Session 27

Properties of the Ambiguity Function (Cont.)

Property 6 *(Quadratic phase shift (chirp) property)* Let

$$v(t) = s(t)e^{i\pi\alpha t^2},$$

where $\alpha \in \mathbf{R}$. Then

$$\beta_v(\tau,\nu) = e^{-i\pi\alpha\tau^2}\beta_s(\tau,\nu-\alpha\tau).$$

Proof: We note that $\beta_s(\tau,\nu)$ can be written as

$$\begin{aligned}
\beta_v(\tau,\nu) &= \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)e^{i\pi\alpha t^2}[s(t-\tau)e^{i\pi\alpha(t-\tau)^2}]^* \cdot e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)e^{i\pi\alpha t^2}s^*(t-\tau)e^{-i\pi\alpha(t^2-2t\tau+\tau^2)}e^{-i2\pi\nu t}\,dt, \\
&= e^{-i\pi\alpha\tau^2}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi(\nu-\alpha\tau)t}\,dt, \\
&= e^{-i\pi\alpha\tau^2}\beta_s(\tau,\nu-\alpha\tau).
\end{aligned}$$

Example of Quatratic Phase Shift Property (Prop. 6)

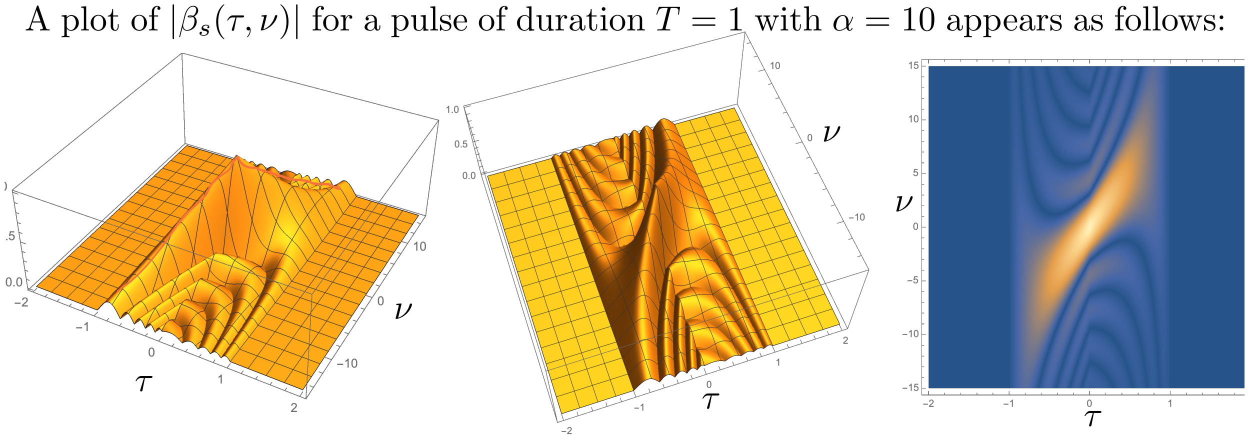
If $s(t) = 1_{[0,T]}(t)$, then

$$v(t) = s(t)e^{i\pi\alpha t^2} = e^{i\pi\alpha t^2} \cdot 1_{[0,T]}(t),$$

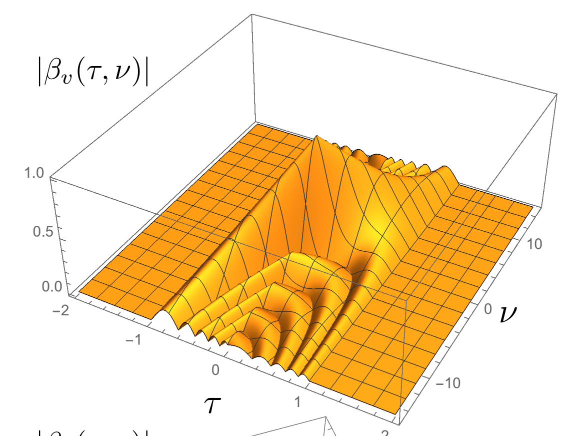
where $\alpha \in \mathbf{R}$. Then

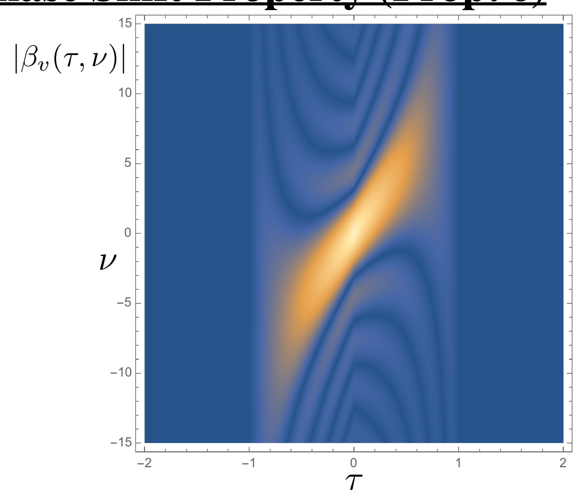
$$\begin{aligned}\beta_v(\tau,\nu) &= e^{-i\pi\alpha\tau^2}\beta_s(\tau,\nu-\alpha\tau) \\ &= e^{-i\pi(T+\tau)}(T-|\tau|)\frac{\sin\pi(\nu-\alpha\tau)(T-|\tau|)}{\pi(\nu-\alpha\tau)(T-|\tau|)} \cdot 1_{[-T,T]}(\tau).\end{aligned}$$

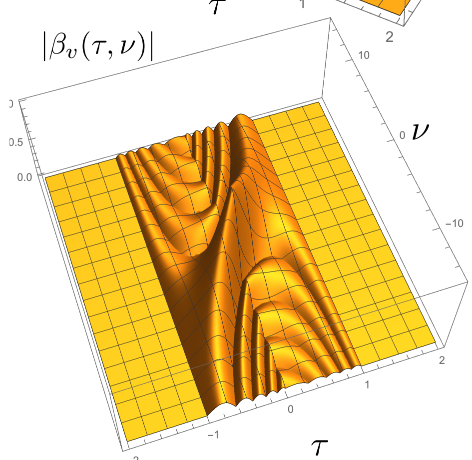
A plot of $|\beta_s(\tau,\nu)|$ for a pulse of duration $T=1$ with $\alpha=10$ appears as follows:

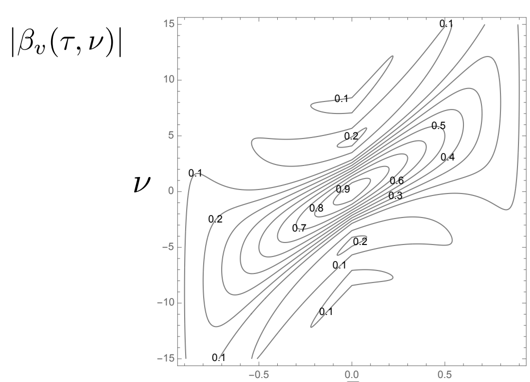


Example of Quatratic Phase Shift Property (Prop. 6)



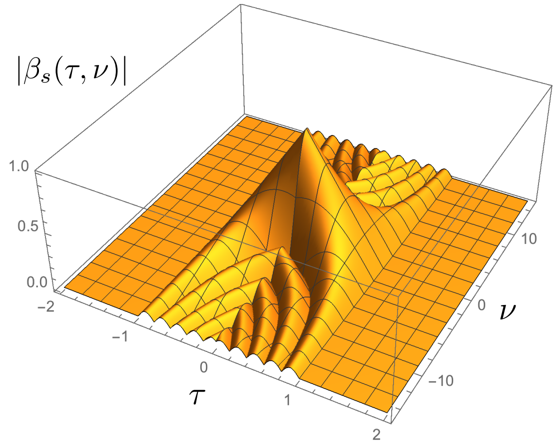


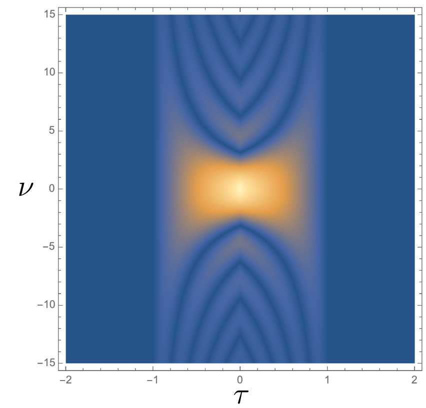


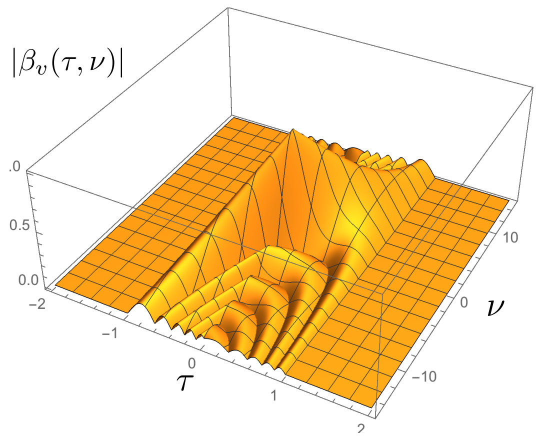


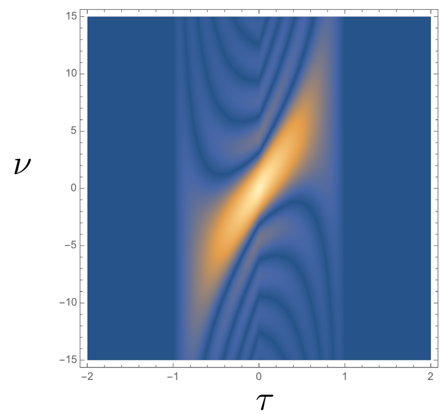
27.4

Example of Quatratic Phase Shift Property (Prop. 6)



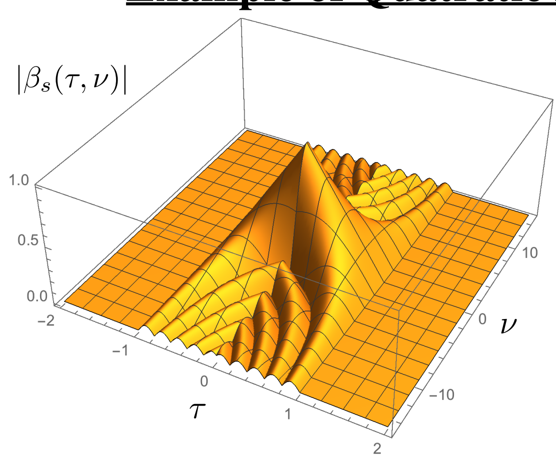




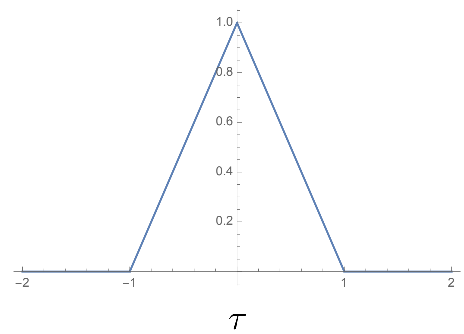


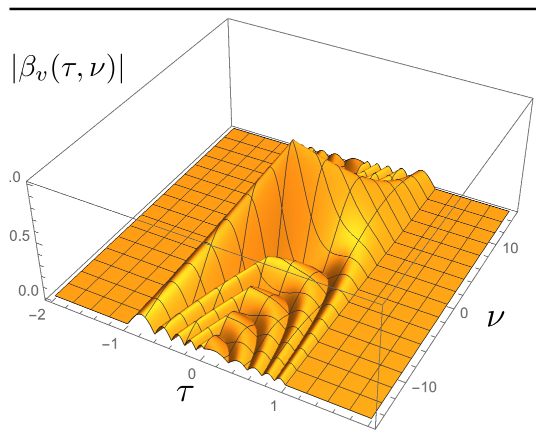
27.5

Example of Quatratic Phase Shift Property (Prop. 6)

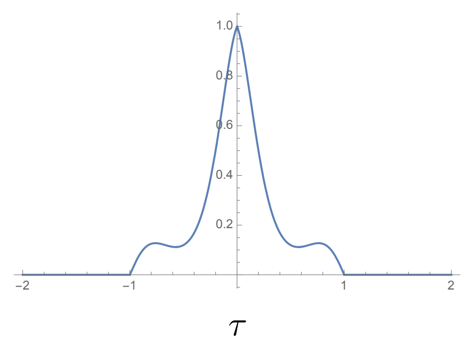


$\nu = 0$ cut $|\beta_s(\tau, 0)|$





$\nu = 0$ cut $|\beta_v(\tau, 0)|$



Properties of the Ambiguity Function (Cont.)

Property 7 *(Volume property)*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\beta_s(\tau,\nu)|^2\, d\tau\, d\nu = E_s^2.$$

Proof: We know we can write $\beta_s(\tau,\nu)$ in two different ways:

$$\beta_s(\tau,\nu) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi\nu t}\, dt,$$

and

$$\beta_s(\tau,\nu) = \int_{-\infty}^{\infty} S(f+\nu)S^*(f)e^{+i2\pi f\tau}\, df.$$

Thus we have

$$\begin{aligned}\int_{-\infty}^{\infty} |\beta_s(\tau,\nu)|^2\, d\tau\, d\nu &= \int_{-\infty}^{\infty} \beta_s(\tau,\nu)\cdot\beta_s^*(\tau,\nu)\, d\tau\, d\nu \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)S^*(f+\nu)S(f)\, dt\, df\, d\tau\, d\nu.\end{aligned}$$

Properties of the Ambiguity Function (Cont.)

Interchanging the order of integration and identifying the following Fourier transforms:

$$\int_{-\infty}^{\infty} s(t-\tau)e^{-i2\pi f\tau}\, d\tau = S^*(f)e^{-i2\pi ft},$$

and

$$\int_{-\infty}^{\infty} s(t-\tau)e^{-i2\pi f\tau}\, d\tau = S^*(f)e^{-i2\pi ft},$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty} |\beta_s(\tau,\nu)|^2\, d\tau\, d\nu &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |s(t)|^2\cdot|S(f)|^2\, dt\, df \\ &= \int_{-\infty}^{\infty} |s(t)|^2\, dt\cdot\int_{-\infty}^{\infty} |S(f)|^2\, df \\ &= E_s\cdot E_s \\ &= E_s^2.\end{aligned}$$

Other Forms of the Ambiguity Function

A number of other slightly different forms of the ambiguity have been used in the literature.

1. Woodward, who introduced the ambiguity function in radar, used the form

$$W_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t) s^*(t+\tau) e^{-i2\pi\nu t}\, dt.$$

2. Levanon and Rihaczek in their widely used radar textbooks use the form

$$R_s(\tau, \nu) = \int_{-\infty}^{\infty} s(t) s^*(t-\tau) e^{+i2\pi\nu t}\, dt.$$

   (note the "+" sign in the complex exponential when compared $\beta_s(\tau, \nu)$, while Levanon uses the form of $\chi_s(\tau, \nu)$ in his more recent book on radar signals.

3. Nathanson, in his widely used book on radar design, uses the form

$$\mathcal{N}_s(\tau . \nu) = \int_{-\infty}^{\infty} s(t) s^*(t+\tau) e^{+i2\pi\nu t}\, dt.$$

Other Forms of the Ambiguity Function (Cont.)

Almost any combination of sign in the delay $\pm\tau$ and Doppler shift $\pm\nu$ has been used in the definition of the ambiguity function. Fortunately it is relatively easy to convert from one form to another. However, in this text we will use $\beta_s(\tau, \nu)$ for the asymmetric ambiguity function and $\Gamma_s(\tau, \nu)$ for the symmetric ambiguity function. With the symmetric ambiguity function there is really no issue; the only form that is ever used is $\Gamma_s(\tau, \nu)$. However for the asymmetric form the choice is not so clear. However we choose $\beta_s(\tau, \nu)$ for two main reasons:

1. The form $\beta_s(\tau, \nu)$ shows up directly in the mismatched response of the matched filter

$$\mathcal{O}_{T+\tau}(\tau, \nu) = e^{-i2\pi(\nu-\nu_0)\tau_0} \cdot \beta_s(\tau - \tau_0, \nu - \nu_0).$$

   If we use the (perhaps) more common form $\chi_s(\tau, \nu)$ we would have to write this in the less direct, slightly akward form

$$\mathcal{O}_{T+\tau}(\tau, \nu) = e^{-i2\pi(\nu-\nu_0)\tau_0} \cdot \chi_s(\tau - \tau_0, -(\nu - \nu_0)).$$

2. When electrical engineers (and hence most radar engineers) write the Fourier transform from time to temporal-frequency, it is customary to use a transform kernel with a negative sign up in the exponential (*e.g.* $e^{-i2\pi ft}$ or $e^{-i\omega t}$) rather than a plus sign such as $e^{+i2\pi\nu t}$ as appears in the definition of $\chi_s(\tau, \nu)$. Using the minus sign as is done in $\beta_s(\tau, \nu)$ is more consistent with standard electrical engineering conventions and standard practice in modern time-frequency analysis.

27.10

Properties of the Ambiguity Function (Cont.)

Note:

1. From Property 4, we have

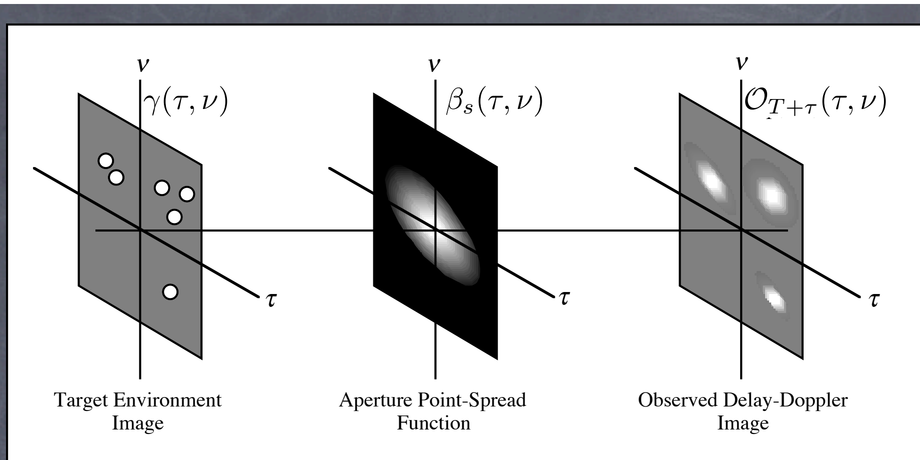
$$|\beta_s(\tau,\nu)|^2 \le |\beta_s(0,0)|^2 = E_s^2.$$

2. From Property 7, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\beta_s(\tau,\nu)|^2 \, d\tau \, d\nu = E_s^2.$$

This limits the possible shape of the ambiguity function. For example, $\beta_s(\tau,\nu)$ cannot be a 2-Dim Dirac delta function centered at the origin of the (τ,ν)-plane (*i.e.* the ideal "thumbtack" ambiguity function.)

27.11



- If we think of a radar as an imaging system, then the ambiguity function behaves much like the <u>point-spread function</u> or <u>impulse response</u> of the system.

- High-resolution images require sharp ambiguity functions => waveform design prob.

The Imaging Interpretation of the Ambiguity Function

- Recall that the response of a matched filter matched to $s(t-\tau)e^{i2\pi\nu t}$ to a signal $s(t-\tau_0)e^{i2\pi\nu_0 t}$ is

$$\mathcal{O}_{T+\tau}(\tau,\nu) = e^{-i2\pi(\nu-\nu_0)\tau_0} \cdot \beta_s(\tau-\tau_0,\nu-\nu_0).$$

- Now consider two scatterers having returns with complex amplitudes c_1 and c_2, delays τ_1 and τ_2, and Doppler shifts ν_1 and ν_2, respectively. Then the output of a matched filter matched to the delay-Doppler pair (τ,ν) (ignoring multiple reflections between the two targets) is

$$\mathcal{O}_{T+\tau}(\tau,\nu) = c_1 e^{-i2\pi(\nu-\nu_1)\tau_1}\beta_s(\tau-\tau_1,\nu-\nu_1)+c_2 e^{-i2\pi(\nu-\nu_2)\tau_2}\beta_s(\tau-\tau_2,\nu-\nu_2).$$

- More generally, for M targets, it can be shown that

$$\mathcal{O}_{T+\tau}(\tau,\nu) = \sum_{m=1}^{M} c_m e^{-i2\pi(\nu-\nu_m)\tau_m}\beta_s(\tau-\tau_m,\nu-\nu_m).$$

- So the overall response is a linear combination of phase-shifted ambiguity functions centered about the target locations $(\tau_1,\nu_1),\ldots,(\tau_M,\nu_M)$.

- This looks a lot like convolution of the ambiguity function with impulses at the target locations, but the phase terms make this not quite right.