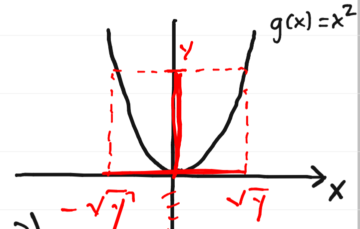
Session 14

Recall...

Ex. B $\quad Y = g(X) = X^2 \Rightarrow g(x) = x^2$ 14.1

Note immediately $F_Y(y) = 0,\ y < 0.$



For $y > 0$:

$$F_Y(y) = P(\{Y \le y\}) = P(\{X^2 \le y\})$$
$$= P(\{-\sqrt{y} \le X \le \sqrt{y}\})$$
$$= P(\{-\sqrt{y} < X \le \sqrt{y}\})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\therefore \quad F_Y(y) = [F_X(\sqrt{y}) - F_X(-\sqrt{y})] \cdot 1_{(0,\infty)}(y)$$

14.2

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\left(\frac{-1}{2\sqrt{y}}\right)$$

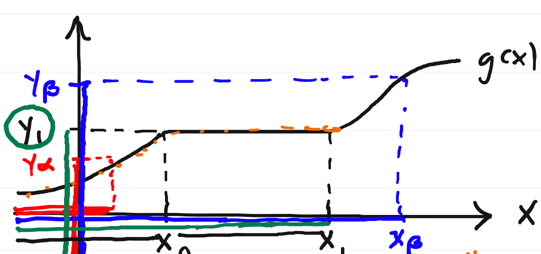
$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right], \quad y > 0$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] \cdot \mathbb{1}_{(0,\infty)}(y)$$

14.3

Ex. 2 Suppose that $g(x)$ is constant across an interval $[x_0, x_1]$

$$F_Y(y) = P(\{Y \le y\})$$



(i) $y < y_1$:

$$F_Y(y_\alpha) = P(\{Y \le y_\alpha\})$$

$$= P(\{X \le x_\alpha\}) = P(\{X \le \text{"}g^{-1}(y_\alpha)\text{"}\}) = F_X(\text{"}g^{-1}(y_\alpha)\text{"})$$

(ii) $y > y_1$: $F_Y(y_\beta) = P(\{Y \le y_\beta\}) = P(\{X \le x_\beta\})$

$$= P(\{X \le \text{"}g^{-1}(y_\beta)\text{"}\}) = F_X(\text{"}g^{-1}(y_\beta)\text{"})$$

14.4

what about $F_Y(y_1)$?

$$F_Y(y_1) = P(\{Y \le y_1\}) = \underline{P(\{X \le x_1\})}$$

$$= P(\{X \le x_0\} \cup \{x_0 < X \le x_1\})$$

disjoint

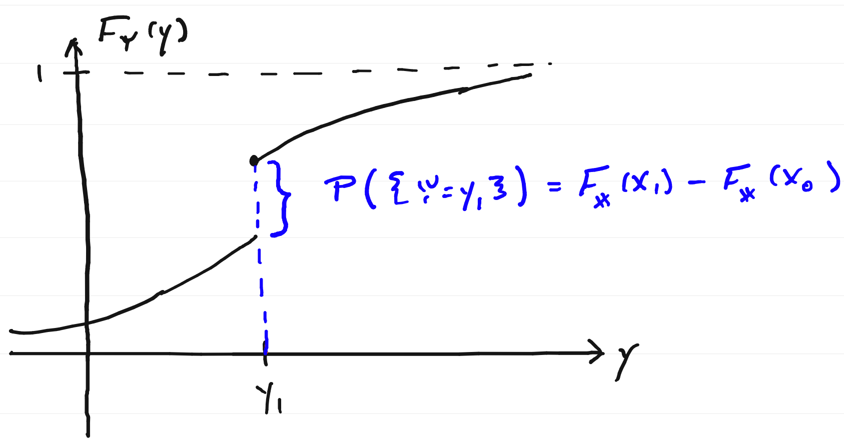
$$= P(\{X \le x_0\}) + \underline{P(\{x_0 < X \le x_1\})}$$

$$= F_X(x_0) + \underbrace{\left[F_X(x_1) - F_X(x_0)\right]}_{\geq 0}$$

$$= F_X(x_1)$$

14.5

$$\therefore \; F_Y(y) = \begin{cases} F_X(g^{-1}(y)) & , \; y < y_1 \\ F_X(x_1) & , \; y = y_1 \\ F_X(g^{-1}(y)) & , \; y > y_1 \end{cases}$$

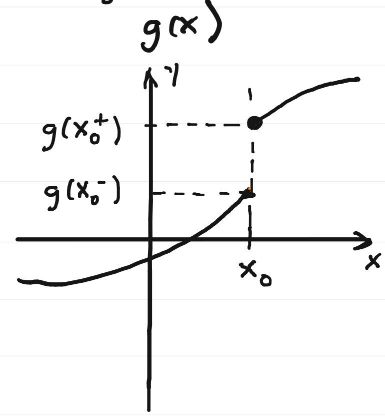


Ex. 3 Assume that $g(x)$ has a jump discontinuity at x_0

$$g(x_0^+) \neq g(x_0^-)$$

Assume $g(x) < g(x_0^-),\ x < x_0$

$g(x) > g(x_0^+),\ x > x_0$



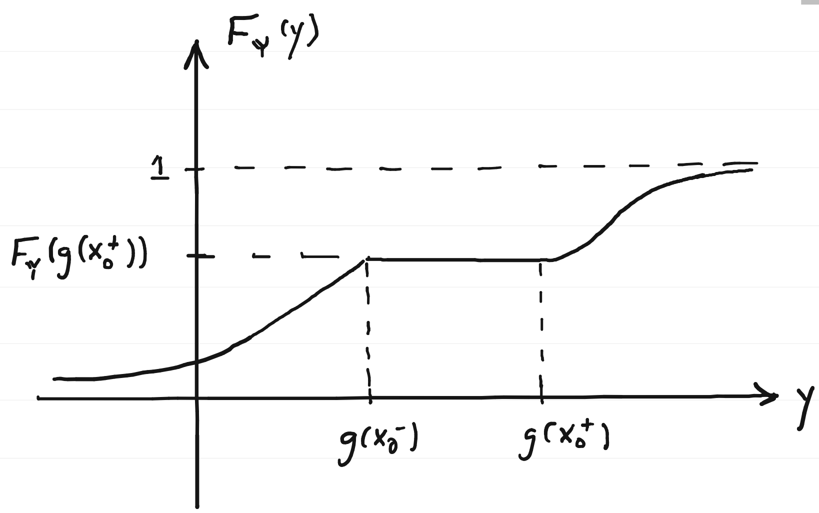
If y: $g(x_0^-) \leq y \leq g(x_0^+)$:

$$F_Y(y) = P(\{Y \leq y\}) = P(\{X \leq x_0^+\}) = P(\{X \leq x_0^-\})$$

$$= F_X(x_0^+) = F_X(x_0^-)$$

if X is absolutely continuous.

$\therefore$ $F_Y(y)$ appears as follows:



The Direct pdf Method

14.8

Suppose $Y = g(X)$, where $g: \mathbb{R} \to \mathbb{R}$ such that the inverse $g^{-1}(\cdot)$ exists,

(i.e., is $y = g(x) \Rightarrow x = g^{-1}(y)$ is unique.),

and assume that $x(y) = g^{-1}(y)$

$$\frac{dx}{dy} = \frac{d g^{-1}(y)}{dy} \quad \text{exists}$$

Then

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d g^{-1}(y)}{dy} \right|$$

$$= f_X(x(y)) \cdot \left| \frac{dx(y)}{dy} \right|, \quad \text{where } x(y) = g^{-1}(y).$$

Example Let $X \sim U[0,1]$ and let

14.9

$$Y = g(X) = \sqrt{X}.$$

Find $f_Y(y)$. $\qquad f_X(x) = \mathbb{1}_{[0,1]}(x)$

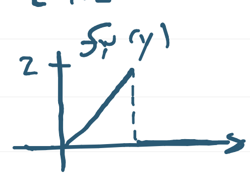
Solution: $y = g(x) = \sqrt{x} \Rightarrow x = y^2$

$x(y) = y^2$

$$\frac{dx(y)}{dy} = 2y$$

$$f_Y(y) = f_X(x(y)) \cdot |2y| = \mathbb{1}_{[0,1]}(y^2) \cdot 2y$$

$$= 2y \cdot \mathbb{1}_{[0,1]}(y)$$



12.19

14.10

Example: Let X be a Gaussian RV with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Let $Y = aX + b$. Find $f_Y(y)$.

Solution: $Y = aX + b$

$$f_Y(y) = f_X(x(y)) \cdot \left| \frac{dx(y)}{dy} \right|, \quad y = ax+b, \quad x = \frac{y-b}{a}$$

$$x(y) = \frac{y-b}{a}$$

$$\frac{dx(y)}{dy} = \frac{1}{a}$$

$$\therefore \quad f_Y(y) = f_X(x(y)) \left| \frac{dx(y)}{dy} \right| = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-b)^2}{2a^2}\right) \cdot \frac{1}{|a|}$$

$$= \frac{1}{\sqrt{2\pi}\,|a|} \exp\left\{ \frac{-(y-b)^2}{2a^2} \right\}$$

Mean, Variance and Expectation

14.11

Defn: The mean or expected value of a RV X with pdf $f_X(x)$ is

$$E[X] \triangleq \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

What about discrete RVs?

n.b. The definition above also applies to discrete RVs if we write their pdf using δ-functions:

If $P(\{X = x_k\}) = P_X(x_k) = p_k$ over the discrete index set, then

$$f_X(x) = \sum_k P_X(x_k)\,\delta(x - x_k) = \sum_k p_k\,\delta(x - x_k)$$

and

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x \left(\sum_k p_k\,\delta(x - x_k)\right) dx$$

$$= \sum_k p_k \underbrace{\int_{-\infty}^{\infty} x\,\delta(x - x_k)\,dx}_{x_k} = \sum_k p_k x_k$$

$$= \sum_k P_X(x_k) \cdot x_k$$

$\therefore$ For a discrete RV X, we have

$$\boxed{E[X] = \sum_k x_k P_X(x_k).}$$

If you know about Riemann-Stiltjes integrals, you can write

$$E[X] = \int_{-\infty}^{\infty} x\,dF_X(x)$$

$$= \begin{cases} \sum_k x_k P_X(x_k) & (\text{discrete RV } X) \\ \int_{-\infty}^{\infty} x f_X(x)\,dx & (\text{continuous RV } X) \end{cases}$$

Defn: Let X be a RV on $(\mathcal{S}, \mathcal{F}, P)$ and let $M \in \mathcal{F}$. Then the conditional mean of X conditioned on M is

$$E[X|M] \triangleq \int_{-\infty}^{\infty} x\, f_X(x|M)\, dx.$$

(n.b. If X is discrete, we have the conditional pmf $p_X(x_k|M) = P(\{X = x_k\}|M)$, and then

$$E[X|M] = \int_{-\infty}^{\infty} x\, f_X(x|M)\, dx = \int_{-\infty}^{\infty} x \Big(\sum_k p_X(x_k|M)\, \delta(x - x_k) \Big) dx$$

$$= \sum_k p_X(x_k|M) \cdot \underbrace{\int_{-\infty}^{\infty} x\, \delta(x - x_k)\, dx}_{x_k} = \sum_k x_k\, p_X(x_k|M) \Big)$$

Recall...

Defn: The mean or expected value of a RV X with pdf $f_X(x)$ is

$$E[X] \triangleq \int_{-\infty}^{\infty} x\, f_X(x)\, dx.$$

Defn: Let X be a RV on $(\mathcal{S}, \mathcal{F}, P)$ and let $M \in \mathcal{F}$. Then the conditional mean of X conditioned on M is

$$E[X|M] \triangleq \int_{-\infty}^{\infty} x\, f_X(x|M)\, dx.$$

Example: Let X be an exponentially distributed RV with pdf

$$f_X(x) = \frac{1}{\mu} \exp\left\{-\frac{x}{\mu}\right\} \mathbb{1}_{[0,\infty)}(x), \quad \mu > 0$$

What is $E[X]$?

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{0}^{\infty} x \cdot \frac{1}{\mu} e^{-x/\mu}\, dx$$

$$\underset{\text{(exercise)}}{\overset{\text{int. by parts}}{=\cdots=}} \left[-xe^{-x/\mu} - \mu e^{-x/\mu}\right]\Big|_0^\infty = \boxed{\mu}$$

Now let's consider the conditional mean $E[X \mid M]$, where $M = \{X > \mu\}$

$$E[X \mid \{X > \mu\}] = \int_{-\infty}^{\infty} x\, f_X(x \mid \{X > \mu\})\, dx$$

It is straightforward to show that

$$f_X(x \mid \{X > \mu\}) = \frac{\frac{1}{\mu} \exp\left\{\frac{-x}{\mu}\right\} \cdot \mathbb{1}_{(\mu,\infty)}(x)}{e^{-\mu/\mu}} \quad \text{(exercise)}$$

$$= \frac{1}{\mu} \exp\left\{-\frac{(x-\mu)}{\mu}\right\} \mathbb{1}_{(\mu,\infty)}(x)$$

$$\therefore \quad E[X \mid \{X > \mu\}] = \int_{-\infty}^{\infty} x\, f_X(x \mid \{X > \mu\})\, dx$$

$$= \int_{\mu}^{\infty} x \cdot \frac{1}{\mu} \exp\left\{-\frac{(x-\mu)}{\mu}\right\} dx$$

$$\left.\begin{array}{l} \text{let } r = x - \mu \\ \Rightarrow x = r + \mu \\ \Rightarrow dx = dr \end{array}\right\} \Rightarrow \quad = \int_{0}^{\infty} (r+\mu) \frac{1}{\mu} \exp\left\{\frac{-r}{\mu}\right\} dr = \cdots$$

14.18

$$= \underbrace{\int_0^\infty \frac{r}{\mu} \exp\left(\frac{-r}{\mu}\right) dr}_{\mu} + \int_0^\infty \frac{\mu}{\mu} \exp\left(\frac{-r}{\mu}\right) dr$$

$$= 2\mu$$

n.b. $E[X] \neq E[X \mid \{X > \mu\}]$

More generally

$$E[X] \neq E[X / M], \quad M \in \mathcal{F}.$$

14.19

Suppose we have a RV X defined on $(\mathcal{S}, \mathcal{F}, P)$ and having pdf $f_X(x)$

Now suppose I have a new RV

$$Y = g(X)$$

where $g: \mathbb{R} \to \mathbb{R}$. What is $E[Y]$?

It appears we must first find $f_Y(y)$ and then compute

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy.$$

While this will work, there is an <u>easier</u> way!

Fact: Let X be a RV and define the new RV $Y = g(X)$.

Then

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

n.b. $\int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$

Proof: Outlined in Papoulis.