbiased) estimate

$$\hat{\mu}_0 = \frac{1}{N} \sum_{i=1}^{N} X_i$$

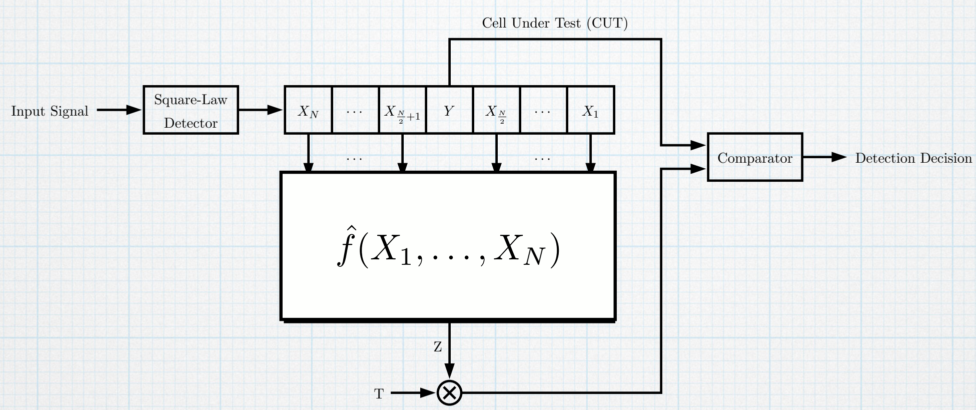
in place of μ_0.

Recall...

- In a "typical" radar scenario, targets are sparsely located against a background of noise and clutter.
- There tends to be regions of local statistical homogeneity in this noise/clutter background because the physical environment giving rise to it often has homogeneous statistics.
- However, there can be significant changes in the local scattering characteristics as you move through the scattering environment.
- There can be sharp boundaries between scattering regions.

- This suggests that one approach to estimating the background noise power for target detection in a particular resolution cell is to average the measured noise power in surrounding resolution cells.
- This is an example of a class of detection techniques called *Constant False Alarm Rate* (CFAR) techniques.
- We will see where the term *Constant False Alarm Rate* comes from, but more important than the constant false alarm rate is a robustness to changes in the average noise power.

A Generic CFAR Processor



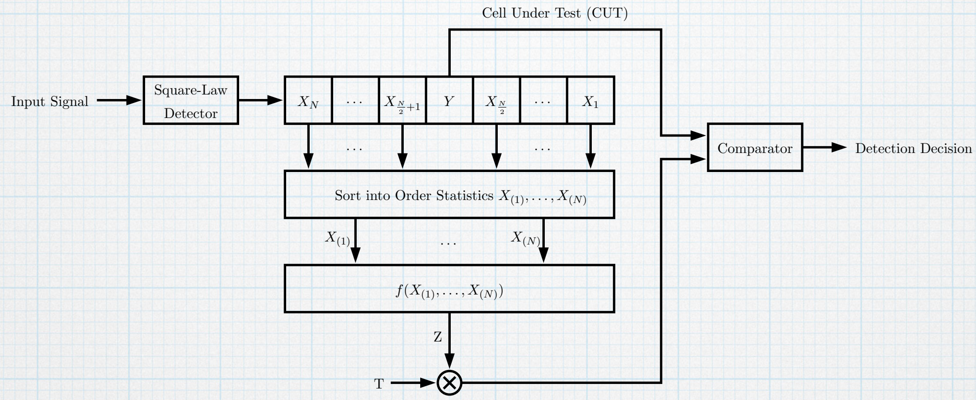
The resolution *cell under test* (CUT) with measurement Y is tested for the presence of the target using a threshold computed using neighboring resolution cell measurements $X_1, \ldots, X_N$.

The statistic $Z = \hat{f}(X_1, \ldots, X_N)$ is an estimate of the noise power.

The threshold scaling factor T sets the threshold level by scaling the statistic the statistic Z. This works because the threshold is the product of a constant and the average noise power.

33.10

A More Specific Class of CFAR Processors

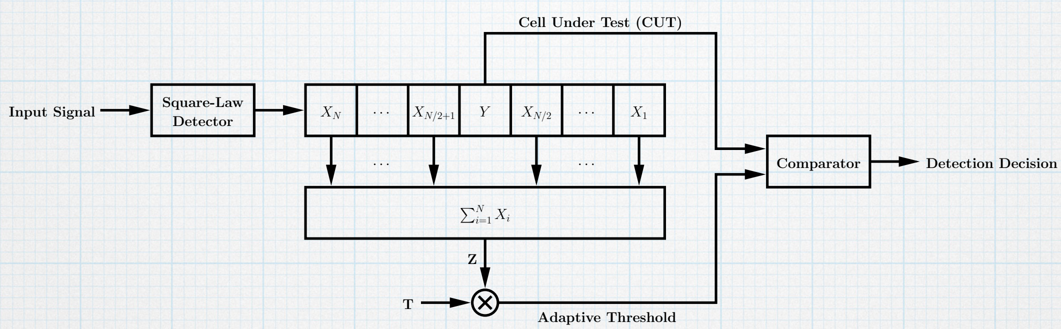


This processor can compute

- Mean
- Median
- Arbitrary Order Statistics
- Linear Combination of Order Statistics.

33.11

Cell-Averaging CFAR (CA-CFAR)



In CA-CFAR, we have that the statistic Z/N is just the sample mean.

It can be shown that Z/N is the maximum-likelihood estimate of μ_0. (It is also the *minimum variance unbiased estimate* (MVUE) of μ_0 and an *efficient estimate*—satisfying the Cramer-Rao lower bound.)

If we assume that $X_1, \ldots, X_N$ are i.i.d exponential with mean μ (drop subscript for simplicity) we have

$$f_{X_i}(x) = \frac{1}{\mu} e^{-x/\mu} \cdot 1_{[0,\infty)}(x).$$

The moment generating function of each X_i is

$$\Phi_{X_i}(s) = \left(\frac{1}{1-\mu s}\right).$$

$\Phi_{X_i}(s) = E\left[e^{sX}\right]$, where $s \in \mathbb{R}$ (or $s \in \mathbb{C}$). Closely related to char. ftn $\phi_X(\omega) = E[e^{i\omega X}]$, $\omega \in \mathbb{R}$.

The moment generating function of Z is

$$\Phi_Z(s) = \left(\frac{1}{1-\mu s}\right)^N.$$

Because the test is a threshold test comparing the CUT Y to the threshold TZ, the probability of false alarm is

$$\begin{aligned}
\alpha &= \mathrm{E}_Z\left[P[Y > TZ|H_0]\right] \\
&= \mathrm{E}_Z\left[\int_{TZ}^{\infty} \frac{1}{\mu} e^{-y/\mu} dy\right] \\
&= \mathrm{E}_Z\left[\exp(-TZ/\mu)\right] \\
&= \int_{-infty}^{\infty} e^{-\frac{TZ}{\mu}} f_Z(z) dz \\
&= \Phi_Z\left(-\frac{T}{\mu}\right),
\end{aligned}$$

$\Phi_Z(s) = E[e^{sZ}]$ = moment generating function of Z

where $\mathrm{E}_Z[\cdot]$ denotes expectation w.r.t. Z.

Substituting this into the expression for $\Phi_Z(s)$, the false alarm probability is

$$\alpha = (1+T)^{-N}.$$

Note that the false alarm rate is not a function of the mean noise power μ. Hence the term *constant false-alarm rate*.

The threshold scaling factor yielding a size α test is

$$T = (\alpha)^{-1/N} - 1.$$

Similarly, the detection probability can be calculated under the alternative hypothesis H_1 and given by

$$\begin{aligned}
\beta &= \mathrm{E}_Z\left[P[Y > TZ|H_1]\right] \\
&= \mathrm{E}_Z\left[\int_{TZ}^{\infty} \frac{1}{\mu(1+S)} e^{-y/\mu(1+S)} dy\right] \\
&= \mathrm{E}_Z\left[\exp(-TZ/\mu)\right] \\
&= \Phi_Z\left(-\frac{T}{\mu(1+S)}\right) \\
&= \left[1 + \frac{T}{(1+S)}\right]^{-N} \\
&= \left(\frac{1+S}{1+T+S}\right)^{N}.
\end{aligned}$$

Combining these results, we find that

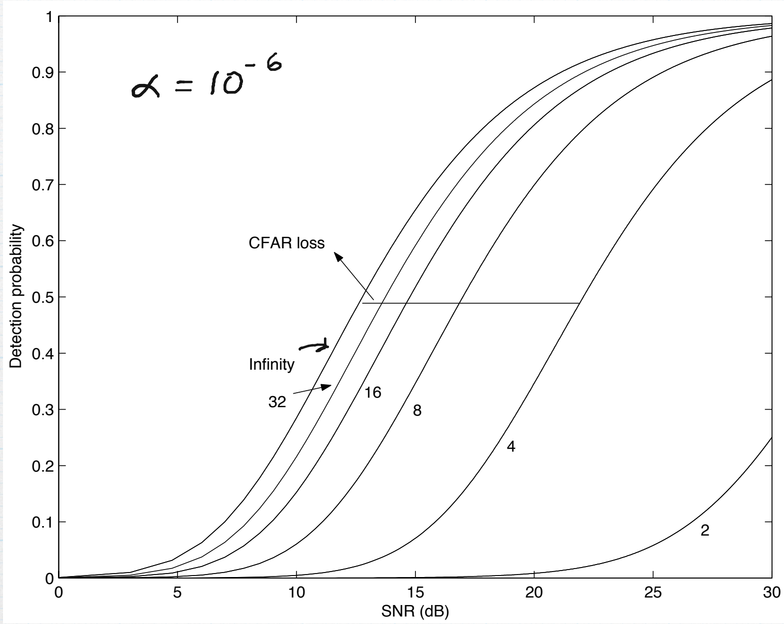
$$\beta = \left(\frac{1+S}{\alpha^{-1/N}+S}\right)^{N}.$$

In the limit, as $N \to \infty$, we have

$$\begin{aligned}
\lim_{N\to\infty} \alpha &= \lim_{N\to\infty} (1+\epsilon/N)^{-N} \\
&= \exp\{-\epsilon\} \\
\lim_{N\to\infty} \beta &= \lim_{N\to\infty} (1+\epsilon/N(1+S))^{-N} \\
&= \exp\{-\epsilon/(1+S)\} \\
\Rightarrow \quad \beta &\to \alpha^{1/(1+S)}, \text{ as } N \to \infty
\end{aligned}$$

33.16

CA-CFAR Detection Performance



33.17

CA-CFAR P_d versus N and SNR (dB) for a desired $P_{fa} = 1 \times 10^{-6}$

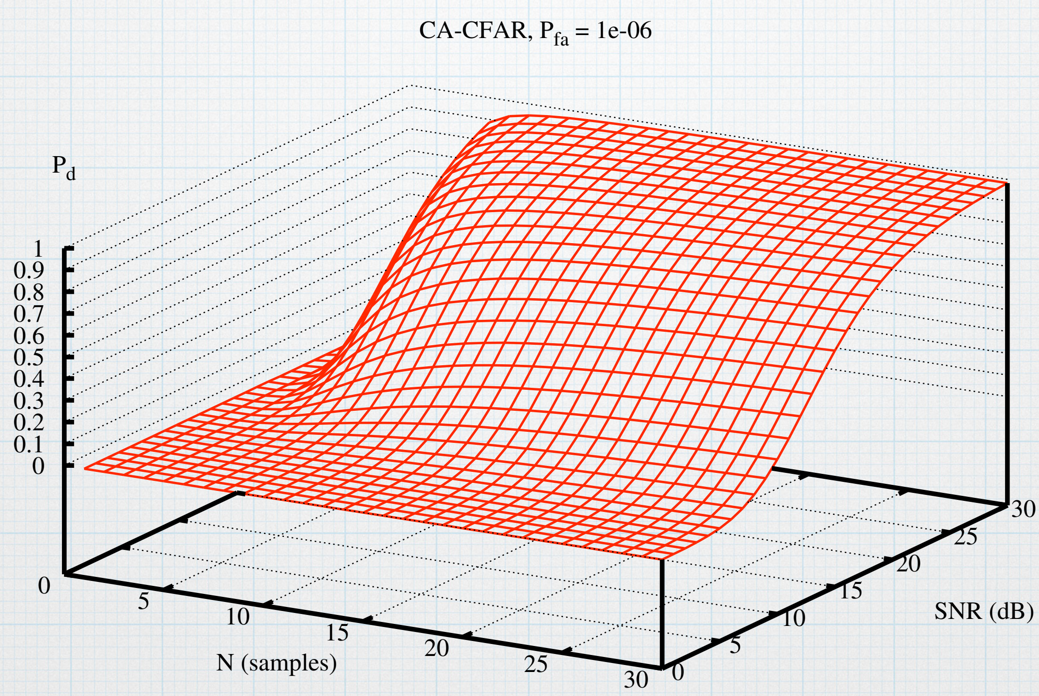


Figure from: Michael F. Rimbert, *Constant False Alarm Rate Detection Techniques Based on Empirical Distribution Function Statistics*, Ph.D Thesis, School of Electrical and Computer Engineering, Purdue University, August 2005.