Session 29

29.1

Cross-Ambiguity Functions

Definition:(Asymmetric cross-ambiguity function) The *asymmetric cross-ambiguity function* of two finite-energy signals $r(t)$ and $s(t)$ is defined as

$$\beta_{rs}(\tau, \nu) = \int_{-\infty}^{\infty} r(t)s^*(t-\tau)e^{-i2\pi\nu t}\, dt.$$

Definition:(Symmetric cross-ambiguity function) The *symmetric cross-ambiguity function* of two finite-energy signals $r(t)$ and $s(t)$ is defined as

$$\Gamma_{rs}(\tau, \nu) = \int_{-\infty}^{\infty} r(t+\tau/2)s^*(t-\tau/2)e^{-i2\pi\nu t}\, dt.$$

We will focus primarily on the asymmetric form, but the symmetric form is of interest in signal theory.

Cross-Ambiguity Functions (Cont.)

It can also be shown that the symmetric cross-ambiguity function can be written in terms of the Fourier transforms $R(f)$ of $r(t)$ and $S(f)$ of $s(t)$ as

$$\Gamma_{rs}(\tau,\nu) = \int_{-\infty}^{\infty} R(f+\nu/2)S^*(f-\nu/2)e^{i2\pi\tau f}\,df,$$

and the asymmetric cross-ambiguity function can be written as

$$\beta_{rs}(\tau,\nu) = \int_{-\infty}^{\infty} R(f+\nu)S^*(f)e^{i2\pi\tau f}\,df.$$

It can also be shown that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\beta_{rs}(\tau,\nu)|^2\,d\tau\,d\nu &= \int_{-\infty}^{\infty} |r(t)|^2\,dt \int_{-\infty}^{\infty} |s(t)|^2\,dt \\ &= E_r \cdot E_S.\end{aligned}$$

Cross-Ambiguity Functions (Cont.)

The ability to distinguish a signal $s_1(t)$ from a time-delayed, Doppler shifted version of a signal $s_2(t)$ is given by the metric

$$\begin{aligned}d_{\tau,\nu}(s_1(t), s_2(t)) &= \int_{-\infty}^{\infty} |s_1(t) - s_2(t-\tau)e^{i\pi\nu t}|^2\,dt \\ &= \ldots = E_{s_1} + E_{S_2} - 2\mathrm{Re}\left\{\beta_{s_1 s_2}(\tau,\nu)\right\}.\end{aligned}$$

The *short-time Fourier transform*

$$S(f,t) = \int_{-\infty}^{\infty} s(x)g^*(x-t)e^{-i2\pi f x}\,dx$$

or equivalently

$$S(\nu,\tau) = \int_{-\infty}^{\infty} s(t)g^*(t-\tau)e^{-i2\pi\nu t}\,dt$$

with window function $g(\cdot)$ bears a strong resemblance to the cross-ambiguity function. In fact it is the cross-ambiguity function between the signal $s(t)$ and the window $g(t)$.

Cross-Ambiguity Functions (Cont.)

In radar and sonar, one of the functions in $\beta_{rs}(\tau,\nu)$ is often known, while the other one is a random process:

$$\underline{\text{Transmit}}:\ s(t) \quad \Rightarrow \quad \text{AAF}\{s(t)\} = \beta_s(\tau,\nu).$$

$$\underline{\text{Receive}}:\ r(t) = a\cdot s(t-\tau_0)e^{i2\pi\nu_0 t} + n(t).$$

The cross-ambiguity function between $r(t)$ and $s(t)$ is

$$\begin{aligned}
\beta_{rs}(\tau,\nu) &= \int_{-\infty}^{\infty} r(t)s^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} \left[as(t-\tau_0)e^{+i2\pi\nu_0 t} + n(t)\right] s^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= a\int_{-\infty}^{\infty} s(x)s^*(x-[\tau-\tau_0])e^{-i2\pi(\nu-\nu_0)(x+\tau_0)}\,dx \\
&\qquad\qquad + \int_{-\infty}^{\infty} n(t)s^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= ae^{-i2\pi(\nu-\nu_0)\tau_0}\cdot\beta_s(\tau-\tau_0,\nu-\nu_0) \ + \ \mathcal{M}(\tau,\nu).
\end{aligned}$$

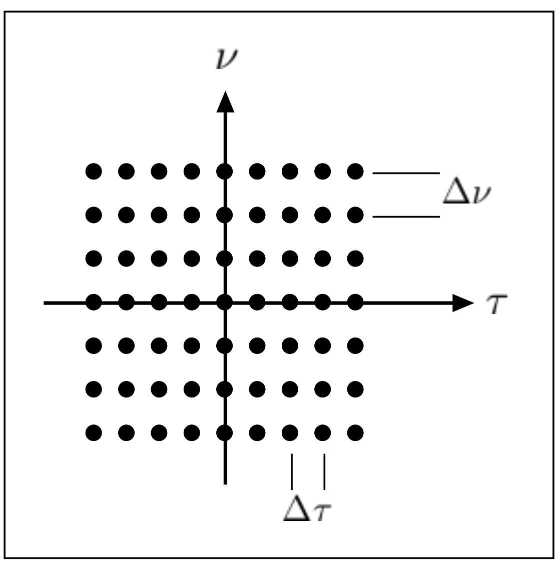
ambiguity function Response

zero-mean noise term

Cross-Ambiguity Functions (Cont.)

What most real radars do is compute $\beta_{rs}(\tau,\nu)$ on a grid of (τ,ν) values using a bank of matched filters:

$$\{(\tau_m,\nu_n) : \tau_m = m\Delta\tau, \quad \nu_n = n\Delta\nu;\ m,n \ \text{ integers}\}$$



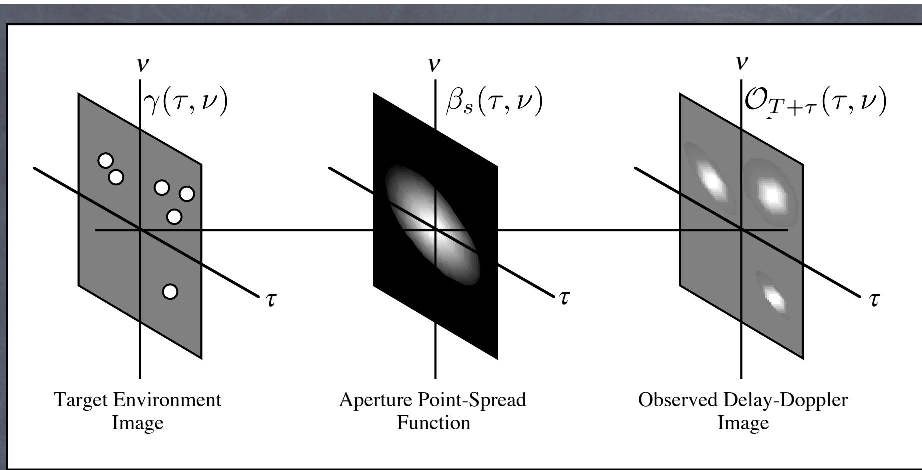
For $\tau_m \approx \tau_0$ and $\nu_n \approx \nu_0$, the response is approximately

$\beta_s(0,0) \approx E_s$

$$\beta_{rs}(\tau_m,\nu_n) \approx aE_s + \mathcal{M}(\tau_m,\nu_n).$$

By sampling finely enough, the system generates an "image" in the coordinates (τ,ν).

29.5



- If we think of a radar as an imaging system, then the ambiguity function behaves much like the point-spread function or impulse response of the system.
- High-resolution images require sharp ambiguity functions => waveform design prob.

The Ambiguity Function of a Pulse Train

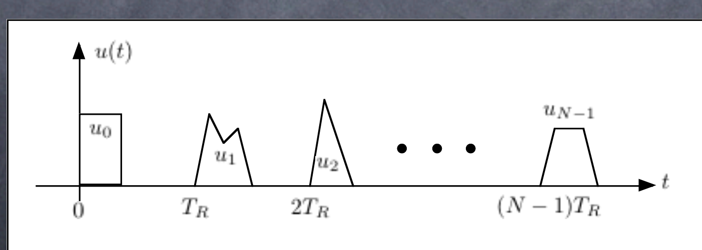
29.6

Consider a passband signal

$$s(t) = \mathrm{Re}\{u(t)e^{i2\pi f_c t}\},$$

where the complex baseband signal $u(t)$ is of the form

$$u(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u_n(t - nT_R),$$



29.7

The Ambiguity Function of a Pulse Train (Cont.)

The asymmetric ambiguity function of $u(t)$ is

$$\begin{aligned}
\beta_u(\tau,\nu) &= \int_{-\infty}^{\infty} u(t)u^*(t-\tau)\, e^{-i2\pi\nu t}\, dt \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} u_n(t-nT_R)\right]\left[\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} u_m(t-mT_R-\tau)\right]^* e^{-i2\pi\nu t}\, dt \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\int_{-\infty}^{\infty} u_n(t-nT_R)u_m^*(t-mT_R-\tau)e^{-i2\pi\nu t}\, dt \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} e^{-i2\pi\nu nT_R}\int_{-\infty}^{\infty} u_n(x)u_m^*(x-[\tau-(n-m)T_R])e^{-i2\pi\nu x}\, dx \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} e^{-i2\pi\nu nT_R}\beta_{n,m}(\tau-(n-m)T_r,\nu),
\end{aligned}$$

where

$$\beta_{n,m}(\tau,\nu) = \int_{-\infty}^{\infty} u_n(t)u_m^*(t-\tau)\, e^{-i2\pi\nu t}\, dt.$$

29.8

If all of the $u_n(t)$ are identical:

$$u_n(t) = c(t), \quad n = 0, 1, 2, \dots N-1,$$

then

$$\beta_u(\tau,\nu) = \frac{1}{N}\sum_{p=-(N-1)}^{N-1} e^{-i\pi\nu(N-1+p)T_R}\cdot\beta_c(\tau-pT_R,\nu)\frac{\sin[\pi\nu(N-|p|)T_R]}{\boxed{\sin(\pi\nu T_R)}}.$$

If the duration of the pulse $t_p < T_R/2$, the pulses do not overlap in the ambiguity function, and $|\beta_u(\tau,\nu)|$ simplfies to

$$|\beta_u(\tau,\nu)| = \frac{1}{N}\sum_{p=-(N-1)}^{N-1} |\beta_c(\tau-pT_R,\nu)|\cdot\left|\frac{\sin[\pi\nu(N-|p|)T_R]}{\boxed{\sin(\pi\nu T_R)}}\right|.$$

Corrections

If we take $c(t)$ to be a unit energy rectangular pulse of duration t_p:

$$c(t) = \frac{1}{t_p}\cdot 1_{[0,t_p]}(t), \quad \text{where} \quad t_p < T_R/2,$$

a plot of $|\beta_u(\tau,\nu)|$ appears as follows:

29.9

Ambiguity function of a pulse-train made up of 5 rectangular pulses

$$|\beta_u(\tau,\nu)| = \frac{1}{N}\sum_{p=-(N-1)}^{N-1} |\beta_c(\tau - pT_R,\nu)| \cdot \left|\frac{\sin\left[\pi\nu(N-|p|)T_R\right]}{\pi\nu(N-|p|)T_R}\right|$$

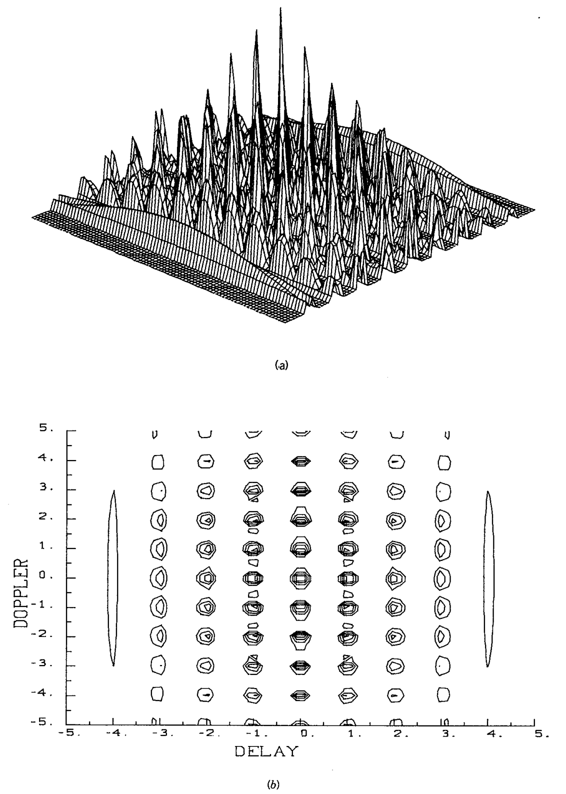


***Figure** 7.7 The ambiguity function of five coherent pulses ($T_R = 1$, $t_P = 0.2$): (a) 3-D view. (b) Contour plot.*