Session 8

8.1

Radar Return From Random Scatterers

Consider a radar illuminating a large number of random scatterers.

The received electric field due to the j-th scatterer is

$$E_j = A_j e^{i\phi_j}$$

where

A_j = the magnitude of the field from the j-th scatterer

ϕ_j = phase of the field from the j-th scatterer

Assume the collection of scatterers have the following properties:

1. The size of any individual scatterer is independent of all others.
2. The locations of the scatterers are independent of each other.
3. Scatterer size is independent of position.

n.b. This would be satisfied by a nonhomogeneous marked Poisson Process.

For this situation, we can model $\{A_1, A_2, \ldots, A_n\}$ and $\{\phi_1, \phi_2, \ldots, \phi_n\}$ as jointly distributed random variables with the following properties:

1. $A_1, \ldots, A_n$ are independent.
2. $\phi_1, \phi_2, \ldots, \phi_n$ are independent.
3. $A_1, \ldots, A_n, \phi_1, \ldots, \phi_n$ are independent.

Note that properties 1 and 2 follow from property 3. We list separately for physical intuition.

We further assume:

4. $A_1, \ldots, A_N$ are *independent, identically distributed* (i.i.d.) random variables with mean μ_A and variance σ_A^2.

5. $\phi_1, \ldots, \phi_n$ are i.i.d. random variables uniformly distributed on the interval $[0, 2\pi)$.

These assumptions are reasonable if:

(a) We have no reason to believe any particular scatterer should be larger than any other.

(b) Scatterers are randomly distributed over a region many wavelengths long.

The total electric field at the receive antenna is the superposition of the individual E_j:

$$E = \sum_{j=1}^{n} E_j = \sum_{j=1}^{n} A_j e^{i\phi_j} = V e^{i\theta} = X + iY,$$

where

$$X = V \cos\theta = \sum_{j=1}^{n} A_j \cos\phi_j = \sum_{j=1}^{n} X_j$$

$$Y = V \sin\theta = \sum_{j=1}^{n} A_j \sin\phi_j = \sum_{j=1}^{n} Y_j,$$

and

$$X_j = A_j \cos\phi_j$$

$$Y_j = A_j \sin\phi_j$$

Note that

$$E[X_j] = E[A_j \cdot \cos\phi_j] = E[A_j] \cdot E[\cos\phi_j] = E[A_j] \cdot 0 = 0$$

$$E[Y_j] = E[A_j \cdot \sin\phi_j] = E[A_j] \cdot E[\sin\phi_j] = E[A_j] \cdot 0 = 0$$

and

$$\mathrm{var}[X_j] = E[X_j^2] = E[A_j^2] \cdot E[\cos^2\phi_j] = (\mu_A^2 + \sigma_A^2) \cdot \frac{1}{2}$$

$$= \frac{\mu_A^2 + \sigma_A^2}{2}$$

and similarly

$$\mathrm{var}[Y_j] = E[Y_j^2] = \cdots = (\mu_A^2 + \sigma_A^2) \cdot \frac{1}{2} = \frac{\mu_A^2 + \sigma_A^2}{2}$$

$$j = 1, \ldots, n$$

By the **Central Limit Theorem**,

$$X = \sum_{j=1}^{n} X_j \qquad \text{and} \qquad Y = \sum_{j=1}^{n} Y_j$$

are asymptotically Gaussian as $n \to \infty$.

Actually, $n \approx 15$–20 is pretty good.

Furthermore

$$E[XY] = E\left[\left(\sum_{j=1}^{n} X_j\right)\left(\sum_{k=1}^{n} Y_k\right)\right]$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} E[A_j A_k] \cdot E[\cos\phi_j \sin\phi_k] = 0$$

Because $E[\cos\phi_j \sin\phi_k] = 0$, for all $j, k = 1, \ldots, n$.

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So

$$\mathrm{E}[XY] = \mathrm{E}[X] \cdot \mathrm{E}[Y]$$

$\Longrightarrow$ X and Y are *uncorrelated*

Recall, X and Y are also Gaussian

Gaussian AND Uncorrelated $\Rightarrow$ **Independent!**

X and Y are independent jointly-distributed Gaussians

$$\mathrm{E}[X] = \mathrm{E}[Y] = 0$$

$$\mathrm{var}[X] = \mathrm{var}[Y] = \sigma^2 = n(\mu_A^2 + \sigma_A^2)/2$$

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So we have

$$Ve^{i\Theta} = X + iY,$$

where

$$V = \sqrt{X^2 + Y^2},$$

$$\Theta = \arctan(Y, X) \quad \text{(four-quadrant arctan)}$$

Define the received power as

$$P = V^2$$

To find $f_P(p)$, compute

$$f_{P\Theta}(p, \theta) = f_{XY}(x, y) \left| \frac{\partial(x, y)}{\partial(p, \theta)} \right|$$

and then

$$f_P(p) = \int_{-\infty}^{\infty} f_{P\Theta}(p, \theta)\, d\theta.$$

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$$f_{P\Theta}(p,\theta) = f_{XY}(x,y)\left|\frac{\partial(x,y)}{\partial(p,\theta)}\right|$$

Now

$$f_{XY}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(x^2+y^2)}{2\sigma^2}\right\}$$

$$x(p,\theta) = \sqrt{p}\cdot\cos\theta$$

$$y(p,\theta) = \sqrt{p}\cdot\sin\theta$$

$$\frac{\partial(x,y)}{\partial(p,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial y}{\partial p} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}p^{-1/2}\cos\theta & \frac{1}{2}p^{-1/2}\sin\theta \\ -p^{1/2}\sin\theta & p^{1/2}\cos\theta \end{vmatrix} = \frac{1}{2}\left(\cos^2\theta+\sin^2\theta\right) = \frac{1}{2}$$

So

$$f_{P\Theta}(p,\theta) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(p\cos^2\theta+p\sin^2\theta)}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)\cdot 1_{[0,2\pi)}(\theta)\cdot\left|\frac{1}{2}\right|$$

$$= \frac{1}{4\pi\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)\cdot 1_{[0,2\pi)}(\theta)$$

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$$f_{P\Theta}(p,\theta) = \frac{1}{4\pi\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)\cdot 1_{[0,2\pi)}(\theta)$$

It follows that

$$f_P(p) = \int_{-\infty}^{\infty} f_{P\Theta}(p,\theta)\,d\theta = \frac{1}{2\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)$$

An exponential pdf with mean $2\sigma^2$

$$f_{P\Theta}(p,\theta) = f_{XY}(x,y)\left|\frac{\partial(x,y)}{\partial(p,\theta)}\right|$$

Now

$$f_{XY}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(x^2+y^2)}{2\sigma^2}\right\}$$

$$x(p,\theta) = \sqrt{p}\cdot\cos\theta$$

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$$\frac{\partial(x,y)}{\partial(p,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial y}{\partial p} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}p^{-1/2}\cos\theta & \frac{1}{2}p^{-1/2}\sin\theta \\ -p^{1/2}\sin\theta & p^{1/2}\cos\theta \end{vmatrix} = \frac{1}{2}(\cos^2\theta + \sin^2\theta) = \frac{1}{2}$$

So

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$$= \frac{1}{4\pi\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)\cdot 1_{[0,2\pi)}(\theta)$$

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It follows that

$$f_P(p) = \int_{-\infty}^{\infty} f_{P\Theta}(p,\theta)\,d\theta = \boxed{\frac{1}{2\sigma^2}\exp\left\{-\frac{p}{2\sigma^2}\right\}\cdot 1_{[0,\infty)}(p)}$$

An exponential pdf with mean $2\sigma^2$

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It can also be shown that

$$f_\Theta(\theta) = \int_{-\infty}^{\infty} f_{P\Theta}(p,\theta)\,dp = \boxed{\frac{1}{2\pi} \cdot 1_{[0,2\pi)}}$$

Uniform distribution on $[0, 2\pi)$

Summarizing, we have the well known results

$$f_P(p) = \frac{1}{2\sigma^2} \exp\left\{-\frac{p}{2\sigma^2}\right\} \cdot 1_{[0,\infty)}(p),$$

$$f_\Theta(\theta) = \frac{1}{2\pi} \cdot 1_{[0,2\pi)}.$$

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Sometimes, in addition to the complex random field return $Ve^{i\Theta}$, there is a constant component a as well:

$$W = |Ve^{i\Theta} + ae^{i\Psi}|$$

Resultant Magnitude (W) — Random Component ($Ve^{i\Theta}$) — Constant Component ($ae^{i\Psi}$)

$a =$ a non-random constant,

$\Psi =$ a random variable uniformly distributed on $[0, 2\pi)$.

It can be shown that the amplitude W has pdf

$$\boxed{f_W(w) = \frac{w}{\sigma^2} \exp\left\{-\frac{(w^2+a^2)}{2\sigma^2}\right\} I_0\left(\frac{wa}{\sigma^2}\right) \cdot 1_{[0,\infty)}(w)}$$

Rician Amplitude Distribution

Received Power: $P = V^2$

It can be shown that

$$f_P(p) = \left(\frac{1+m^2}{\mu_p}\right) e^{-m^2} \exp\left\{-p\left(\frac{1+m^2}{\mu_p}\right)\right\} I_0\left(2m\sqrt{\frac{p(1+m^2)}{\mu_p}}\right) \cdot 1_{[0,\infty)}(p),$$

Rician Power Distribution

where

$$m = \sqrt{\frac{a^2}{2\sigma^2}}$$

and

$$I_0(\cdot) = \textit{modified Bessel function of order zero.}$$

RCS of Corner Reflectors

- A corner reflector is a radar target constructed by letting a number of planar surfaces come together, forming a corner.
- Corner reflectors tend to have the property that they reflect strongly back in the direction of the incident wave.
- For this reason, they are also called retro-reflectors.
- One common corner reflector is the "corner cube," used to construct optical bicycle reflectors.

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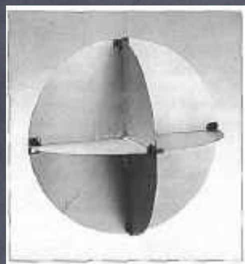
Another Recreational Use...



A typical sailboat corner reflector
(hung from mast)

A typical situation where they are useful!

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Common Corner Reflector Geometries