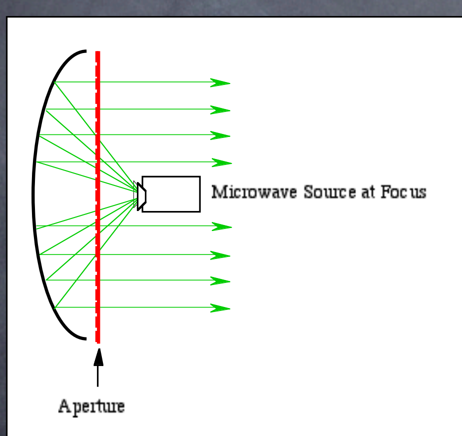
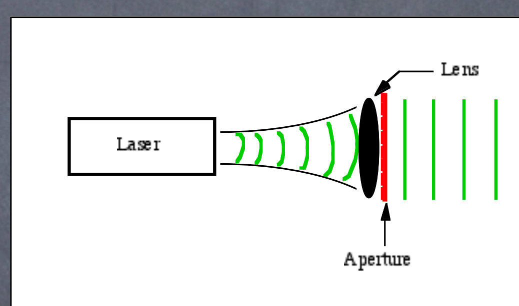
Session 4

4.1

Antenna Apertures

An antenna aperture is a surface of constant phase near the "face" of the antenna.





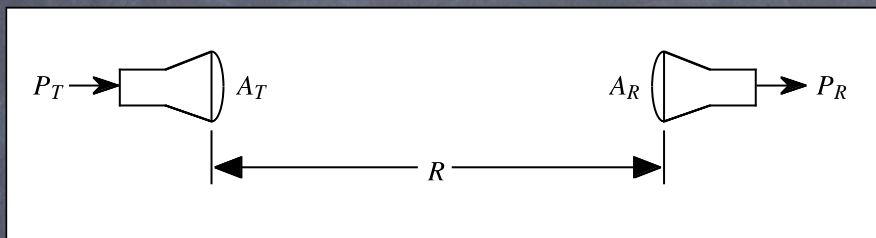
The aperture of an antenna has an area A.

This area characterizes the antenna's behavior.

4.2

The Friis Equation

> Suppose we have two antennas "pointing at each other" a large distance R apart.



If

$$P_T = \text{transmitted power}$$
$$P_R = \text{received power}$$

then

$$\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}. \qquad \text{(Friis Equation)}$$

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$$\boxed{\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}}$$

We must be in the "far field" and looking at small angles off "boresight":

$$R \gg \frac{D_{\text{max}}^2}{4\lambda}$$

The aperture must be large enough for scalar diffraction to be accurate:

$$D_{\text{min}} \gg \lambda$$

4.4

The Friis Equation

$$\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}$$

Intuitively

$$\frac{P_R}{P_T} = \left(\frac{A_T}{\lambda^2}\right)(A_R)\left(\frac{1}{R^2}\right)$$

Large A_T w.r.t. λ more concentrated beam

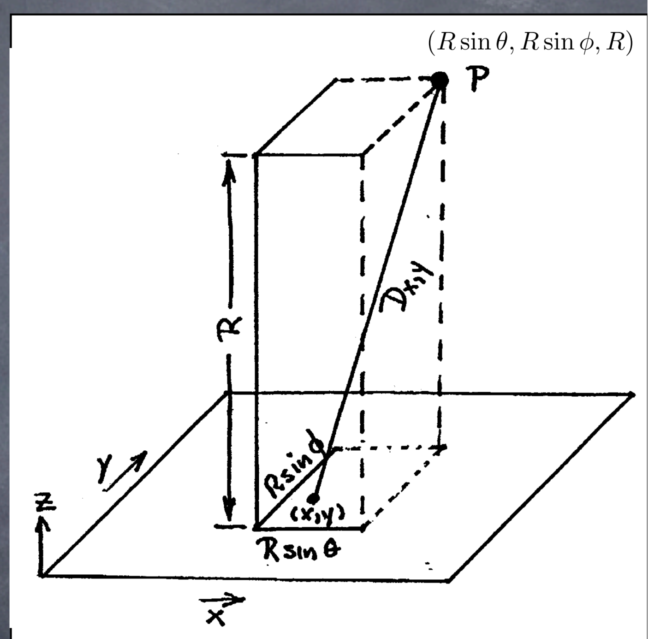
Large A_R, larger collection area

Inverse square law

4.5

Derivation of Friis Equation

- Consider a point P in the z=R plane in Cartesian 3D space
- Consider the electric field at P resulting from a differential element dx dy centered at (x,y) in the z=0 plane, having field



$$dF_{x,y}(P) = \frac{C}{D_{x,y}} \exp\left\{-i\frac{2\pi}{\lambda}D_{x,y}\right\}E(x,y)\,dx\,dy,$$

4.6

So we have

$$dF_{x,y}(P) = \frac{C}{D_{x,y}} \exp\left\{-i\frac{2\pi}{\lambda}D_{x,y}\right\}E(x,y)\,dx\,dy,$$

where

$$D_{x,y} = \sqrt{(R\sin\theta - x)^2 + (R\sin\phi - y)^2 + R^2}$$

$$= \text{distance from } (x,y,0) \text{ to } P.$$

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$$D_{x,y} = \sqrt{(R\sin\theta - x)^2 + (R\sin\phi - y)^2 + R^2}$$

Because R is arbitrarily large, while x and y are fixed and $x, y \ll R$, we can ignore the x^2 and y^2 terms in $D_{x,y}$.

$$D_{x,y} = \sqrt{R^2 + R^2\sin^2\theta - 2xR\sin\theta + R^2\sin^2\phi - 2yR\sin\phi}$$

Furthermore, if we use $\sqrt{1-\epsilon} \approx 1 - \epsilon/2$, we can write

$$D_{x,y} = R'\left(1 - \frac{x\sin\theta + y\sin\phi}{R(1+\sin^2\theta+\sin^2\phi)}\right)$$

$$R' = R\left(1 + \frac{1}{2}(\sin^2\theta + \sin^2\phi)\right)$$

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$$D_{x,y} = R'\left(1 - \frac{x\sin\theta + y\sin\phi}{R(1+\sin^2\theta+\sin^2\phi)}\right)$$

Because θ and ϕ are small, $|\sin\theta| \ll 1$ and $|\sin\phi| \ll 1$

substitute R'

$$D_{x,y} \cong R'\left(1 - \frac{x\sin\theta + y\sin\phi}{R\left(1+\frac{1}{2}(\sin^2\theta+\sin^2\phi)\right)}\right)$$

$$= R'\left(1 - \frac{x\sin\theta + y\sin\phi}{R'}\right)$$

$$= R' - x\sin\theta - y\sin\phi$$

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$$D_{x,y} = R' - x\sin\theta - y\sin\phi$$

$$dF_{x,y}(P) = \frac{C}{D_{x,y}}\exp\left\{-i\frac{2\pi}{\lambda}D_{x,y}\right\}E(x,y)\,dx\,dy,$$

$$\Downarrow$$

$$dF_{x,y}(P) = \frac{C}{D_{x,y}}\exp\left\{-i\frac{2\pi}{\lambda}R'\right\}\exp\left\{\frac{i2\pi}{\lambda}[x\sin\theta+y\sin\phi]\right\}\cdot E(x,y)\,dx\,dy$$

$\Downarrow$ Replace $D_{x,y}$ by R

$$dF_{x,y}(P) = \frac{C}{R}\exp\left\{-i\frac{2\pi}{\lambda}R'\right\}\exp\left\{\frac{i2\pi}{\lambda}[x\sin\theta+y\sin\phi]\right\}\cdot E(x,y)\,dx\,dy$$

So we have

$$dF_{x,y}(P) = \frac{C}{R} \exp\left\{-i\frac{2\pi}{\lambda}R'\right\} \exp\left\{\frac{i2\pi}{\lambda}\left[x\sin\theta + y\sin\phi\right]\right\} \cdot E(x,y)\,dx\,dy$$

We get the total field $F(P)$ at P

by integrating over $z = 0$ plane w.r.t. x and y:

$$F(P) = \frac{C}{R} \exp\left\{\frac{-i2\pi}{\lambda}R'\right\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y) \exp\left\{\frac{i2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\} dx\,dy.$$

$$F(P) = \frac{C}{R} \exp\left\{\frac{-i2\pi}{\lambda}R'\right\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y) \exp\left\{\frac{i2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\} dx\,dy.$$

Replace R' by R—interpreting as
field at point P on a sphere of radius R
centered at $(0, 0, 0)$

$$F(P) = \frac{C}{R} \exp\left\{\frac{-i2\pi}{\lambda}R\right\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y) \exp\left\{\frac{i2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\} dx\,dy.$$

This amounts to moving the points in slightly,
but still gives the correct answer at P.

Now consider a square aperture (wxw) uniformly illuminated by a field Eo in the z=0 plane (centered at origin)

$$F(P) = \frac{C}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y)\exp\left\{\frac{i2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\}dx\,dy.$$

$$= \frac{C}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\int_{-W/2}^{W/2}\int_{-W/2}^{W/2} E_0\exp\left\{i\left(\frac{2\pi}{\lambda}\right)x\sin\theta\right\}\exp\left\{i\left(\frac{2\pi}{\lambda}\right)y\sin\phi\right\}dx\,dy$$

$$= \frac{E_0 C W^2}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\left[\frac{\sin(\pi W\sin\theta/\lambda)}{\pi W\sin\theta/\lambda}\right]\left[\frac{\sin(\pi W\sin\phi/\lambda)}{\pi W\sin\phi/\lambda}\right]$$

$$\approx \frac{E_0 C W^2}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]\left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right] \quad \text{(for small } \theta \text{ and } \phi\text{)}$$

To find the power density at P, we compute

$$F(P)F(P)^* = \frac{|E_0|^2 C^2 W^4}{R^2}\left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2\left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$

To find the total power in a region of the z=R plane at small angles about (0,0,R), we integrate this power density over that region w.r.t. P

Physical reasoning tells us that most of the power is at small angles if

$$W \gg \lambda$$

If we integrate over the whole sphere of radius R, we only get significant contributions at small angles

$$P_{\text{Total}} = \int\int_{S_R} F(P)F(P)^* \, dP$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|E_0|^2 C^2 W^4}{R^2} \left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2 d(R\sin\theta)\, d(R\sin\phi)$$

note extension of limits!

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|E_0|^2 C^2 W^4}{R^2} \left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2 R^2\, d\theta\, d\phi$$

(noting that $d(R\sin\theta)\,d(R\sin\phi) = R^2\cos\theta\cos\phi\, d\theta\, d\phi \approx R^2\, d\theta\, d\phi$ for small angles)

$$= \frac{\lambda^2 |E_0|^2 C^2 W^4}{W^2} \left(\int_{-\infty}^{\infty} \left[\frac{\sin \pi u}{\pi u}\right]^2 du\right) \left(\int_{-\infty}^{\infty} \left[\frac{\sin \pi v}{\pi v}\right]^2 dv\right)$$

(Here $u = W\theta/\lambda$ and $v = W\phi/\lambda$)

Both integrals in parenthesis equal 1!

So

$$P_{\text{Total}} = \lambda^2 |E_0|^2 C^2 W^2$$

But the total power leaving the transmit aperture is

$$P_{\text{Total}} = |E_0|^2 W^2$$

So by conservation of energy (power)

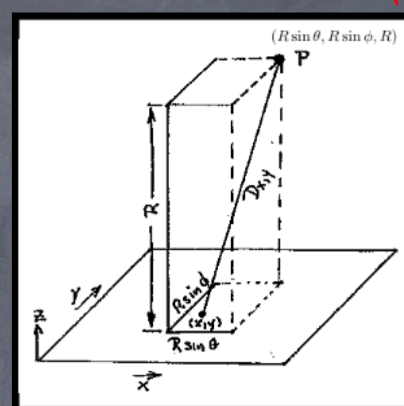
$$|E_0|^2 W^2 = \lambda^2 |E_0|^2 C^2 W^2$$

and thus

$$C = \frac{1}{\lambda}$$

Substituting this C into out previous power density expression...

$$F(P)F(P)^* = \frac{|E_0|^2 W^4}{\lambda^2 R^2}\left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2\left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$



Recall that we showed that the power density at point P is

$$F(P)F(P)^* = \frac{|E_0|^2 C^2 W^4}{R^2}\left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2\left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$

and that the unknown constant C is

$$C = 1/\lambda$$

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$$F(P)F(P)^* = \frac{|E_0|^2 C^2 W^4}{R^2} \left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$

$$\Downarrow$$

$C = 1/\lambda$ Substituting

$$\Downarrow$$

$$F(P)F(P)^* = \frac{|E_0|^2 W^4}{\lambda^2 R^2} \left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$

A transmit antenna with area A_R located at $(0, 0, R)$ will receive power

$$A_R F(P)F^*(P) = \frac{|E_0|^2 A_R W^4}{\lambda^2 R^4} = \frac{A_R A_T^2 |E_0|^2}{\lambda^2 R^2}$$

4.19

So the received power is

$$P_R = \frac{A_R A_T^2 |E_0|^2}{\lambda^2 R^2},$$

and the transmitted power is

$$P_T = |E_0|^2 A_T.$$

$$\Downarrow$$

$$\frac{P_R}{P_T} = \frac{\frac{A_R A_T^2 |E_0|^2}{\lambda^2 R^2}}{|E_0|^2 A_T} = \boxed{\frac{A_T A_R}{\lambda^2 R^2}}$$

It appears we have derived this result for square transmit apertures and arbitrary receive apertures.

Reciprocity Theorem: For linear antenna systems, P_R/P_T remains the same when the roles of the transmit and receive antennas are reversed.

$\Longrightarrow$ P_R/P_T proportional to uniformly illuminated area A_T regardless of shape.

$$\therefore \quad \boxed{\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}}$$