Session 42

42.1

Order-Statistic CFAR (OS-CFAR)

Herman Rohling, "Radar CFAR Thresholding in Clutter and Multiple Target Situations," *IEEE Transactions on Aerospace and Electronic Systems,* vol. 19, pp. 608–621, 1983.

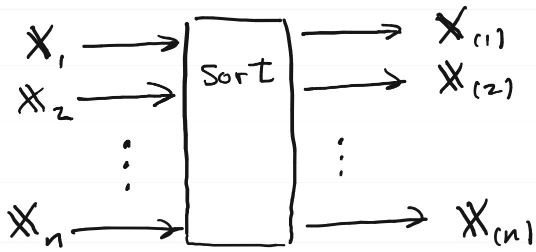
- In OS-CFAR, the average noise power in a region is estimated using an *order statistic*, or ranked sample of the noise power samples in the reference window.
- For example, we might use the *sample median* instead of the *sample mean* to estimate the average noise power.
- While an order statistic estimate is not the maximum likelihood estimate if the samples are independent and statistically homogeneous (i.i.d.), order statistics (e.g., the sample median) are much more robust to deviations from this ideal.

Order Statistics

42.2

Given i.i.d. RVs $X_1, \ldots, X_n$, we can order these as

$$X_{(1)} \le X_{(2)} \le X_{(3)} \le \cdots \le X_{(n)}$$



We call this ordering of the RVs $X_1, \ldots, X_n$ the order statistics of $X_1, \ldots, X_n$.

42.3

Suppose $X_1, \ldots, X_n$ are i.i.d. RVs with pdf $f_X(x)$ and $F_X(x)$.

Assume the X_k are absolutely continuous RVs.

What is the p.d.f. of the k-th order statistic $X_{(k)}$?

Let's call the pdf of $X_{(k)}$

$$f_k(x).$$

We want to find $f_k(x)$.

$$f_k(x)\,\underbrace{dx}_{\Delta x} = P(\{x < X_{(k)} \le x+dx\})$$

a small positive number

$$= P(B_k)$$

where

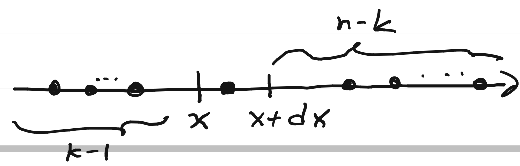
$$B_k = \{x < X_{(k)} \le x+dx\}.$$

The event B_k occurs iff

(i) $k-1$ RVs are less than x

(ii) one RV is in $(x, x+dx]$

(iii) The remaining $n-k$ RVs take on values greater than $x+dx$



For any of the n RVs $X_1, \ldots, X_n$
(denote the one selected by X)
Define the events

$$A_1 \triangleq \{X \le x\}$$

$$A_2 \triangleq \{x < X \le x+dx\}$$

$$A_3 \triangleq \{X > x+dx\}$$

n.b $\{A_1, A_2, A_3\}$ is a partition of $\mathbb{R}$

42.6

For any of the n i.i.d. RVs $\mathbb{X}_1, \ldots, \mathbb{X}_n$:

$$P(A_1) = P(\{\mathbb{X} \le x\}) = F_{\mathbb{X}}(x)$$

$$P(A_2) = P(\{x < \mathbb{X} \le x + dx\}) = f_{\mathbb{X}}(x) \cdot dx$$

$$P(A_3) = P(\{\mathbb{X} > x + dx\}) = 1 - F_{\mathbb{X}}(x+dx)$$

$$\simeq 1 - F_{\mathbb{X}}(x), \quad F_{\mathbb{X}}(x) \text{ is continuous}$$

42.7

For the n i.i.d. RVs $\mathbb{X}_1, \ldots, \mathbb{X}_n$, we know that B_k occurs iff

1. A_1 occurs $k-1$ times,
2. A_2 occurs once,
3. A_3 occurs $n-k$ times.

We can compute the probability of B_k using the multinomial distribution for 3 events.

42.8

$$P(B_k) = \frac{n!}{(k-1)!\,1!\,(n-k)!} P(A_1)^{k-1} P(A_2)^{1} P(A_3)^{n-k}$$

$$= \frac{n!}{(k-1)!\,(n-k)!} \left[F_X(x)\right]^{k-1} \left[f_X(x)\,dx\right] \left[1-F_X(x)\right]^{n-k}$$

$$= f_k(x)\,dx$$

$$\therefore \quad f_k(x) = \frac{n!}{(k-1)!\,(n-k)!} F_X^{k-1}(x) \left[1-F_X(x)\right]^{n-k} f_X(x)$$

42.9

When n is odd, we can set $k = \frac{n+1}{2}$, and we get the order statistic called the sample median $X_{\left(\frac{n+1}{2}\right)}$ of $X_1, \ldots, X_n$.

Equal numbers of RVs lie above and below the sample median.

Order statistics are used in

(1) Median Filters

(2) Order Statistic Filters

(3) OS CFAR Processors

Example: i.i.d Cauchy Random Variables

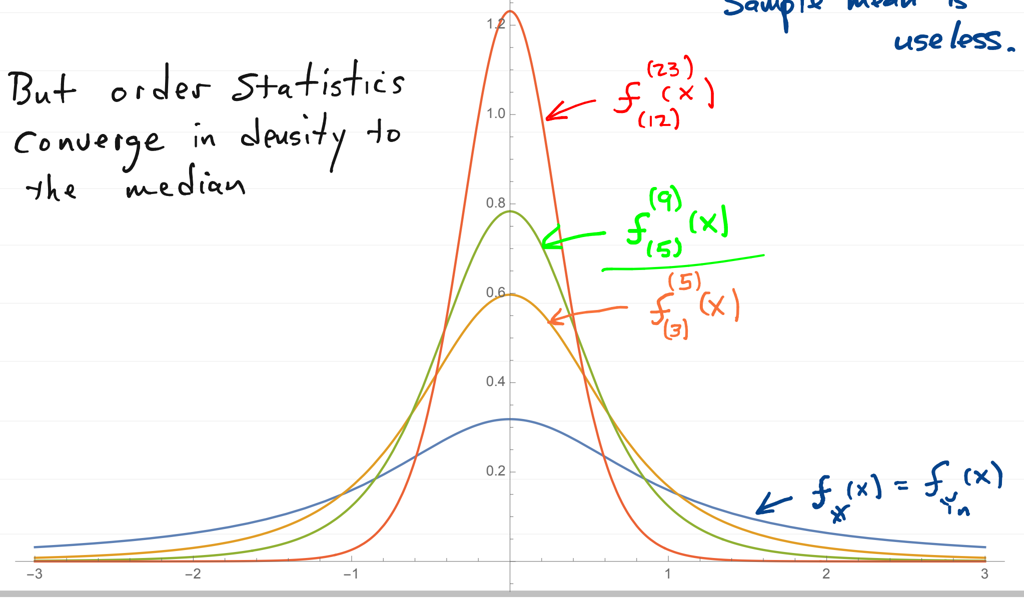
42.10

$$f_X(x) = f_{Y_n}(x) = \frac{1}{\pi(1+x^2)}, \quad Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j$$

$$f_{Y_n}(y) = \frac{1}{\pi(1+x^2)} \text{ for all } n = 1, 2, 3, \ldots$$

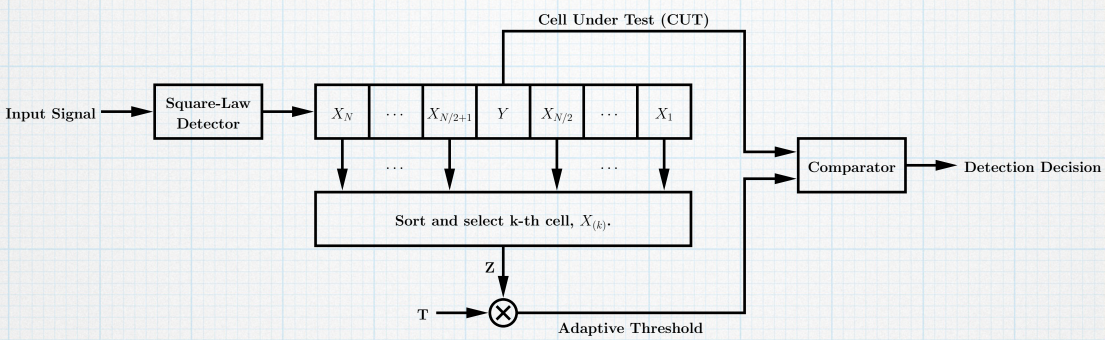
Computing the Sample mean is useless.

But order Statistics converge in density to the median



42.11

Order-Statistic CFAR (OS-CFAR)



In OS-CFAR, the reference noise samples $X_1, \ldots, X_N$ are sorted from smallest to largest and designated

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(N)}.$$

The k-th order statistic $X_{(k)}$—or some scaled version of it—can then be used as the mean power estimate.

While the median seems like a logical choice, selecting values of k in the area of $3N/4$ to $4N/5$ have been shown to work well.*

* See: Michael F. Rimbert, *Constant False Alarm Rate Detection Techniques Based on Empirical Distribution Function Statistics*, Ph.D Thesis, School of Electrical and Computer Engineering, Purdue University, August 2005.

Behavior of OS-CFAR

- OS-CFAR is robust to outliers deviating from a set of homogeneous i.i.d. samples in the reference window because order statistics—especially central order statistics near the median— are robust to outliers.
- This is in fact why statistician John W. Tukey developed and advocated statistical estimation techniques based on them.
- More general results from the theory of *order statistic filters* may also yield interesting new CFAR techniques.
- How well do they behave compared to optimal CA-CFAR when the noise reference samples are i.i.d. ?

Analysis of OS-CFAR

Assume that $X_1, \ldots, X_N$ are i.i.d. samples from a common pdf $f(x)$ having corresponding cdf $F(x)$. If we form the order statistics

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(N)},$$

it can be shown that (See Papoulis, Ch. 8)* the pdf of $X_{(k)}$ is

$$\begin{aligned} f_k(x) &= \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1} f(x)[1-F(x)]^{N-k} \\ &= k\binom{n}{k}[F(x)]^{k-1} f(x)[1-F(x)]^{N-k}. \end{aligned}$$

* We just showed this.

Now if the X_i are i.i.d. with pdf

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \cdot 1_{[0,\infty)}(x),$$

as we have been assuming under H_0, then this becomes

$$\begin{aligned} f_k(x) &= \frac{n!}{(k-1)!(n-k)!}[1-e^{-x/\mu}]^{k-1} \cdot \frac{1}{\mu} e^{-x/\mu} \cdot 1_{[0,\infty)}(x) \cdot [e^{-y/\mu}]^{N-k} \cdot 1_{[0,\infty)}(x) \\ &= \frac{k}{\mu}\binom{n}{k}[e^{-x/\mu}]^{N-k+1}[1-e^{-x/\mu}]^{k-1} \cdot 1_{[0,\infty)}(x). \end{aligned}$$

Thus the pdf of the OS-CFAR statistic $Z = X_{(k)}$ is Equivalently, the p.d.f. of Z is given by

$$f_Z(z) = \frac{k}{\sigma^2}\binom{n}{k}[e^{-z/\mu}]^{N-k+1}[1-e^{-z/\mu}]^{k-1} \cdot 1_{[0,\infty)}(z).$$

Thus the probability of false alarm for OS-CFAR is given by

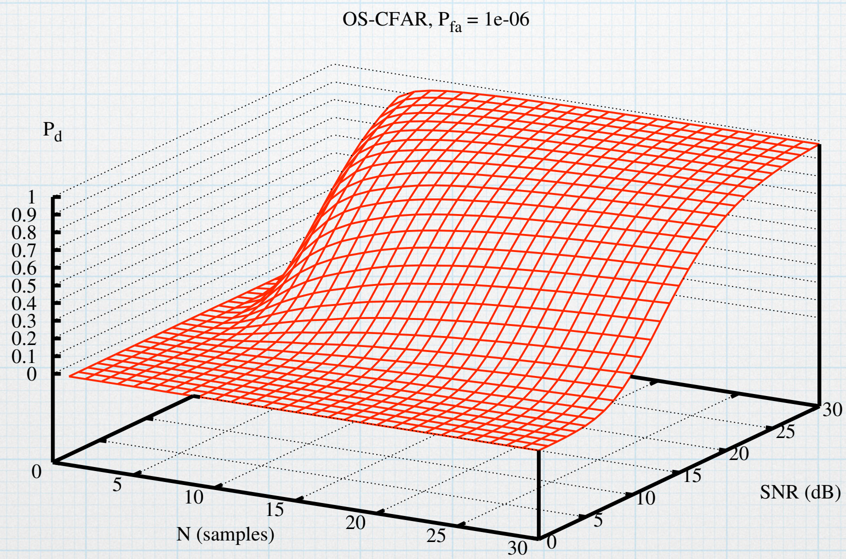
$$\begin{aligned} \alpha &= \mathrm{E}_Z\left[P[Y > TZ|H_0]\right] \\ &= \mathrm{E}_Z\left[\exp\{-TZ/\mu\}\right] \\ &= \frac{k}{\mu}\binom{N}{k}\int_0^\infty e^{(-T+N-k+1)z/\mu}(1-e^{-z/\mu})^{k-1}dz \\ &= k\binom{N}{k}\frac{(k-1)!(T+N-k)!}{(T+N)!} \\ &= \prod_{i=0}^{k-1}\left(\frac{N-i}{N-i+T}\right). \end{aligned}$$

n.b. If T is not an integer, use the fact that
$x! = \Gamma(x+1)$
to evaluate non-integer factorials

Similarly, we can derive the probability of detection as

$$\begin{aligned} P_D &= \mathrm{E}_Z[P[Y > TZ|H_1]\} \\ &= \mathrm{E}_Z[\exp(-TZ/\mu(1+S))\} \\ &= \prod_{i=0}^{k-1}\left(\frac{N-i}{N-i+T/(1+S)}\right). \end{aligned}$$

P_D as a function of N and SNR for desired $P_{FA} = 1 \times 10^{-6}$

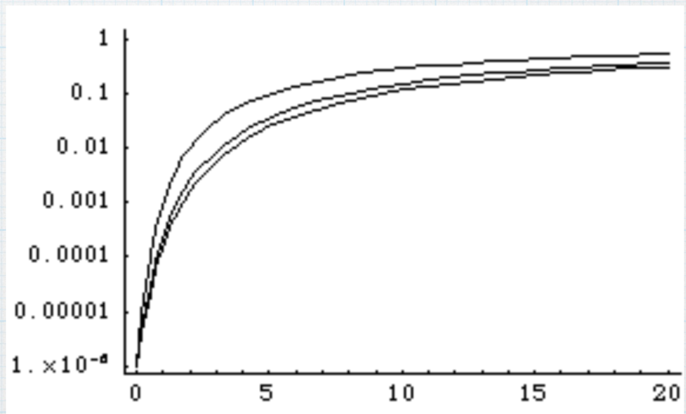


Mathematica Code to Generate Plots

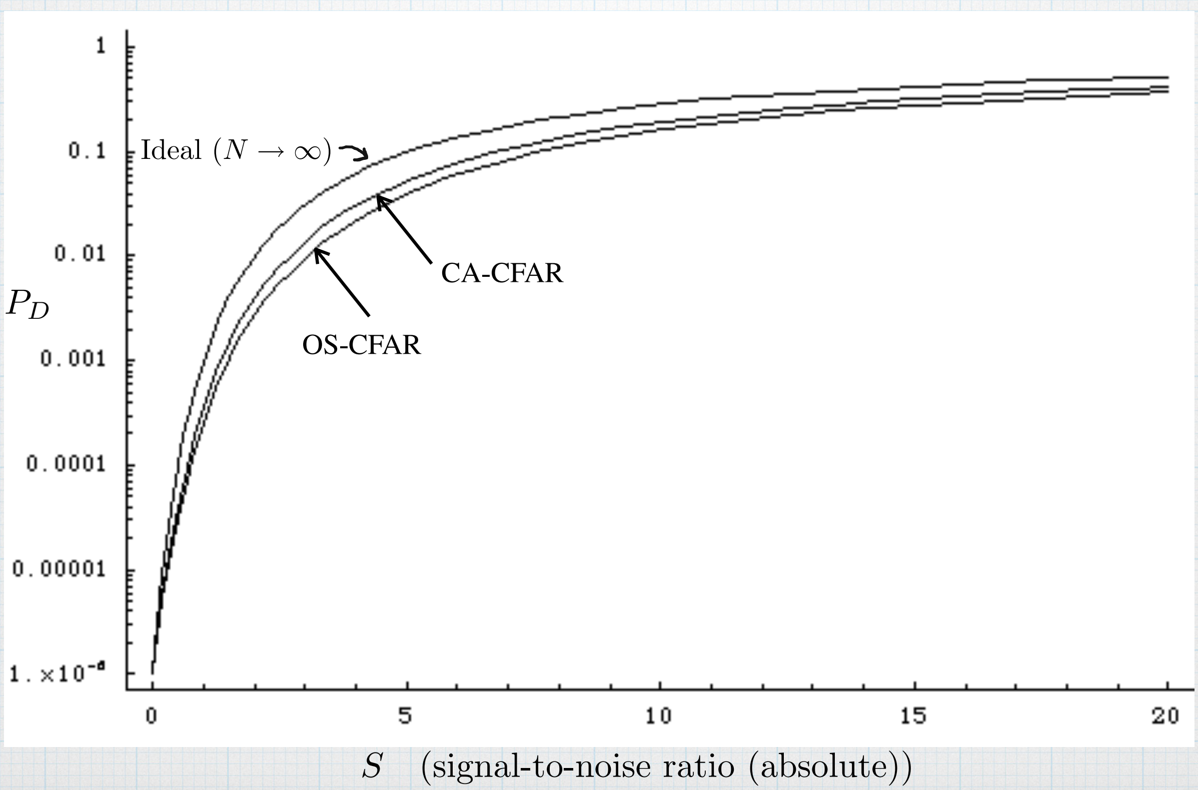
```
In[1]:=osPfa[n_,k_,t_]:=Module[{i},
    Product[(n-i)/(n-i+t),{i,0,k-1}]]

In[2]:=osPd[n_,k_,t_,s_]:=Module[{i},
    Product[(n-i)/(n-i+t/(1+s)),{i,0,k-1}]]

In[3]:=findOS[n_,quantile_]:=Module[{}, Round[n*quantile]]

In[4]:=findT[n_,k_,pfa_]:=Module[{sol,r},
    sol=FindRoot[osPfa[n,k,r]==0.000001,{r,1}];
    r/.sol]

In[5]:=caPd[n_,pfa_,s_]:=Module[{}, ((1+s)/(pfa^(-1/n)+s))^n]
```

A Typical Run

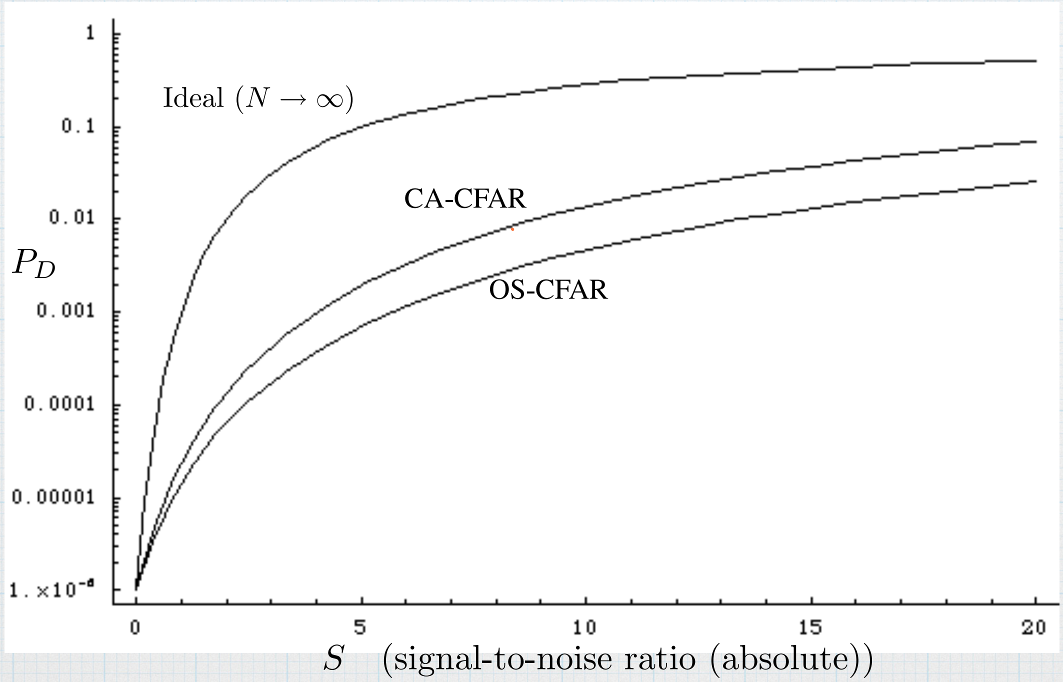
```
n=16

k=findOS[n,4/5]
Out[7]=13

t=findT[n,k,0.000001]
Out[8]=16.9527

LogPlot[{0.000001^(1/(1 + s)), caPd[n, 0.000001, s], osPd[n,
    k, t, s]}, {s, 0, 20}]
```



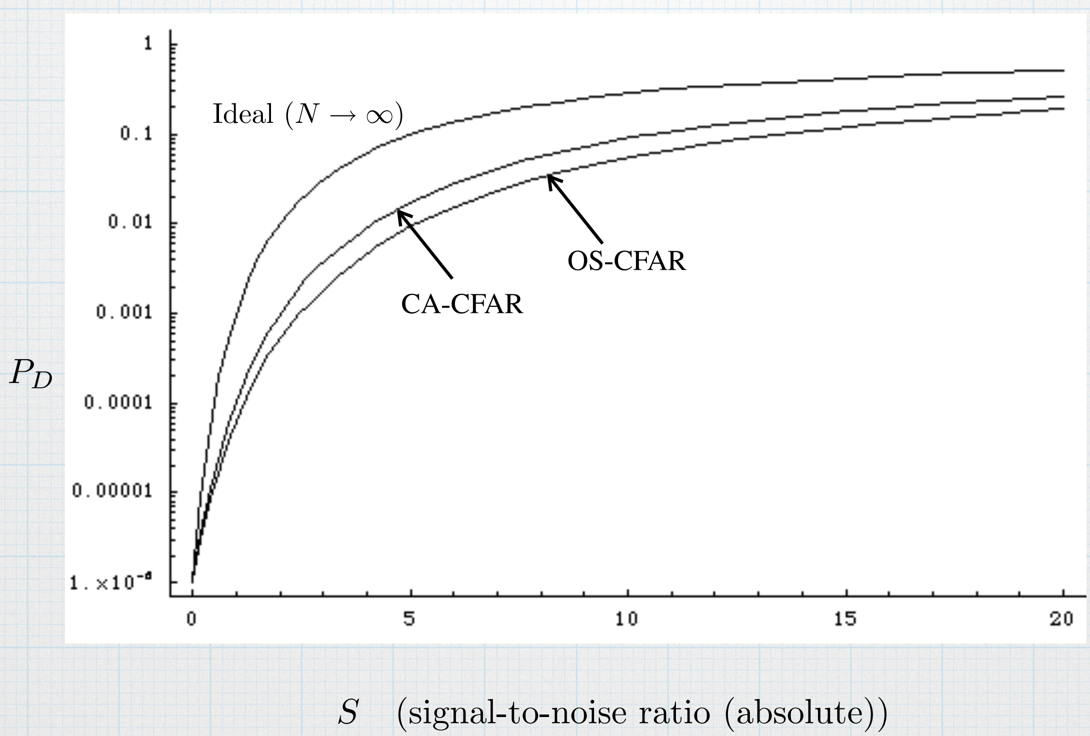
OS-CFAR/CA-CFAR Comparison (N = 24, k = 19, α = 10^(-6))



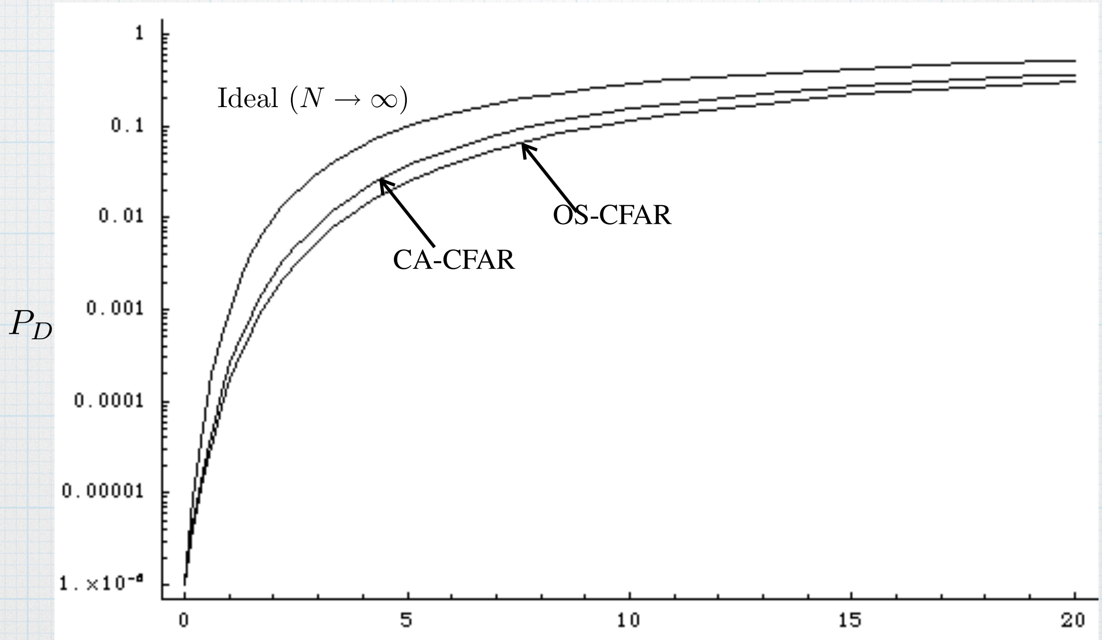
OS-CFAR/CA-CFAR Comparison ($N = 5$, $k = 4$, α = 10^(-6))



OS-CFAR/CA-CFAR Comparison ($N = 10$, $k = 8$, α = 10^(-6))



OS-CFAR/CA-CFAR Comparison ($N = 16$, $k = 13$, $\alpha = 10$^(-6))



S (signal-to-noise ratio (absolute))

CFAR Processor Performance Comparison

Problem	**Processor**			
	CA-CFAR	**GO-CFAR**	**SO-CFAR**	**OS-CFAR**
Clutter Edges	Poor	Good	Poor	Good
Interfering Targets	Poor	Poor	Good	Good

Figure from: Michael F. Rimbert, *Constant False Alarm Rate Detection Techniques Based on Empirical Distribution Function Statistics*, Ph.D Thesis, School of Electrical and Computer Engineering, Purdue University, August 2005.

Stretch Processing for Low-Complexity Radar Signal Processing

Wideband waveforms using LFM, phase coding or frequency coding are used in radar when high range resolution is needed.

These waveforms are often processed using a digital matched filter.

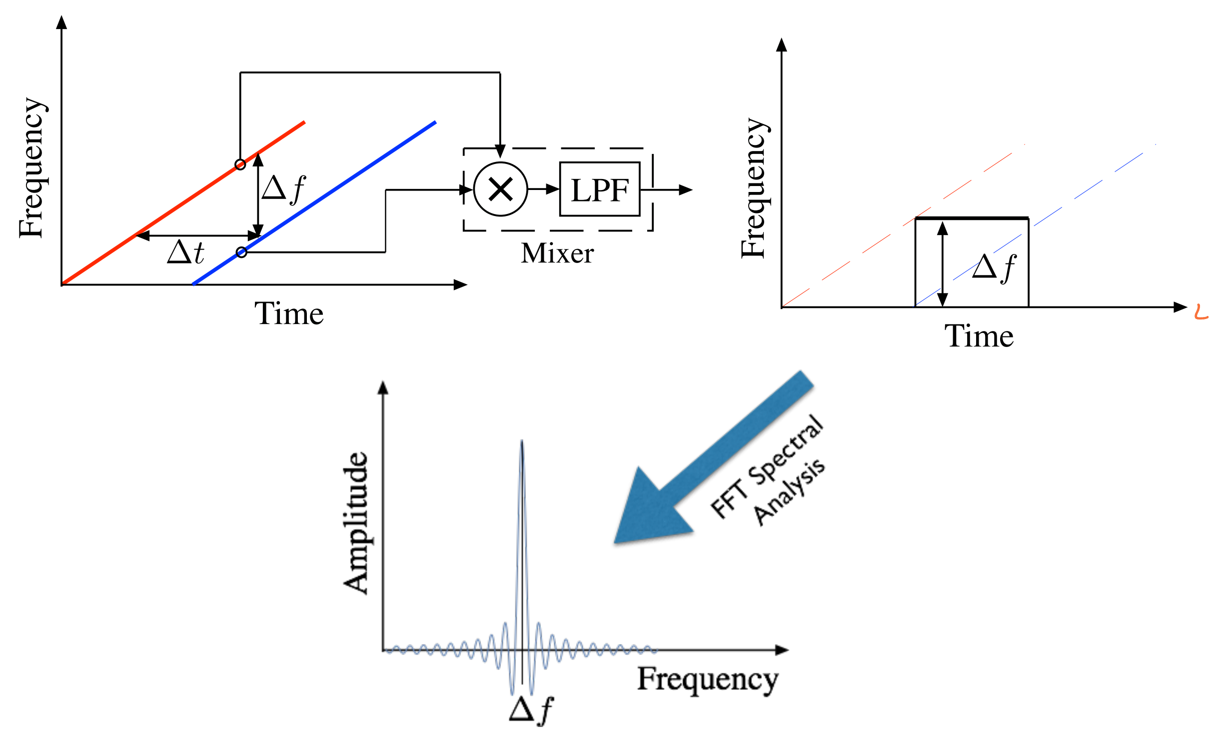
This requires sampling of the received signal at very high sampling rates (twice the waveform bandwidth.)

Stretch-processing makes use of the distinct structure of LFM waveforms to reduce the required sampling rate.

Stretch Processing allows for high resolution LFM measurements over a restricted range interval with much lower sampling rates than sampling the waveform bandwidth at the Nyquist rate as required by a digital matched filter.

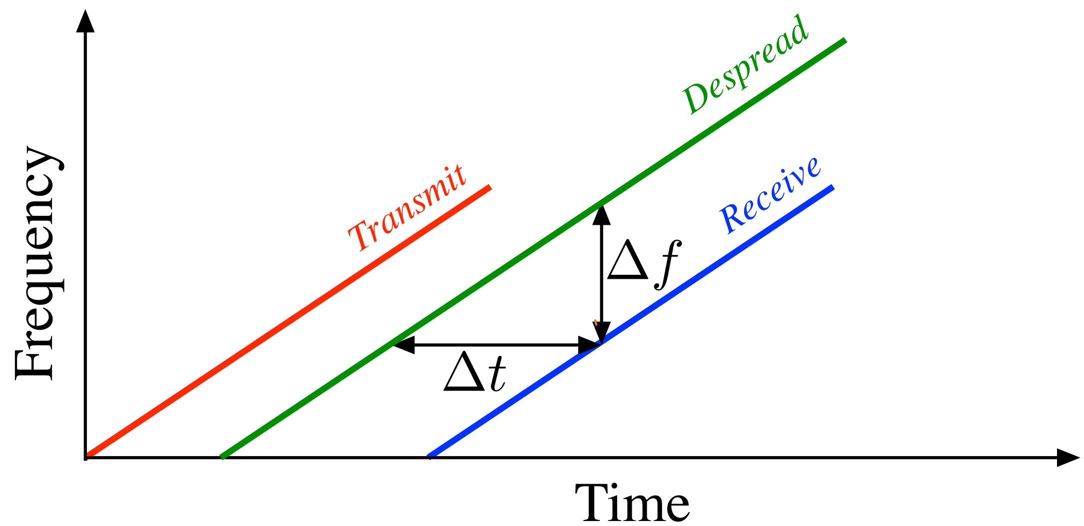
Properly implemented, the range resolution is equivalent to that of the matched filter and the SNR is equivalent to the matched filter response of the LFM waveform.

The Basic Idea of Stretch Processing



42.26

Stretch Processing can be Modified for Additional Flexibility



42.27

Automotive Radar Example (Rohling)

An example from real world radar waveform:

A chirp waveform of 2.5milliseconds and 150MHz bandwidth is considered here. Chirp slope $k = \dfrac{150\text{Mhz}}{2.5\text{ms}}$.(data taken from 'waveform design principles for Automotive Radar System' by Hermann Rohling)

Match Filter: Sampling Frequency = 2x150Mhz = $\boxed{\text{300Mhz}}$

#samples/chirp = 2.5ms × 300Mhz = 750,000 samples

Stretch Processing: Consider maximum tracking distance 400 meters, (At highway speed of 65 mph, recommended 3 second seperation means range of 100 meters.) $\Delta f = k\Delta t = \dfrac{150\text{Mhz}}{2.5\text{ms}} \times \dfrac{2\times 400}{3\times 10^{8}} = 80Khz$

$\Rightarrow$ Sampling Frequency = 2x80Khz = $\boxed{\text{160Khz}}$