Session 43

ECE 678 Final Exam

Monday, December 9, 2024
8:00 – 10:00 am
FRNY G124

- You may bring in 2 pages of notes

- 5 problems covering beginning of course through SAR.

43.1

Radar Target Detection

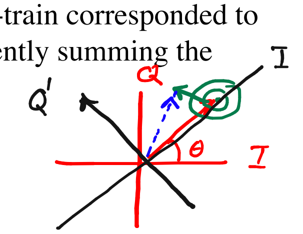
ECE678: Radar Engineering

Prof. Mark R. Bell

43.2

Nonideal Aspects of Radar Target Detection

- When we derived the matched-filter, we assumed we had complete knowledge of the received signal.
- In most radar detection problems, there is at the very least uncertainty in the phase of the received signal, as round-trip motion of the target though one wavelength corresponds to a 2π change in phase.
- We rarely have *a priori* knowledge of the target position down to a wavelength, so we must consider the effect of this on detection performance.
- In the ideal case of known, constant, received phase and system coherence between pulses, we saw that the matched-filter of a pulse-train corresponded to matched-filtering of the individual pulses and then coherently summing the results. This assumes:
  - knowledge of target-return phase
  - coherence of the radar system from pulse-to-pulse
  - constant target return (in amplitude and phase) from pulse-to-pulse



- In practice, any or all of these may not hold.

- We have seen that optimal detection of a known target in additive white Gaussian noise is provided by the matched filter.
- This corresponds to the following hypothesis testing problem:

$$H_0: \quad r(t) = n(t)$$

versus

$$H_1: \quad r(t) = s(t) + n(t)$$

white Gaussian noise

signal

- But two-way motion of one wavelength results in a phase shift of 2π radians in the passband signal (the carrier).
- So in general, we have a complex factor $e^{i\theta}$ applied to the received complex baseband signal.
- Here, θ is unknown, and is often modeled as a random variable uniformly distributed on the interval $[0, 2\pi)$.

- Thus the received signal becomes

$$r(t) = e^{i\theta} s(t) + n(t),$$

- The new hypothesis testing problem to be considered becomes

$$H_0: \quad r(t) = n(t)$$

versus

$$H_1: \quad r(t) = e^{i\theta} s(t) + n(t),$$

complex white Gaussian noise

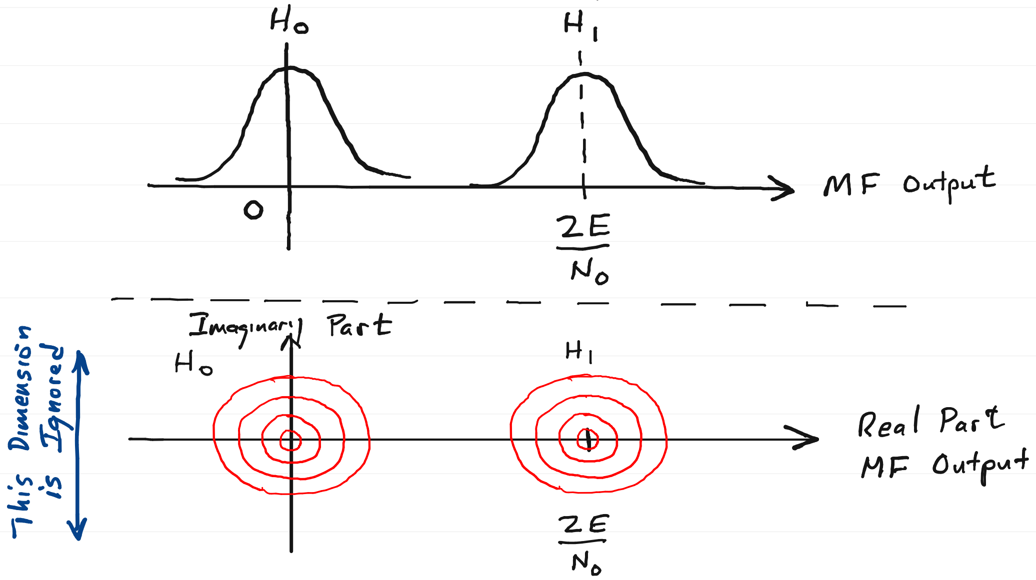
signal

where θ is unknown and usually modeled as uniformly distributed on $[0, 2\pi)$.

- Also, we assume that $n(t)$ is independent of θ.

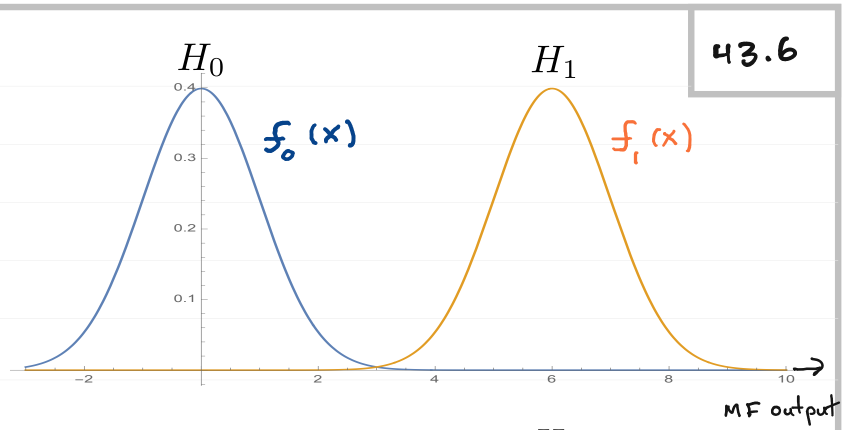
43.5

We know that in the case where the phase is known, we effectively have a one-dimensional problem:

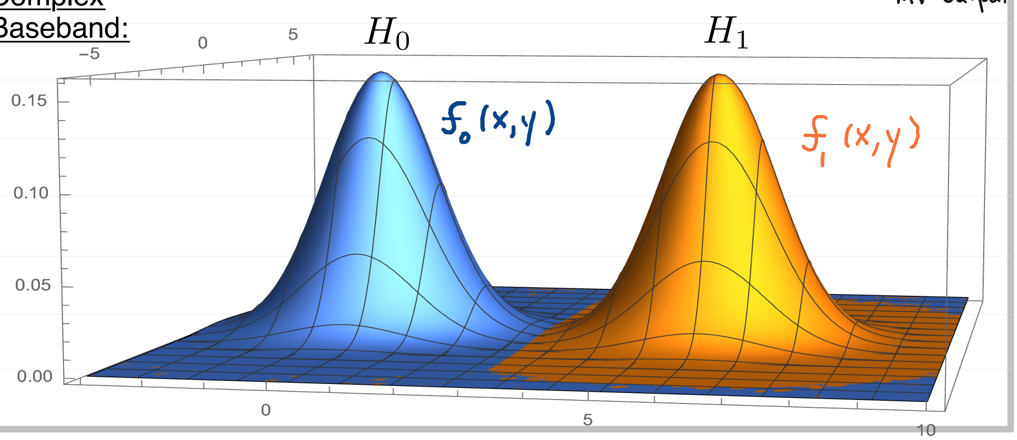


43.6

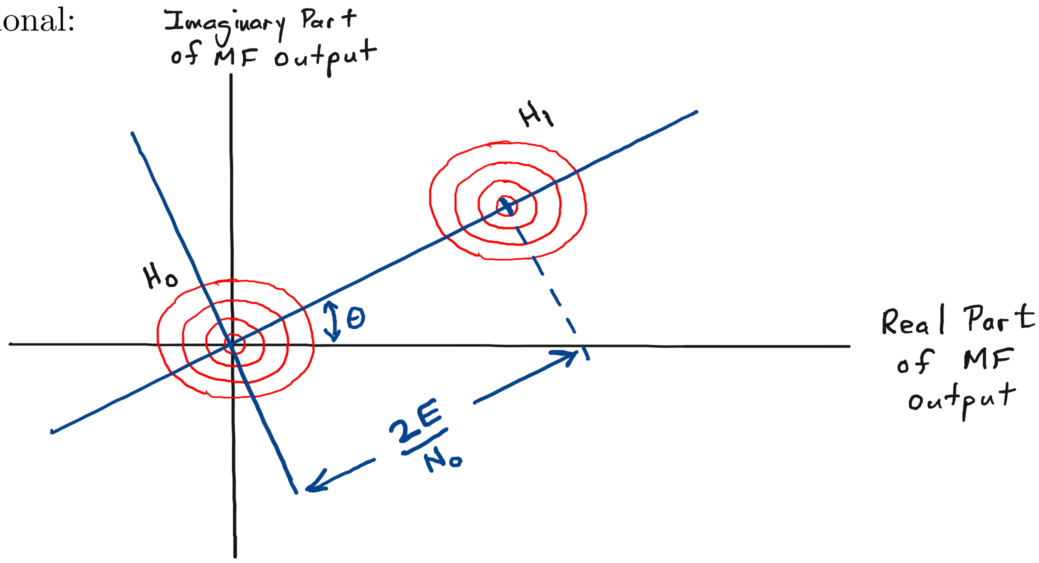
Real Part of Complex Baseband:



Complex Baseband:



However, in the case where the phase θ is unknown, the problem is intrinsically 2-dimensional:



When we do not know the phase θ, we must consider both dimensions (the real and imaginary part of the matched-filter output.) There is i.i.d. noise in both dimensions, and we get a contribution from both noise components.

The problem of detecting a known signal $s(t)$ with unknown phase θ (i.e., the problem of detecting $e^{i\theta}s(t)$) is called the **non coherent pulse detection problem**.

Noncoherent Pulse Detection

In detecting a known pulse with an unknown phase factor $e^{i\theta}$, the detection statistic is the magnitude of the complex matched filter output.

If: $s(t) \longrightarrow \boxed{MF} \text{ — } \underset{t=T}{\times} \text{ — } x(T)$

Then: $e^{i\theta}s(t) \longrightarrow \underbrace{\boxed{MF}}_{\text{L.T.I.}} \text{ — } \underset{t=T}{\times} \text{ — } \underbrace{e^{i\theta}x(T)}_{\text{by linearity of MF.}}$

If we assume $\theta \sim U[0,2\pi)$, there is no prefered direction for the signal or the noise.

$\Rightarrow$ The phase of the M.F. output is irrelevant.

The magnitude $\sqrt{x_R^2(T)+x_I^2(T)}$ is all that matters as a detection statistic, where

$$\underbrace{x(T) = x_R(T) + i\,x_I(T)}_{\text{Complex Baseband MF Output}}$$

43.9

The in-phase (real) and quadrature (imaginary) noise components in the complex baseband matched filter output are i.i.d, zero mean RVs with variance $\sigma^2 = \frac{2E}{N_0}$

They together can be viewed as a complex "circular Gaussian" RV

$$Z = X + iY \quad , \quad X, Y \sim N[0, \sigma^2], \; X \perp\!\!\!\perp Y.$$

Since the magnitude of the noise is given by $R = \sqrt{X^2 + Y^2}$

and $\Theta \sim \tan^{-1}(Y, X)$

43.10

- The output of the matched filter can be shown to have pdf

$$f_{R\Theta}(r, \theta) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \cdot 1_{[0,\infty)}(r) \cdot 1_{[0,2\pi)}(\theta).$$

- Integrating w.r.t. θ, we get

$$f_R(r) = \int_{-\infty}^{\infty} f_{R\Theta}(r, \theta)\, d\theta = \underbrace{\frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \cdot 1_{[0,\infty)}(r)}_{\text{Rayleigh pdf}}$$

pdf under H_0

Under H_1 (Signal + noise):

The output of the matched filter, for a fixed $\Theta = \theta_0$, can be described by a two-dimensional Gaussian pdf with non-zero means $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{2E}{N_0} \cos \theta_0$$

and

$$\bar{y} = \frac{2E}{N_0} \sin \theta_0 .$$

We can write the means in the form

$$\bar{x} = A \cos \theta ,$$

$$\bar{y} = A \sin \theta .$$

Then we have

$$f_{Re}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ - \frac{(x - A\cos\theta)^2}{2\sigma^2} \right\}$$

$$f_{Im}(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ - \frac{(y - A\sin\theta)^2}{2\sigma^2} \right\}$$

Then letting

$$\mathbb{R} = \sqrt{\mathbb{X}^2 + \mathbb{Y}^2}$$

and

$$\Phi = \tan^{-1}(\mathbb{Y}, \mathbb{X}),$$

we can write the joint pdf of $\mathbb{R}$ and Φ as

$$f_{\mathbb{R}\Phi}(r,\phi) = \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{(r^2 - 2Ar\cos(\theta-\phi) + A^2)}{2\sigma^2}\right\} \mathbb{1}_{[-\pi,\pi)}(\theta)\cdot \mathbb{1}_{[0,\infty)}(r).$$

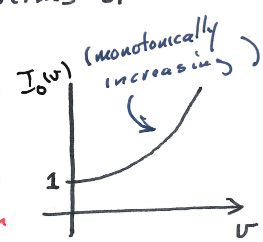
Integrating over ϕ, we get

$$f_{\mathbb{R}}(r) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \cdot e^{-A^2/2\sigma^2} \cdot \int_{-\pi}^{\pi} \underbrace{e^{Ar\cos(\theta-\phi)/2\sigma^2}}_{\text{periodic in } \phi.} d\phi$$

periodic in ϕ. Integrated over exactly one period regardless of value of θ.

This integral is independent of θ because it extends over exactly one period of $\cos(\theta - \phi)$ regardless of the value of θ. So the value of the integral is independent of θ and can be expressed in terms of the zero-order modified Bessel function

$$I_0(v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{v\cos\phi}\, d\phi \;\underbrace{= \frac{1}{\pi}\int_0^{\pi} e^{v\cos\phi}\, d\phi}_{\text{This is how it is often written in tables.}}$$



Thus it follows that under H_1, we can write

$$f_{\mathbb{R}}(r) = \frac{r}{\sigma^2}\exp\left\{\frac{-(r^2+A^2)}{2\sigma^2}\right\}\cdot I_0\left(\frac{rA}{\sigma^2}\right), \quad r \geq 0$$

where

$$A = \frac{2E}{N_0}$$

and

$$\sigma^2 = \frac{2E}{N_0}$$

$$= \frac{r}{\mathcal{S}}\exp\left\{\frac{-(r^2+\mathcal{S}^2)}{2\mathcal{S}}\right\} I_0(r), \quad r \geq 0$$

where $\mathcal{S} = \frac{2E}{N_0}$

Under H_0:

$$f_{R,0}(r) = \frac{r}{\sigma^2} \exp\left\{\frac{-r^2}{2\sigma^2}\right\} \cdot \mathbb{1}_{[0,\infty)}(r)$$

$$= \frac{r}{\mathcal{S}} \exp\left\{\frac{-r^2}{2\mathcal{S}}\right\} \cdot \mathbb{1}_{[0,\infty)}(r)$$

Under H_1:

$$f_{R,1}(r) = \frac{r}{\mathcal{S}} \exp\left\{-\frac{(r^2+\mathcal{S}^2)}{2\mathcal{S}}\right\} I_0(r) \cdot \mathbb{1}_{[0,\infty)}(r)$$

Computing the log-likelihood ratio, we have

$$\ell(r) = \ln\left(\frac{f_{R,1}(r)}{f_{R,0}(r)}\right) = \ln I_0(r) - \frac{\mathcal{S}^2}{2\mathcal{S}} \underset{H_0}{\overset{H_1}{\gtrless}} \ln L_0$$

Because

$\ell(r)$ = monotonically increasing function of r

(n.b. $I_0(r)$ is monotonically increasing
$\ln(\cdot)$ is monotonically increasing
$\Rightarrow \ln(I_0(r))$ is monotonically increasing.)

the likelihood ratio test reduces to a threshold test on r:

$$\phi(R) = \begin{cases} 1, & R > \lambda_0 \\ 0, & R \le \lambda_0 \end{cases}$$

The size and power of this test is given by

$$\alpha = \int_{\lambda_0}^{\infty} f_{R,0}(r)\, dr$$

$$\beta = \int_{\lambda_0}^{\infty} f_{R,1}(r)\, dr$$

Now if we let $z = \frac{r}{\sigma} = \frac{r}{\sqrt{\mathcal{S}}} \Rightarrow r = z\sqrt{\mathcal{S}} \Rightarrow dr = \sqrt{\mathcal{S}}\, dz$
and let $\gamma_0 = \frac{\lambda_0}{\sqrt{\mathcal{S}}}$, we can write

$$\alpha(\gamma_0) = \int_{\gamma_0}^{\infty} z e^{-z^2/2}\, dz = e^{-\gamma_0^2/2}$$

from which it follows that the threshold γ_0 yielding a size α test is

$$\gamma_0 = \sqrt{-2 \ln \alpha}$$

Again, with $z = \frac{r}{\sqrt{\mathcal{S}}}$ and threshold $\gamma_0 = \frac{\lambda_0}{\sqrt{\mathcal{S}}}$ on z, we have

$$\beta(\gamma_0) = \int_{\lambda_0}^{\infty} f_{R,1}(r)\, dr = \int_{\lambda_0}^{\infty} \frac{r}{\mathcal{S}} \exp\left(\frac{-r^2}{2\mathcal{S}}\right) \exp\left(\frac{-\mathcal{S}}{2}\right) I_0(r)\, dr$$

let $z = \frac{r}{\sqrt{\mathcal{S}}} \Rightarrow r = z\sqrt{\mathcal{S}} \Rightarrow dr = \sqrt{\mathcal{S}}\, dz$

$$= \int_{\gamma_0}^{\infty} \frac{z}{\sqrt{\mathcal{S}}} \exp\left(-\frac{z^2}{2}\right) \exp\left(\frac{-\mathcal{S}}{2}\right) I_0(z\sqrt{\mathcal{S}}) \sqrt{\mathcal{S}}\, dz$$

$$= \int_{\gamma_0}^{\infty} z \exp\left\{-\frac{(z^2 + (\sqrt{\mathcal{S}})^2)}{2}\right\} I_0(z\sqrt{\mathcal{S}})\, dz$$

$$= Q(\sqrt{\mathcal{S}}, \gamma_0) = Q\left(\sqrt{\frac{2E}{N_0}}, \gamma_0\right)$$

where

$$Q(\alpha, \gamma) \triangleq \int_{\gamma}^{\infty} x \exp\left\{-\frac{(x^2+\alpha^2)}{2}\right\} I_0(\alpha x)\, dx$$

= "Marcum Q-function"

So the ROC for the non-coherent pulse detection problem is

$$\boxed{\beta(\alpha) = Q\left(\sqrt{\frac{2E}{N_0}}, \sqrt{-2\ln\alpha}\right).}$$

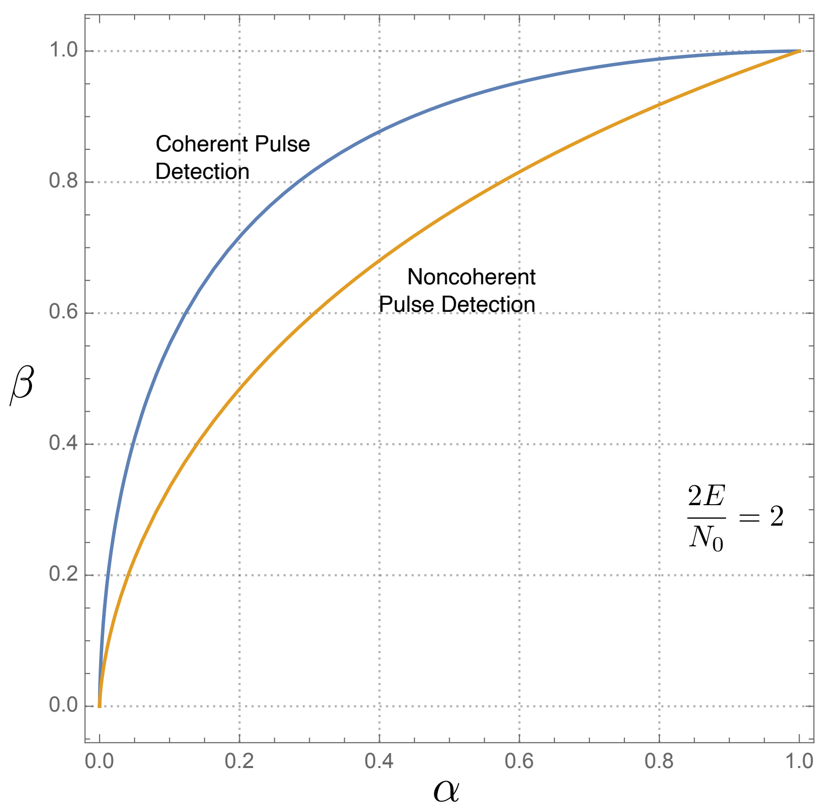
ROC for Non-coherent Pulse Detection

The ROC for coherent detection of the same pulse with known phase is

$$\boxed{\beta(\alpha) = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{\frac{2E}{N_0}}\right)}$$

ROC for Coherent Pulse Detection

Coherent and Noncoherent Pulse Detection ROCs



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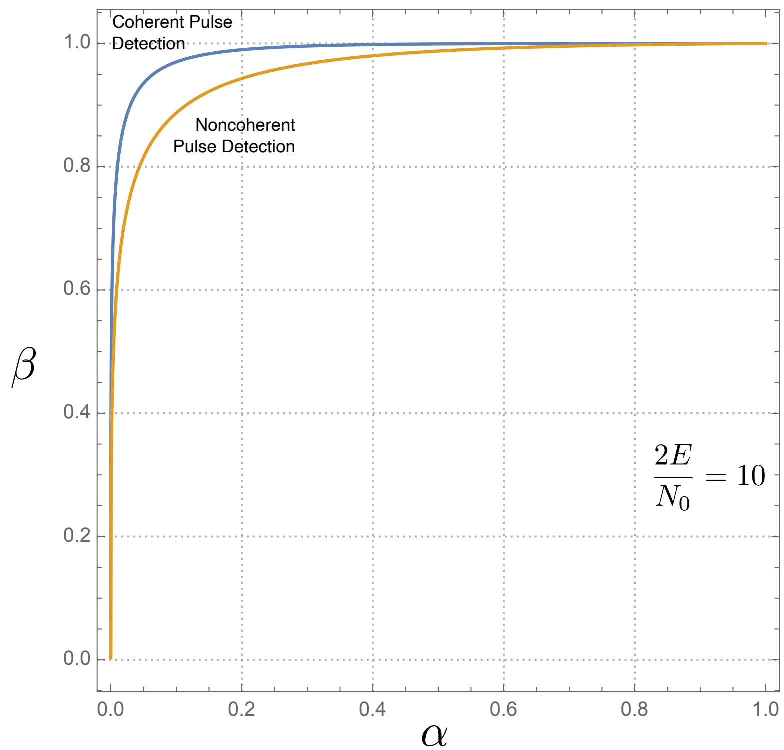
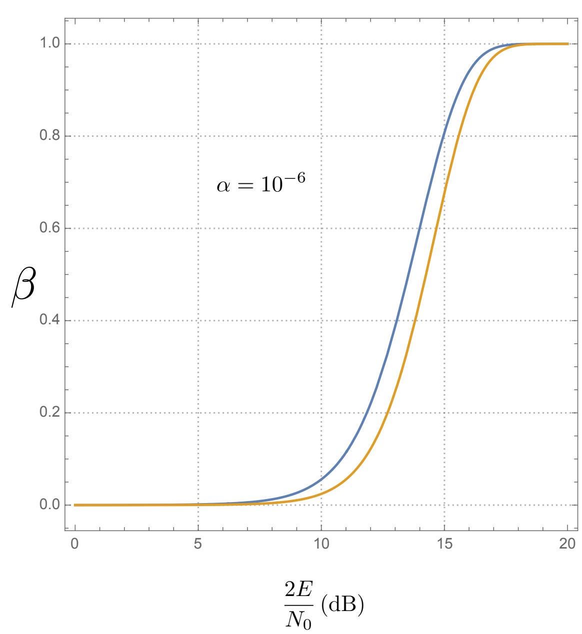
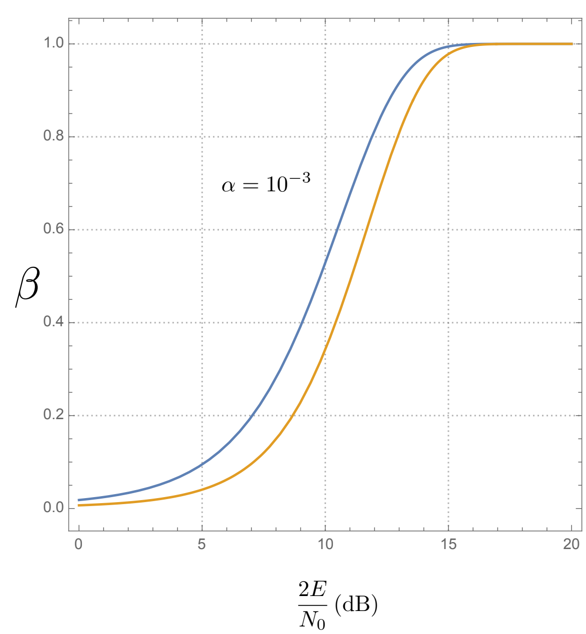
Coherent and Noncoherent Pulse Detection ROCs

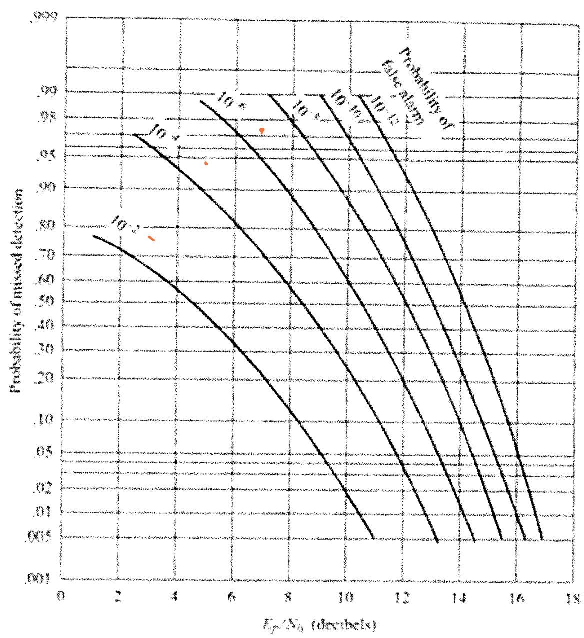


43.22

Probability of Detection as a Function of SNR







igure 12.2 Performance of a coherent detector in gaussian noise

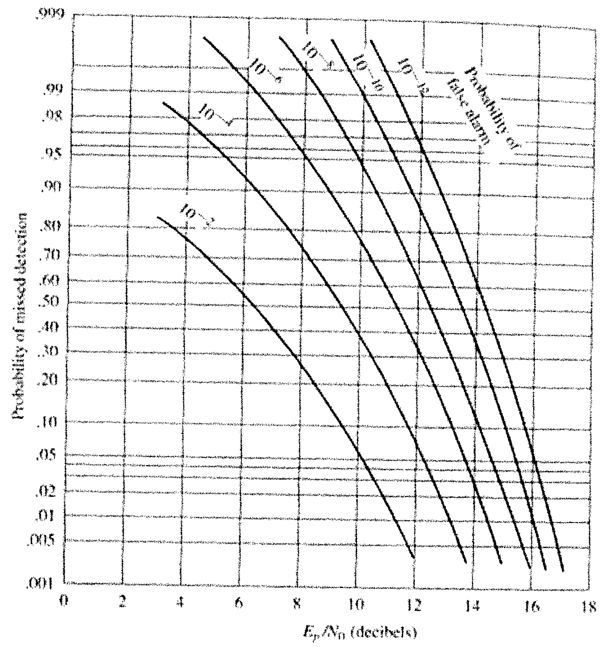
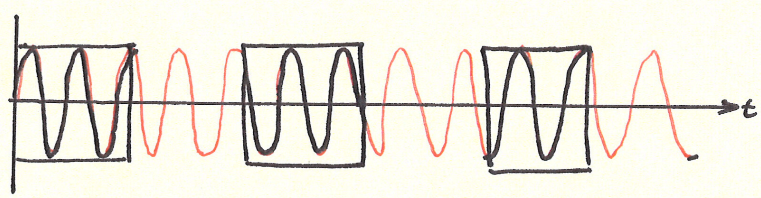


Figure 12.3 Performance of a noncoherent detector in gaussian noise

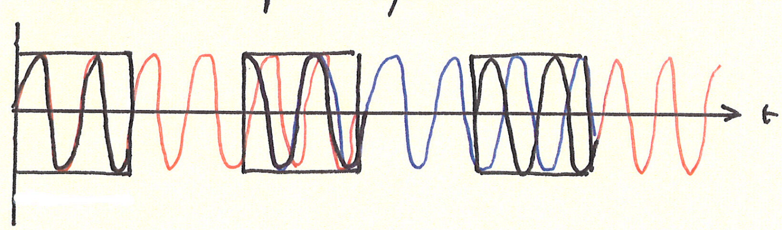
Detection Using Multiple Observations (Pulses)

When we talk about detection of multiple pulses, there are two cases we must consider:

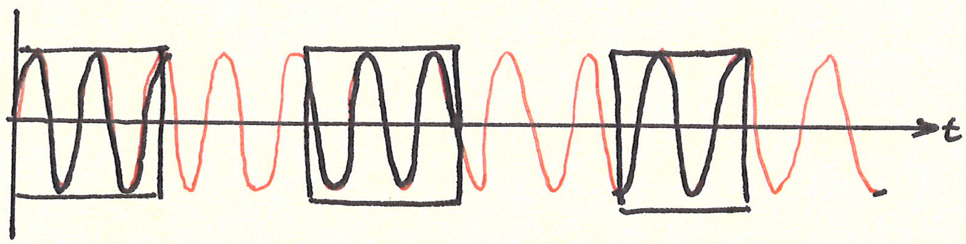
1. A coherent-pulse system



2. A Noncoherent-pulse system

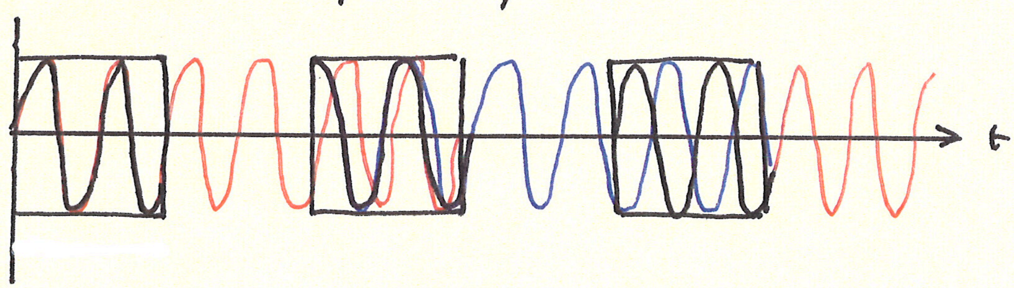


1. A coherent-pulse system



Case 1 in which pulse-to-pulse coherence is maintained is essential for pulse-Doppler radar operation. It allows us to view the phase shift from pulse-to-pulse caused by target motion.

2. A Noncoherent-pulse system



In case 2 – which is used in older and less expensive radar systems, the oscillator is effectively turned on with the transmission of each pulse. This is usually modeled by applying a complex phase factor $e^{i\theta_n}$ to each pulse at baseband, where $\theta_1, \ldots, \theta_N$ are modeled as i.i.d $U[0, 2\pi)$ –

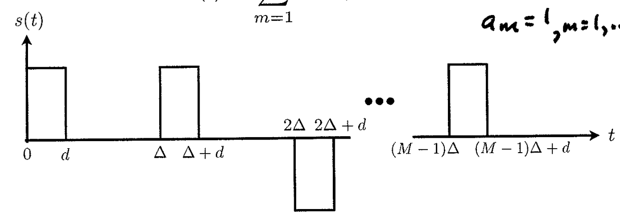
1 Coherent Multipulse Detection Performance with complete knowledge of received pulses

We have already looked at this problem when we consider matched-filtering a pulse train

Consider the waveform $s(t)$ constructed by regularly repeating a basic waveform $p(t)$ with amplitude modulation on each pulse:

$$s(t) = \sum_{m=1}^{M} a_m p(t - (m-1)\Delta).$$

$a_m = 1, m = 1, \ldots, M$



- Here a_m is the amplitude of the m-th pulse.
- The amplitudes could be *binary*(*e.g.*,±1), *real* or *complex*.
- Assume we receive the signal in the presence of zero-mean stationary noise having PSD $S_{nn}(f)$.
- We want to design the *matched filter* to maximze the output SNR at observation time $t = T$.

We know that in this situation, if

$$S(f) = \mathcal{F}\{s(t)\} = \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t}\, dt,$$

Then the matched filter is given by

$$\tilde{H}_M(f) = \frac{S^*(f) e^{-i2\pi f T}}{S_{nn}(f)}.$$

Assume $S_{nn}(f) = N_0/2$

If we take $T = M\Delta$ so that the observation time trailing the last pulse is equal to the observation time between all other pulses, and we note that

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t}\, dt = \sum_{m=1}^{M} a_m P(f) e^{-i2\pi f (m-1)\Delta},$$

where

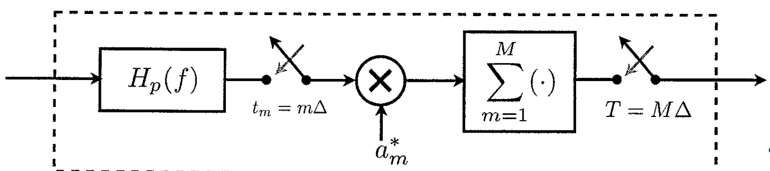
$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-i2\pi f t}\, dt.$$

So the matched filter can be written as

$$\begin{aligned}\tilde{H}_M(f) &= \frac{S^*(f)}{S_{nn}(f)} e^{-i2\pi f M\Delta} \\ &= \sum_{m=1}^{M} a_m^* \frac{P^*(f)}{S_{nn}(f)} e^{-i2\pi f \Delta} e^{-i2\pi (M-m) f\Delta} \\ &= \sum_{m=1}^{M} a_m^* H_p(f) e^{-i2\pi (M-m) f\Delta},\end{aligned}$$

where $H_p(f)$ is the single pulse matched filter sampled at time $t = \Delta$ and given by

$$H_p(f) = \frac{P^*(f)}{S_{nn}(f)} e^{-i2\pi f\Delta}.$$



$\tilde{H}_M(f)$

So we see that the matched-filter for the pulse train is obtained by summing the responses for the matched-filter of each pulse:

For a completely known signal

$$H_0: \quad x(T) \sim N\left[0, \frac{2E_s M}{N_0}\right]$$

Because you are adding M independent zero-mean Gaussian RVs with mean 0 and variance $\frac{2E_s}{N_0}$.

$$H_1: \quad x(T) \sim N\left[\frac{2E_s M}{N_0}, \frac{2E_s M}{N_0}\right]$$

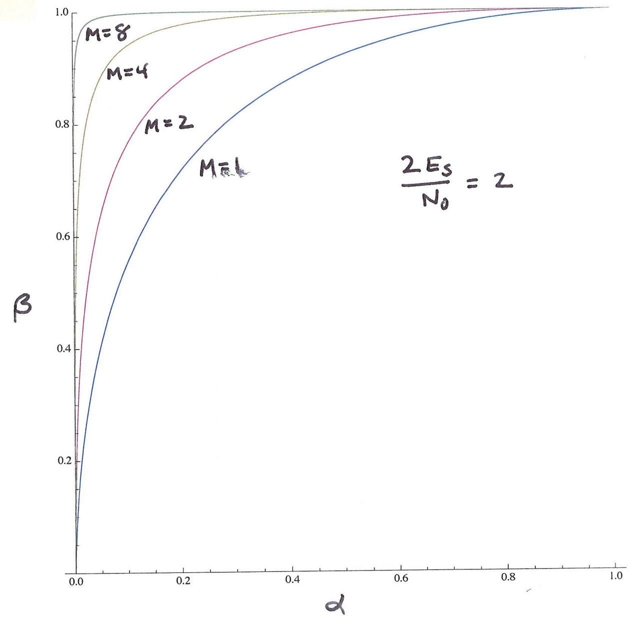
Because you are adding M independent Gaussian RVs with mean $\frac{2E_s}{N_0}$ and variance $\frac{2E_s}{N_0}$.

n.b. The standard deviation of the noise grows like $\sqrt{M}$ while the mean grows like M.
SNR increases by $\left(\frac{M}{\sqrt{M}}\right)^2 = \frac{M^2}{M} = \boxed{M}$.

Thus we know that in this case, the ROC is given by

$$\beta(\alpha) = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{\frac{2ME_s}{N_0}}\right)$$

ROC For Detection of M Completely Known Pulses



Coherent Pulse-Train Detection with Identical Pulses

Unknown identical Phase for Each Pulse

The magnitude of the complex matched-filter output is taken as the decision statistic.

Under H_0: This corresponds to the sum of M independent circular complex Gaussian RVs:

$$x(T) = Z_1 + Z_2 + \cdots + Z_M$$

Under H_1:

$$x(T) = e^{i\theta}\frac{2E_s}{N_0} + Z_1 + e^{i\theta}\frac{2E_s}{N_0} + Z_2 + \cdots + e^{i\theta}\frac{2E_s}{N_0} + Z_M$$

$$= e^{i\theta}\cdot\frac{2ME_s}{N_0} + \underbrace{(Z_1 + Z_2 + \cdots + Z_M)}$$

Zero-mean complex Gaussian with I and Q components having variance $\sigma_M^2 = M\sigma^2 = \frac{2ME_s}{N_0}$.

This looks just like the corresponding single pulse problem with a new mean and variance.

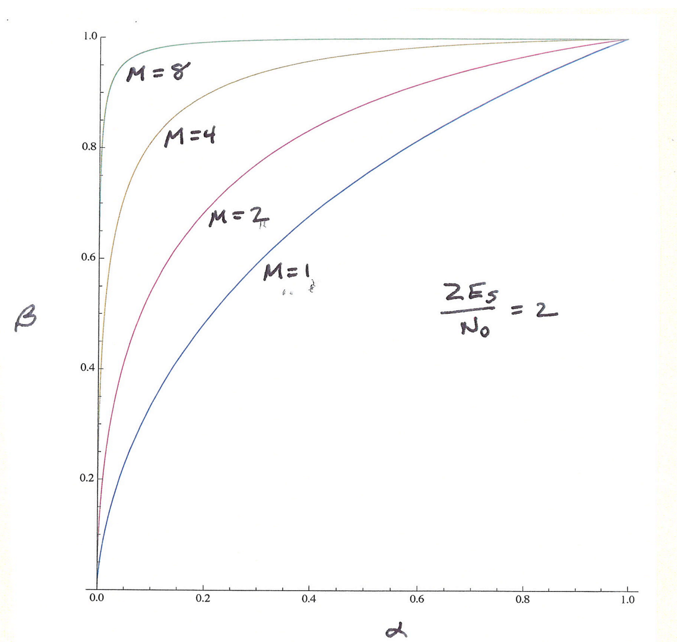
Thus it follows that we can write

$$\alpha(\gamma_0) = \int_{\gamma_0}^{\infty} z e^{-z^2/2}\, dz = e^{-\gamma_0^2/2}$$

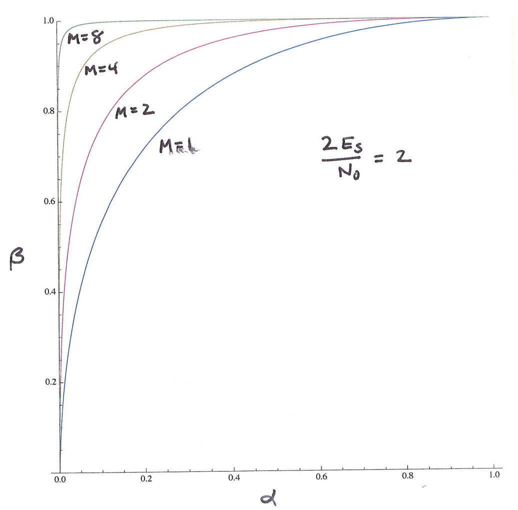
$$\beta(\gamma_0) = \int_{\gamma_0}^{\infty} z e^{-\left(z^2 + \frac{2ME_S}{N_0}\right)} \cdot I_0\left(z\sqrt{\frac{2ME_S}{N_0}}\right) dz$$

$$= Q\left(\sqrt{\frac{2ME_S}{N_0}}, \gamma_0\right)$$

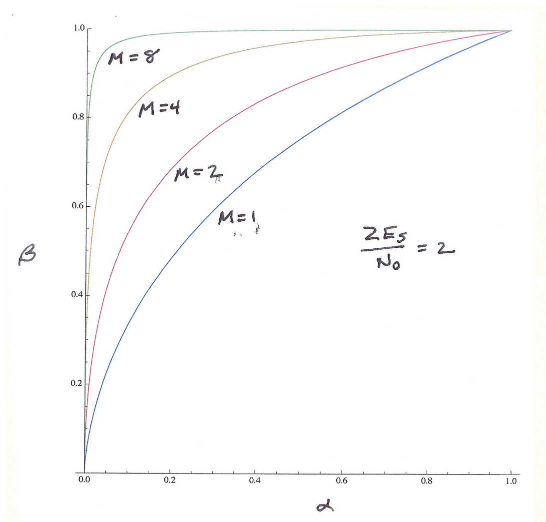
$$\beta(\alpha) = Q\left(\sqrt{\frac{2ME_S}{N_0}}, \sqrt{-2\ln\alpha}\right)$$



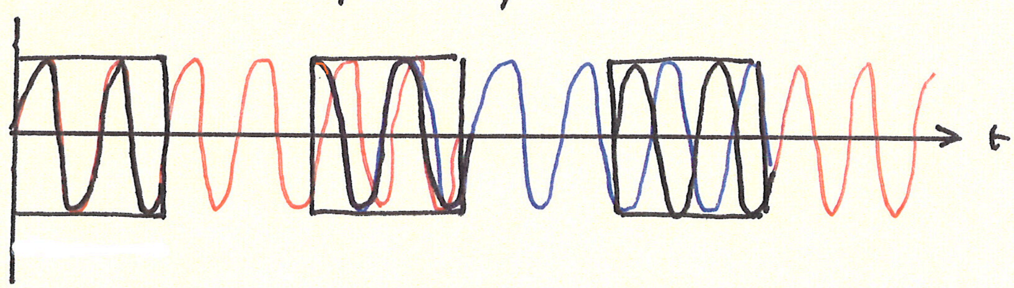
Coherent Pulse Train



Noncoherent Pulse Train



2. A Noncoherent pulse system



In case 2 – which is used in older and less expensive radar systems, the oscillator is effectively turned on with the transmission of each pulse. This is usually modeled by applying a complex phase factor $e^{i\theta_n}$ to each pulse at baseband, where $\theta_1, \ldots, \theta_N$ are modeled as i.i.d $U[0,2\pi)$.

2. Radar Detection with a Noncoherent Pulse Train

- Assume that we once again detect a constant target, this time using an M pulse pulse train, but now assume there is a complete lack of coherence between pulses.
- Assume that once again, each pulse is processed with a complex baseband matched filter.

Under H_0: The complex matched filter output of each pulse is of the form

$$Z_m = X_m + iY_m \,, \quad X_m, Y_m \sim N[0, \sigma^2], \quad X_m \perp\!\!\!\perp Y_m \,, \quad (X_m, Y_m) \perp\!\!\!\perp (X_n, Y_n),\ n \neq m.$$

Under H_1:

$$Z_m = \frac{2E}{N_0} e^{i\theta_m} + \underbrace{X_m + iY_m}_{\text{Circular complex Gaussians as above (under } H_0)}$$

$$\theta_1, \theta_2, \ldots, \theta_M \quad \text{i.i.d.} \quad U[0, 2\pi)$$

- As before, the phase does not contain useful information for detection.
- Thus we base our decision on the conditionally i.i.d. RVs $R_1, R_2, \ldots, R_M$,

where $R_m = |Z_m|$ for $m = 1, 2, \ldots, M$.

As in the single noncoherent pulse case, each individual pulse return has pdf

$R_m = |Z_m|, \; m = 1, \dots, M.$

Under H_0:

$$f_{R_m,0}(r_m) = \frac{r_m}{\sigma^2} e^{-r_m^2/2\sigma^2} \cdot 1_{[0,\infty)}(r_m), \quad m = 1, 2, \dots, M.$$

(n.b., The R_m are conditionally i.i.d.)

Under H_1:

$$f_{R_m,1}(r_m) = \frac{r_m}{\sigma^2} \exp\left\{ -\frac{(r_m^2 + A^2)}{2\sigma^2} \right\} I_0\left(\frac{r_m A}{\sigma^2}\right) \cdot 1_{[0,\infty)}(r_m),$$

$$m = 1, 2, \dots, M$$

Thus the log-likelihood ratio of $\underline{R} = (R_1, R_2, \dots, R_M)$ is

$$\ell(\underline{R}) = \ln\left(\frac{f_{\underline{R},1}(\underline{R})}{f_{\underline{R},0}(\underline{R})}\right) = \ln\left(\frac{f_{R_1,1}(R_1)\cdots f_{R_M,1}(R_M)}{f_{R_1,0}(R_1)\cdots f_{R_M,0}(R_M)}\right)$$

$$= \sum_{m=1}^{M} \ln\left(\frac{f_{R_m,1}(R_m)}{f_{R_m,0}(R_m)}\right) = \sum_{m=1}^{M}\left[\ln I_0\left(\frac{R_m A}{\sigma^2}\right) - \frac{A^2}{2\sigma^2}\right]$$

$$= \sum_{m=1}^{M}\left[\ln\left(I_0\left(\frac{R_m A}{\sigma^2}\right)\right)\right] - \frac{MA^2}{2\sigma^2}$$

Now for small $x \ll 1$, $I_0(x) \cong 1 - x^2$.
For $y \cong 1$, $\ln y \cong y - 1$.

Thus at small signal-to-noise ratios, we have

$$\ln I_0\left(\frac{R_m A}{\sigma^2}\right) \simeq \ln\left(1 + \left[\frac{R_m A}{\sigma^2}\right]^2\right)$$
$$\simeq = R_m^2 \cdot \left(\frac{A}{\sigma^2}\right)^2$$

Thus it follows that

$$l(\underline{R}) \simeq \sum_{m=1}^{M} R_m^2 \cdot \left(\frac{A}{\sigma^2}\right)^2 - \frac{MA^2}{2\sigma^2}$$
$$= \left(\frac{A}{\sigma^2}\right)^2 \sum_{m=1}^{M} R_m^2 - \frac{MA^2}{2\sigma^2}.$$

Thus we can implement the test in the form

$$l(\underline{R}) \cong \left(\frac{A}{\sigma^2}\right)^2 \sum_{m=1}^{M} R_m^2 - \frac{MA^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} l_0$$

$$\Rightarrow \sum_{m=1}^{M} R_m^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{l_0 + \frac{MA^2}{2\sigma^2}}{(A/\sigma^2)^2} = \gamma_0$$

Hence for weak signals, the statistic

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2$$

is used in the threshold test

$$\boxed{\begin{aligned} \phi(\underline{R}) &= \begin{cases} 1, & T(\underline{R}) > \gamma_0 \\ 0, & T(\underline{R}) \le \gamma_0 \end{cases} \\ T(\underline{R}) &= \sum_{m=1}^{M} R_m^2 \end{aligned}}$$

Weak Signal Test

Weak Signal Test:

$$\phi(\underline{R}) = \begin{cases} 1, & T(\underline{R}) > T_0 \\ 0, & T(\underline{R}) \le T_0 \end{cases}$$

where

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2 .$$

(Square-Law Detector)

Alternatively, for $x >> 1$, we have

$$I_0(x) \cong \frac{e^x}{\sqrt{2\pi x}}, \qquad (x >> 1)$$

and thus

$$\ln\left[I_0\left(\frac{R_m A}{\sigma^2}\right)\right] \simeq \ln\left[\exp\left[\frac{R_m A}{\sigma^2}\right]\right] - \frac{1}{2}\ln\left(\frac{2\pi R_m A}{\sigma^2}\right)$$

$$\simeq \frac{R_m A}{\sigma^2}$$

$$\Rightarrow \ell(\underline{R}) \simeq \sum_{m=1}^{M} \frac{R_m A}{\sigma^2} - \frac{MA}{2\sigma^2} = \frac{A}{\sigma^2}\sum_{m=1}^{M} R_m - \frac{MA}{2\sigma^2}$$

$$\Rightarrow \sum_{m=1}^{M} R_m \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ell_0 + \frac{MA}{2\sigma^2}}{A/\sigma^2} =: \mu_0$$

Thus in the strong signal case, we have a threshold test of the form

$$\boxed{\begin{aligned} \phi(\underline{R}) &= \begin{cases} 1, & u(\underline{R}) > u_0 \\ 0, & u(\underline{R}) \leq u_0 \end{cases} \\ \text{where} \\ u(\underline{R}) &= \sum_{m=1}^{M} R_m \end{aligned}}$$

Linear-Law Detector

So we have two integration rules:

Linear Detector: (Strong Signal case)

$$u(\underline{R}) = \sum_{m=1}^{M} R_m \underset{H_0}{\overset{H_1}{\gtrless}} u_0$$

Square-Law Detector: (Weak Signal Case)

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2 \underset{H_0}{\overset{H_1}{\gtrless}} T_0$$

In order to characterize the performance of these detectors, we need to find the distributions of $u(\underline{R})$ and $T(\underline{R})$ under both H_0 and H_1.

Recall..

2. Radar Detection with a Noncoherent Pulse Train

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- Assume that we once again detect a constant target, this time using an M pulse pulse train, but now assume there is a complete lack of coherence between pulses.
- Assume that once again, each pulse is processed with a complex baseband matched filter.

Under H_0: The complex matched filter output of each pulse is of the form

$$Z_m = X_m + iY_m \quad , \quad X_m, Y_m \sim N[0, \sigma^2],$$
$$X_m \perp\!\!\!\perp Y_m \; ,$$
$$(X_m, Y_m) \perp\!\!\!\perp (X_n, Y_n), \; n \neq m.$$

Recall...

Under H_1:

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$$Z_m = \frac{2E}{N_0} e^{i\theta_m} + \underbrace{X_m + iY_m}_{\substack{\text{Circular complex} \\ \text{Gaussians as} \\ \text{above (under } H_0)}}$$

$$\theta_1, \theta_2, \ldots, \theta_M \quad \text{i.i.d.} \quad U[0, 2\pi)$$

- As before, the phase does not contain useful information for detection.
- Thus we base our decision on the conditionally i.i.d. RVs $R_1, R_2, \ldots, R_M$,

where $R_m = |Z_m|$ for $m = 1, 2, \ldots, M$.

Recall ...

As in the single noncoherent pulse case, each individual pulse return has pdf

$R_m = |Z_m|, \quad m = 1, \ldots, M.$

Under H_0:

$$f_{R_m,0}(r_m) = \frac{r_m}{\sigma^2} e^{-r_m^2/2\sigma^2} \cdot 1_{[0,\infty)}(r_m), \quad m = 1, 2, \ldots, M.$$

(n.b., The R_m are conditionally i.i.d.)

Under H_1:

$$f_{R_m,1}(r_m) = \frac{r_m}{\sigma^2} \exp\left\{-\frac{(r_m^2 + A^2)}{2\sigma^2}\right\} I_0\left(\frac{r_m A}{\sigma^2}\right) \cdot 1_{[0,\infty)}(r_m),$$

$$m = 1, 2, \ldots, M$$

Weak Signal Test:

$$\phi(\underline{R}) = \begin{cases} 1, & T(\underline{R}) > T_0 \\ 0, & T(\underline{R}) \leq T_0 \end{cases}$$

where

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2.$$

(Square-Law Detector)

Alternatively, for $x >> 1$, we have

$$I_0(x) \cong \frac{e^x}{\sqrt{2\pi x}}, \qquad (x >> 1)$$

and thus

$$\ln\left[I_0\left(\frac{R_m A}{\sigma^2}\right)\right] \simeq \ln\left[\exp\left[\frac{R_m A}{\sigma^2}\right]\right] - \frac{1}{2}\ln\left(\frac{2\pi R_m A}{\sigma^2}\right)$$

$$\simeq \frac{R_m A}{\sigma^2}$$

$\Rightarrow$

$$l(\underline{R}) \simeq \sum_{m=1}^{M} \frac{R_m A}{\sigma^2} - \frac{MA}{2\sigma^2} = \frac{A}{\sigma^2}\sum_{m=1}^{M} R_m - \frac{MA}{2\sigma^2}$$

$\Rightarrow$

$$\sum_{m=1}^{M} R_m \underset{H_0}{\overset{H_1}{\gtrless}} \frac{l_0 + \frac{MA}{2\sigma^2}}{A/\sigma^2} =: u_0$$

Thus in the strong signal case, we have a threshold test of the form

$$\boxed{\phi(\underline{R}) = \begin{cases} 1, & u(\underline{R}) > u_0 \\ 0, & u(\underline{R}) \le u_0 \end{cases} \quad \text{where} \quad u(\underline{R}) = \sum_{m=1}^{M} R_m}$$

(Linear - Law Detector)

So we have two integration rules:

Linear Detector: (Strong Signal case)

$$u(\underline{R}) = \sum_{m=1}^{M} R_m \underset{H_0}{\overset{H_1}{\gtrless}} u_0$$

Square-Law Detector: (Weak Signal Case)

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2 \underset{H_0}{\overset{H_1}{\gtrless}} T_0$$

In order to characterize the performance of these detectors, we need to find the distributions of $u(\underline{R})$ and $T(\underline{R})$ under both H_0 and H_1.

For the Weak Signal Case (Square-Law Detector)

The distribution of the sum $T(\underline{R})$ can be found as an M-fold convolution of these distributions (or use characteristic fzns.)

Under H_0:

$$f_{T,0}(t) = \frac{t^{(M-1)/2}}{2^M \sigma^{2M} (M-1)!} \exp\left(\frac{-t}{2\sigma^2}\right) 1_{[0,\infty)}(t)$$

Under H_1:

$$f_{T,1}(t) = \frac{1}{2\sigma^2}\left(\frac{t}{MA^2}\right)^{M-1} \cdot \exp\left(-\frac{(t+MA^2)}{2\sigma^2}\right) \cdot I_{M-1}\left(\frac{A\sqrt{t}}{\sigma^2}\right) \cdot 1_{[0,\infty)}(t)$$

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Computing the likelihood ratio, we have

$$L(r) = \frac{f_{T,1}(t)}{f_{T,0}(t)} = \frac{2^M \sigma^2 (M-1)!}{(MA^2)^{M-1}} \exp\left\{-\frac{MA^2}{2\sigma^2}\right\} t^{(M-1)/2} I_{M-1}\left(\frac{A\sqrt{t}}{\sigma^2}\right)$$

$$= \underbrace{K(M,A)}_{\text{contant}} \cdot \underbrace{t^{(M-1)/2}}_{\text{monotone increasing in } t} \cdot \underbrace{I_{M-1}\left(\frac{A\sqrt{t}}{\sigma^2}\right)}_{\text{monotone increasing in } t} \underset{H_0}{\overset{H_1}{\gtrless}} L_0$$

$$\Rightarrow \quad T(R) \underset{H_0}{\overset{H_1}{\gtrless}} T_0$$

So this test is a threshold test on $T(R)$.

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Given this fact and the distributions on $T(\underline{R})$ under H_0 and H_1, we can write

$$\alpha(M,T_0) = P_{FA}(M,T_0) = \int_{T_0}^{\infty} f_{T,0}(t)\, dt$$

$$\beta(M,T_0) = P_D(M,T_0) = \int_{T_0}^{\infty} f_{T,1}(t)\, dt$$

These can be evaluated numerically, and then for fixed M we can plot the ROC as a parametric plot

$$\{(\alpha(M,T_0), \beta(M,T_0)), T_0 \in [0,\infty)\}.$$

Performance of Square-Law Detector

The following numerically integrated curves are from

J.V. DiFranco and W.L. Rubin, *Radar Detection*, Prentice-Hall, 1968.

(Reprinted by Scitech, 2004)

In plots, R_p = Single pulse SNR $= \frac{2E_p}{N_0}$ E_p = energy per pulse.

n.b. n' = "false alarm number" $\triangleq \frac{0.693}{P_{FA}}$

$\Rightarrow P_{FA} = \frac{0.693}{n'}$ (Also $M = N$ in plots.)

Square-Law Detector Noncoherent Pulse Train Performance

R_p = Single pulse SNR $= \frac{2E_p}{N_0}$

$M = 1$ $M = 2$

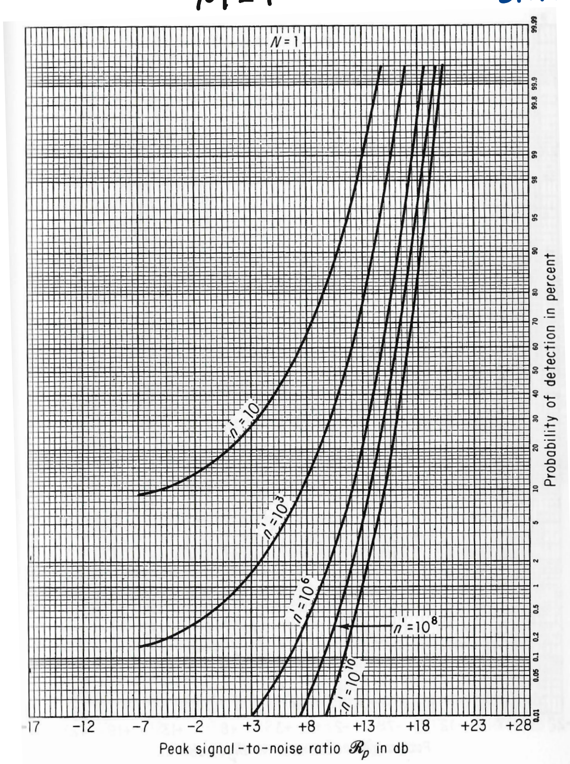

Fig. 10.4-1. Probability of detecting a nonfluctuating target (square-law detector), $N = 1$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

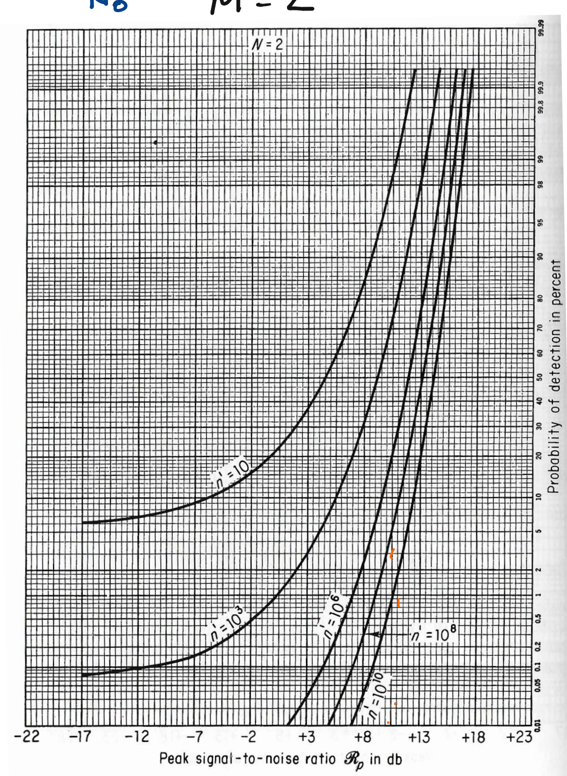

Fig. 10.4-2. Probability of detecting a nonfluctuating target (square-law detector), $N = 2$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

Square-Law Detector Noncoherent Pulse Train Performance

M = 3

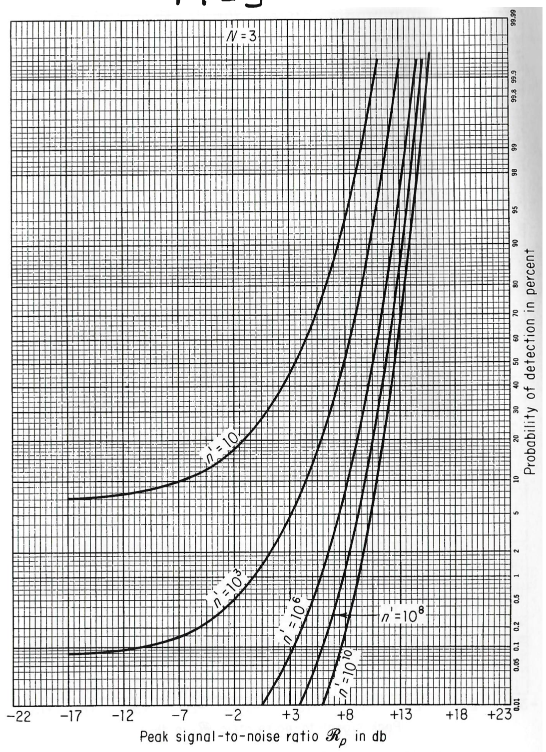

Fig. 10.4-3. Probability of detecting a nonfluctuating target (square-law detector), $N = 3$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

M = 6

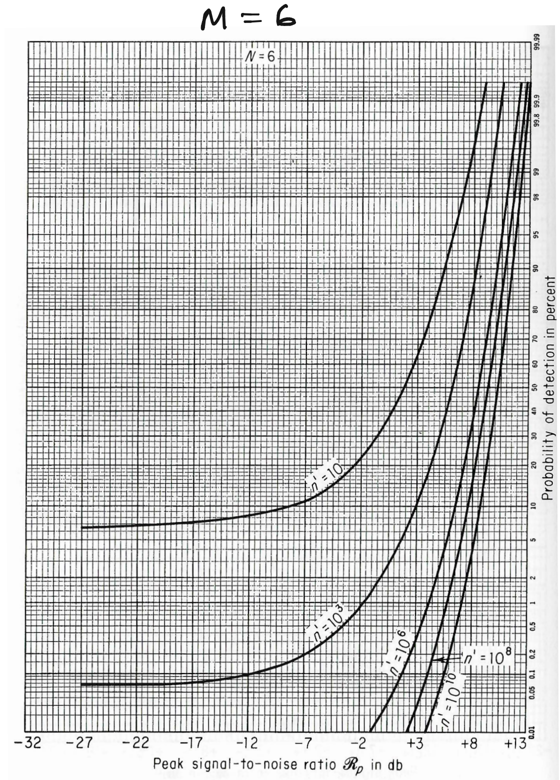

Fig. 10.4-4. Probability of detecting a nonfluctuating target (square-law detector), $N = 6$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

Square-Law Detector Noncoherent Pulse Train Performance

M = 10

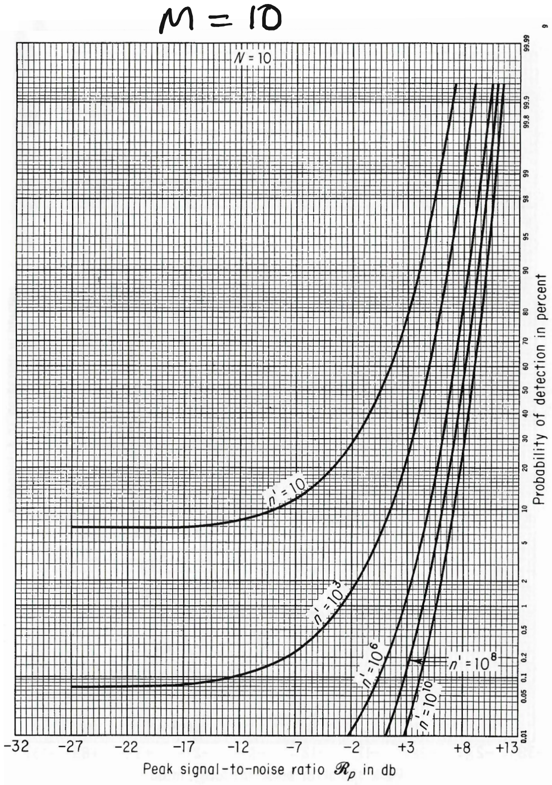

Fig. 10.4-5. Probability of detecting a nonfluctuating target (square-law detector), $N = 10$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

M = 30

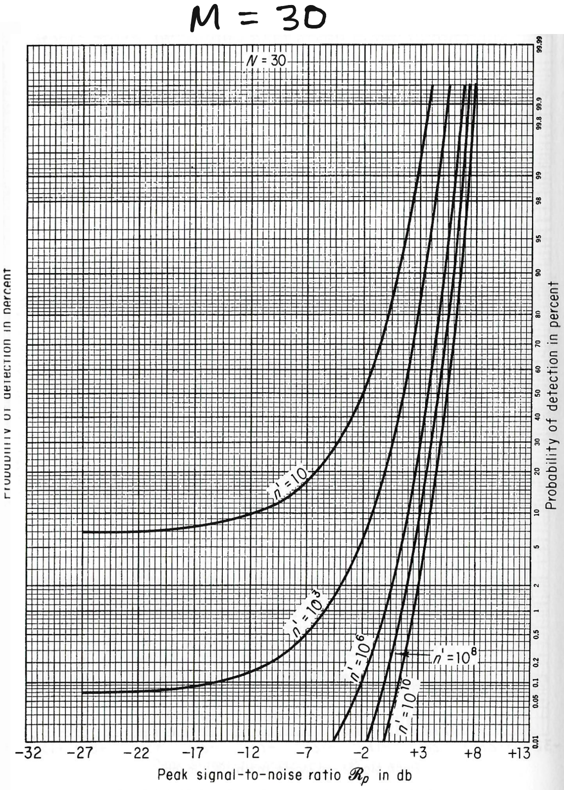

Fig. 10.4-6. Probability of detecting a nonfluctuating target (square-law detector), $N = 30$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

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Square-Law Detector Noncoherent Pulse Train Performance

M = 100

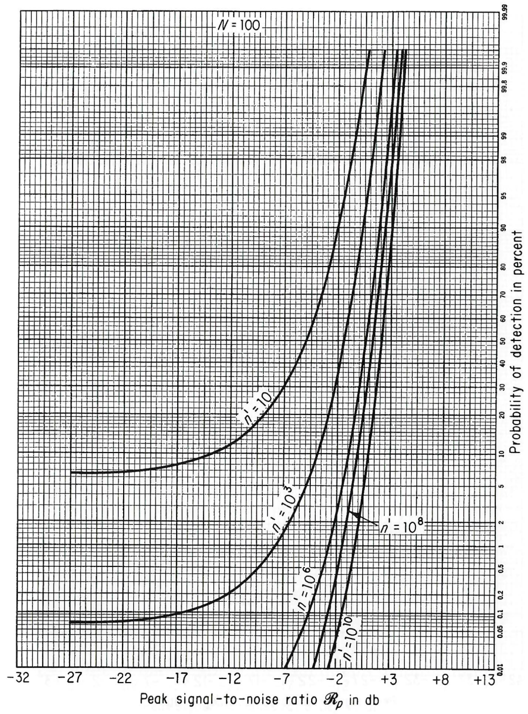

Fig. 10.4-7. Probability of detecting a nonfluctuating target (square-law detector), $N = 100$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

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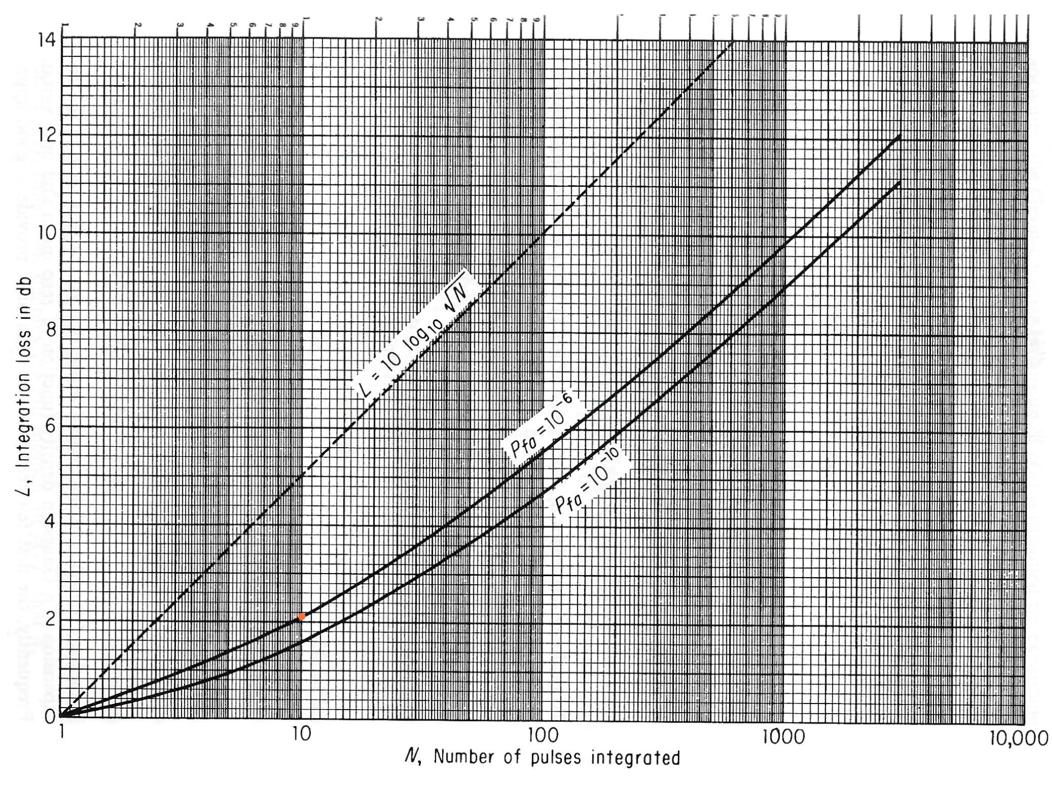
Non-Coherent Integration Loss Compared to Coherent Pulse Integration



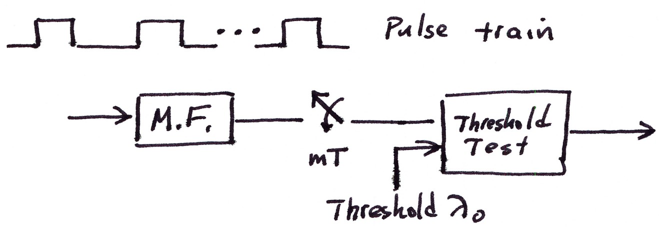
Binary Integration

(See pp. 57–63 of Levanon, *Radar Principles*)

Another form of multi-pulse integration that – while weak – is often used is Binary Integration.

Idea: Send M pulses and detect each pulse separately and independently.

- Whenever a reflected pulse (target) is detected, Assign a 1.
- When no target is detected, assign a 0.



We get a stream of binary numbers.

In binary integration, we declare a target present if there are h or more "hits" (i.e., "1's") in the M pulse detections.

Assuming that the noise is stationary, we have that for each pulse

$$\alpha = P_{FA}^{(1)} = \alpha(\lambda_0)$$

$$\beta = P_D^{(1)} = \beta(\lambda_0)$$

Thus it follows that the overall M-pulse false alarm and detection probabilities are

$$P_{FA}(M) = \sum_{m=h}^{M} \binom{M}{m} \alpha^m (1-\alpha)^{M-m}$$

$$P_D(M) = \sum_{m=h}^{M} \binom{M}{m} \beta^m (1-\beta)^{M-m}$$

So in binary integration, we have two thresholds to select:

1. The single pulse detection threshold λ_0.
2. The "hit" number threshold h.

There are often several (λ_0, h) pairs that will yield a particular false alarm $P_F(M)$.

We will want to select the combination (λ_0, k) that gives the largest $P_D(M)$.

A special case: $h=1 \Rightarrow \underbrace{P_{CD}}_{}$ Analysis

Cumulative Prob. of Detection

If we only need one success, we have

$$P_{FA}(M) = 1-(1-\alpha)^M = 1-(1-\alpha(\lambda_0))^M$$

$$P_D(M) = 1-(1-\beta)^M = 1-(1-\beta(\lambda_0))^M$$

If $\alpha << 1$ (and $\alpha << \beta$), then

$$P_{FA}(M) \simeq M\alpha = M\alpha(\lambda_0)$$

$$P_D(M) = 1-(1-\beta)^M = 1-(1-\beta(\lambda_0))^M.$$

If $\alpha << 1$ and $0.25 < \beta < 0.5$, you can get very reasonable results.

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Single Pulse

Table 3.1 Required Detection Probability for Cumulative Detection Probability of 0.99

M	1	2	3	4	5	6	10	20	100
$P_D \mid_{P_{CD} = 0.99}$	0.99	0.90	0.78	0.68	0.60	0.54	0.37	0.20	0.045