Session 26

ECE678 Radar Engineering
Fall 2024

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Midterm Examination
Due by Midnight (EST), Monday, October 28, 2024

Directions: This is a take-home exam. It is an open-book and open class-note exam. You may use other resources in addition to the course text and notes, but you must list them on a final reference page of your exam if you do. You are not to ask questions or receive assistance from anyone else. Failure to adhere to this rule will result in a failing grade on the exam. If there are any questions, email them to me and I will respond promptly (mrb@purdue.edu). A pdf scan of your exam solutions should be emailed to me by Midnight EDT on Monday, October 28, 2024.

The output of the matched filter at time $t = T+\tau$ is

Recall: $O_{T+\tau}(\tau,\nu) = r(t) * h_{\tau,\nu}(t)\Big|_{t=T+\tau} = s_{\tau_0 \nu_0}(t) * h_{\tau,\nu}(t)\Big|_{t=T+\tau}$

$$= \int_{-\infty}^{\infty} s(p-\tau_0)\, e^{i2\pi\nu_0 t}\, s^*(T+\tau-(t-p)-\tau) \cdot e^{-i2\pi\nu(T+\tau-(t-p))}\, dp \Big|_{t=T+\tau}$$

$$= \int_{-\infty}^{\infty} s(p-\tau_0)\, s^*(p-\tau)\, e^{i2\pi\nu_0 p}\, e^{-i2\pi\nu p}\, dp$$

let $x = p-\tau_0 \Rightarrow p = x+\tau_0 \Rightarrow dp = dx$

$$= \int_{-\infty}^{\infty} s(x)\, s^*(x-(\tau-\tau_0))\, e^{-i2\pi(\nu-\nu_0)(x+\tau_0)}\, dx$$

$$= e^{-i2\pi(\nu-\nu_0)\tau_0} \int_{-\infty}^{\infty} s(x)\, s^*(x-(\tau-\tau_0))\, e^{-i2\pi(\nu-\nu_0)x}\, dx$$

$$= e^{-i2\pi(\nu-\nu_0)\tau_0} \cdot \beta_s(\tau-\tau_0, \nu-\nu_0)$$

where

$$\beta_s(\tau,\nu) \triangleq \int_{-\infty}^{\infty} s(t)\, s^*(t-\tau)\, e^{-i2\pi\nu t}\, dt,$$

which is an ambiguity function of $s(t)$

Ambiguity Function Definitions

Definition:(Asymmetric ambiguity function) The *asymmetric ambiguity function* of a finite energy signal $s(t)$ is defined as

$$\beta_s(\tau,\nu) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi\nu t}\, dt.$$

Definition:(Symmetric ambiguity function) The *symmetric ambiguity function* of a finite energy signal $s(t)$ is defined as

$$\Gamma_s(\tau,\nu) = \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{-i2\pi\nu t}\, dt.$$

- The adjectives *asymmetric* and *symmetric* follow from the way in which the delay τ is distributed in the integrand of the respective definitions.
- The asymmetric ambiguity function is the form most often used by radar engineers, primarily because it arises in determining the response of a matched filter radar as we have seen.
- The symmetric ambiguity function is more often used in theoretical investigations of signal properties because its symmetric form simplifies some derivations, as well as the fact that it is closely related to the widely used Wigner distribution of time-frequency analysis.

Properties of the Ambiguity Function

Property 1 *(Time-shift property) Let* $v(t) = s(t - \Delta)$*, where* $\Delta \in \mathbf{R}$*. Then*

$$\beta_v(\tau, \nu) = e^{-i2\pi\nu\Delta}\beta_s(\tau, \nu).$$

Proof: The ambiguity function of $v(t) = s(t - \Delta)$ can be written as

$$\begin{aligned}
\beta_v(\tau, \nu) &= \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t-\Delta)s^*(t-\Delta-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(x)s^*(x-\tau)e^{-i2\pi\nu(x+\Delta)}\,dx, \qquad \text{letting } x = t - \Delta, \\
&= e^{-i2\pi\nu\Delta}\int_{-\infty}^{\infty} s(x)s^*(x-\tau)e^{-i2\pi\nu x}\,dx \\
&= e^{-i2\pi\nu\Delta}\beta_s(\tau, \nu).
\end{aligned}$$

Properties of the Ambiguity Function (Cont.)

Property 2 *(Frequency-shift property)* Let $v(t) = s(t)e^{i2\pi f t}$, where $f \in \mathbf{R}$. Then

$$\beta_v(\tau, \nu) = e^{-i2\pi f\tau}\beta_s(\tau, \nu).$$

Proof: The ambiguity function of $v(t) = s(t)e^{i2\pi f t}$ can be written as

$$\begin{aligned}
\beta_v(\tau, \nu) &= \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)e^{i2\pi f t}(s(t-\tau)e^{i2\pi f(t-\tau)})^*e^{-i2\pi\nu t}\,dt \\
&= e^{i2\pi f\tau}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi\nu t}\,dt, \\
&= e^{i2\pi f\tau}\beta_s(\tau, \nu).
\end{aligned}$$

Properties of the Ambiguity Function (Cont.)

Property 3 *(Symmetry property)*

$$\beta_s(-\tau,-\nu) = e^{-i2\pi\nu\tau}\beta_s^*(\tau,\nu).$$

Proof: We can evaluate $\beta_s(-\tau,-\nu)$ as

$$\begin{aligned}
\beta_s(-\tau,-\nu) &= \int_{-\infty}^{\infty} s(t)s^*(t-(-\tau))e^{-i2\pi(-\nu)t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)s^*(t+\tau))e^{+i2\pi\nu t}\,dt \\
&= \left(\int_{-\infty}^{\infty} s(t+\tau)s^*(t)e^{-i2\pi\nu t}\,dt\right)^* \\
&= \left(\int_{-\infty}^{\infty} s(x)s^*(x-\tau)e^{-i2\pi\nu(x-\tau)}\,dx\right)^*, \quad \text{(letting } x = t+\tau\text{)} \\
&= e^{-i2\pi\nu\tau}\left(\int_{-\infty}^{\infty} s(x)s^*(x-\tau)e^{-i2\pi\nu x}\,dx\right)^* \\
&= e^{-i2\pi\nu\tau}\beta_s^*(\tau,\nu).
\end{aligned}$$

Property 4 *(Maximum property)* The largest value of $|\beta_s(\tau,\nu)|$ always occurs at the origin $(\tau,\nu) = (0,0)$:

$$|\beta_s(\tau,\nu)| \leq \beta_s(0,0) = E_s,$$

where E_s is the energy in the signal $s(t)$.

Proof: Recall that the *Cauchy-Schwarz Inequality* states that for any two square-integrable functions $a(t)$ and $b(t)$,

$$\left|\int_{-\infty}^{\infty} a(t)b(t)\,dt\right|^2 \leq \int_{-\infty}^{\infty} |a(t)|^2\,dt \cdot \int_{-\infty}^{\infty} |b(t)|^2\,dt,$$

with equality if and only if $b(t) = ka^*(t)$ (a.e.), where k is a constant. It follows that

$$\begin{aligned}
|\beta_s(\tau,\nu)|^2 &= \left|\int_{-\infty}^{\infty} \left[s(t)e^{-i\pi\nu t}\right]\left[s(t-\tau)e^{i\pi\nu t}\right]^*\,dt\right|^2 \\
&\leq \int_{-\infty}^{\infty} \left|s(t)e^{-i\pi\nu t}\right|^2\,dt \cdot \int_{-\infty}^{\infty} \left|s(t-\tau)e^{i\pi\nu t}\right|^2\,dt \\
&= \int_{-\infty}^{\infty} |s(t)|^2\,dt \cdot \int_{-\infty}^{\infty} |s(t-\tau)|^2\,dt \\
&= E_s \cdot E_s \\
&= E_s^2,
\end{aligned}$$

where equality holds if and only if

$$\left[s(t-\tau)e^{i\pi\nu t}\right]^* = k\left[s(t)e^{-i\pi\nu t}\right]^*.$$

This can easily be seen to occur when $(\tau,\nu) = (0,0)$. Thus it follows that

$$|\beta_s(\tau,\nu)| \leq \beta_s(0,0) = E_s.$$

Properties of the Ambiguity Function (Cont.)

Property 5 *(Time-scaling property)* Let $v(t) = s(\alpha t)$, where $\alpha \in \mathbf{R}$. Then

$$\beta_v(\tau,\nu) = \frac{1}{|\alpha|}\beta_s(\alpha\tau, \nu/\alpha).$$

Proof: The proof is left as an exercise.

Properties of the Ambiguity Function (Cont.)

Property 6 *(Quadratic phase shift (chirp) property)* Let

$$v(t) = s(t)e^{i\pi\alpha t^2},$$

where $\alpha \in \mathbf{R}$. Then

$$\beta_v(\tau,\nu) = e^{-i\pi\alpha\tau^2}\beta_s(\tau, \nu - \alpha\tau).$$

Proof: We note that $\beta_s(\tau,\nu)$ can be written as

$$\begin{aligned}
\beta_v(\tau,\nu) &= \int_{-\infty}^{\infty} v(t)v^*(t-\tau)e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)e^{i\pi\alpha t^2}[s(t-\tau)e^{i\pi\alpha(t-\tau)^2}]^* \cdot e^{-i2\pi\nu t}\,dt \\
&= \int_{-\infty}^{\infty} s(t)e^{i\pi\alpha t^2}s^*(t-\tau)e^{-i\pi\alpha(t^2-2t\tau+\tau^2)}e^{-i2\pi\nu t}\,dt, \\
&= e^{-i\pi\alpha\tau^2}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{-i2\pi(\nu-\alpha\tau)t}\,dt, \\
&= e^{-i\pi\alpha\tau^2}\beta_s(\tau,\nu-\alpha\tau).
\end{aligned}$$