Session 28

Recall...

Stationary Random Processes

28.1

"Stationary" $\triangleq$ staying in one place throughout time.

What if the n-th order distributions don't change with a shift in the time origin?

" Probabilistic behavior is the same yesterday, today and tomorrow."

Probabilistic description "stays in one place" throughout time.

Recall ...

28.2

Defn: A random process $X(t)$ is called <u>n-th order stationary</u> if all n-th order cdfs or pdfs are invariant to time shifts in the origin:

$$f_{X(t_1)\cdots X(t_n)}(x_1,\ldots,x_n) = f_{X(t_1+c)\cdots X(t_n+c)}(x_1,\ldots,x_n)$$

for all $c \in \mathbb{R}$ and for any n time instants $t_1,\ldots,t_n$.

Defn: A random process $X(t)$ is called <u>stationary</u>, or <u>strict sense stationary</u> (SSS), if $X(t)$ is n-th order stationary for all $n = 1, 2, 3, \ldots$.

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28.3

Defn: A random process $X(t)$ is called <u>wide-sense stationary</u> (WSS) if

(i) $E[X(t)] = \overline{X} = \text{constant}$

and

(ii) $E[X(t_1)\cdot X(t_2)] = \underbrace{R(t_1 - t_2)}$

i.e., it depends on t_1 and t_2 only through the time difference $t_1 - t_2$.

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28.4

Note:

1. If a random process $X(t)$ is first and second order $(n=1,2)$ stationary, and if $E[X(t_1)]$ and $E[X(t_1)X(t_2)]$ exist, then $X(t)$ is wide-sense stationary. The converse is <u>not</u> true.

2. If a R.P. $X(t)$ is stationary (S.S.S), it is also wide-sense stationary (W.S.S.) if $E[X(t_1)]$ and $E[X(t_1)X(t_2)]$ exist.

3. If $X(t)$ is a W.S.S R.P., $E[X(t_1)X(t_2)]$, $\mathrm{cov}(X(t_1), X(t_2))$, and the correlation coefficient $r_{X(t_1)X(t_2)}$ will depend only on the time difference $t_1 - t_2$.

28.5

Example: Consider a R.P.

$$X(t) = \cos(w_0 t + \Theta)$$

where Θ is a R.V. uniformly distributed on $[0, 2\pi)$:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi \\ 0, & \text{elsewhere} \end{cases}, \quad w_0 = \text{constant}$$

(radian frequency, not outcome)

Is $X(t)$ W.S.S ? Let's check the two defining conditions

28.6

(i) $E[X(t)] = E[\cos(\underbrace{w_0}t + \underbrace{\Theta})]$

$= E[\cos(w_0 t)\cos(\Theta) - \sin(w_0 t)\sin(\Theta)]$

$= \int_{-\infty}^{\infty} (\cos(w_0 t)\cos\theta - \sin(w_0 t)\sin\theta) f_{\Theta}(\theta)\, d\theta$

$= \frac{\cos w_0 t}{2\pi} \int_0^{2\pi} \cancelto{0}{\cos\theta\, d\theta} - \frac{\sin w_0 t}{2\pi} \int_0^{2\pi} \cancelto{0}{\sin\theta\, d\theta}$

$= 0 \quad \therefore \quad E[X(t)] = 0 = \text{constant}$ ☑

n.b. Even easier:

$E[X(t)] = E[\cos(w_0 t + \Theta)] = \frac{1}{2\pi}\underbrace{\int_0^{2\pi} \cos(\underbrace{w_0} t + \theta)\, d\theta}_{\text{integral over one period} = 0}$

28.7

$\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

(ii): $E[X(t_1) X(t_2)]$

$= E[\cos(w_0 t_1 + \Theta) \cdot \cos(w_0 t_2 + \Theta)]$

$= E[\frac{1}{2}[\cos(w_0(t_1+t_2) + 2\Theta) + \cos(w_0(t_1 - t_2))]]$

$= \frac{1}{2}[\cancelto{0}{E[\cos(w_0(t_1+t_2) + 2\Theta)]} + E[\cos(w_0(t_1-t_2))]]$

$= \frac{1}{2}\cos(w_0(t_1 - t_2))$ ☑

$\therefore$

(i) $E[X(t_1)] = 0 = \text{constant}$

(ii) $E[X(t_1)X(t_2)] = \underbrace{\frac{1}{2}\cos(w_0(t_1-t_2))}_{\text{A function of } t_1 \text{ and } t_2 \text{ only through } t_1 - t_2.}$

$\therefore X(t)$ is a W.S.S R.P.

Another Example:

Let $Y(t) = \cos(w_0 t + \phi)$
where $w_0 = \text{constant}$
and ϕ is a uniform R.V. on $[0, \pi)$.

Is $Y(t)$ W.S.S. ?

(i) $E[Y(t)] = E[\cos(w_0 t + \phi)]$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(w_0 t + \phi)\, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} [\cos w_0 t \cos\phi - \sin w_0 t \sin\phi]\, d\phi$$

28.10

$$= \frac{\cos w_o t}{\pi} \int_0^{\pi} \cancel{\cos \phi \, d\phi}^{\,0} + \frac{\sin w_o t}{\pi} \int_0^{\pi} \sin \phi \, d\phi$$

$$= \frac{\sin w_o t}{\pi} \Big[-\cos \phi \Big]_0^{\pi} = \frac{2}{\pi} \sin w_o t$$

$\neq$ constant

$\therefore E[Y(t)] \neq$ constant

$\Rightarrow Y(t)$ is not W.S.S.

28.11

<u>Defn</u>: The <u>mean</u> of a R.P. $X(t)$ is

$$\mu_X(t) \triangleq E[X(t)]$$

<u>Defn</u>: The <u>autocorrelation function</u> of a R.P. $X(t)$ is

$$R_{XX}(t_1, t_2) \triangleq E[X(t_1) X(t_2)]$$

n.b. The autocorrelation function is just a measure of the correlation in $X(t)$ at time t_1 and time t_2.

28.12

n.b. $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

$= E[X(t_2)X(t_1)] = R_{XX}(t_2,t_1)$

for a real random process.

n.b The autocorrelation function is symmetric in its time arguments.

For a complex random process $X(t)$

$$R_{XX}(t_1,t_2) = E[X(t_1)X^*(t_2)] = (E[X(t_2)X(t_1)^*])^*$$
$$= R_{XX}^*(t_2,t_1)$$

28.13

Defn: The autocovariance function of a real random process $X(t)$ is defined as

$$C_{XX}(t_1,t_2) \triangleq E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))]$$

$$\underbrace{\left(= \cdots = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2) \right)}_{\text{can easily show this}}$$

Defn: A random process $X(t)$ is called a **Gaussian random process** if the RVs

$X(t_1), X(t_2), \ldots, X(t_n)$

are jointly Gaussian for any $n \in \mathbb{N}$ and any set of sample times $t_1, \ldots, t_n$.

Fact: The n-th order characteristic function of a Gaussian R.P. $X(t)$ is

$$\Phi_{X(t_1),\ldots,X(t_n)}(\omega_1,\ldots,\omega_n) = \exp\left\{ i \sum_{j=1}^{n} \mu_X(t_j)\,\omega_j \right\} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n}\sum_{k=1}^{n} C_{XX}(t_j,t_k)\,\omega_j\omega_k \right\}$$

This is a complete probabilistic description.

Fact: A Gaussian random process $X(t)$ is completely characterized by

(i) $\mu_X(t) = E[X(t)]$

and

(ii) $C_{XX}(t_1,t_2) = E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))]$

Note: If a R.P $X(t)$ is W.S.S., then

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$$

$$= f(t_1-t_2)$$

$$= \text{"} R_X(t_1-t_2) \text{"}$$

This is also sometimes written as

$$E[X(t+\tau)X(t)] = R_X(\tau)$$

for a W.S.S. random process.

Defn: If $X(t)$ is a W.S.S. random process with autocorrelation function $R_X(\tau)$, then the

<u>Power Spectral Density</u>

of $X(t)$ is defined as

$$S_{XX}(\omega) \triangleq \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} \, d\tau$$

Here ω is radian frequency (not the outcome of experiment.)

$S_{XX}(\omega)$ is a measure of the average distribution of signal power in frequency for the R.P. $X(t)$.

28.18

Systems with Stochastic Inputs

$$X(t) \longrightarrow \boxed{T} \longrightarrow Y(t)$$

$X(t,\omega)$ $\qquad\qquad$ $Y(t,\omega)$

Given a R.P. $X(t)$, if we assign to each sample function $X(t,\omega)$ a new sample function $Y(t,\omega)$, we have a new random process

$$Y(t) = T[X(t)]$$

whose sample functions are

$$Y(t,\omega) = T[X(t,\omega)].$$

28.19

n.b. We will assume that $T[\cdot]$ is deterministic. (not random)

Think of

$X(t)$ = input to a system

$Y(t)$ = output of the system

$$X(t) \longrightarrow \boxed{T} \longrightarrow Y(t)$$

$$X(t,\omega) \longrightarrow \boxed{T} \longrightarrow Y(t,\omega)$$

$$Y(t,\omega) = T[X(t,\omega)], \ \forall \omega \in \mathcal{S}.$$

28.20

- We are interested in finding a statistical description of the output $Y(t)$ in terms of the statistical description of the input $X(t)$ and the system description $T[\,]$

- For general T this is very difficult
- We will look at two special cases:
  1. Linear Time Invariant (L.T.I.) Systems
  2. Memoryless Systems.

28.21

Linear Systems

A linear system $\overset{T}{\downarrow}$ $L[\cdot]$ is a transformation rule satisfying the following two properties:

1. $L[X_1(t) + X_2(t)] = L[X_1(t)] + L[X_2(t)]$

   (Superposition)

2. $L[A \cdot X(t)] = A \cdot L[X(t)]$

   (Homogeneity)

n.b. A can be an R.V. or a constant

Defn: A (linear) system is **time-invariant** if, given response $y(t)$ to input $x(t)$ it has response $y(t+c)$ for input $x(t+c)$, for all $c \in \mathbb{R}$.

A linear time invariant system is characterized by its impulse response $h(t)$:

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$

$$\delta(t-t_0) \qquad\qquad\qquad h(t-t_0)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t).$$