Session 23

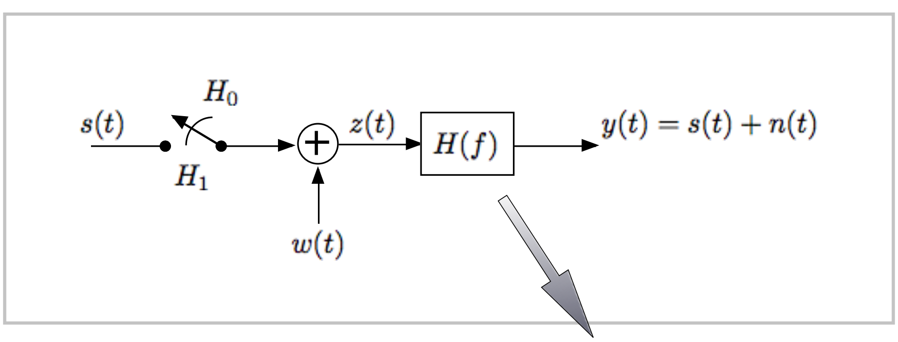
Recall...

Detection of a Known Signal in Additive White Gaussian Noise

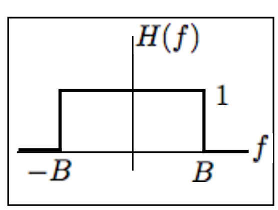
Suppose we have a signal $s(t)$ of known duration T in the interval $[0, T]$ such that

$$s(t) = 0, \quad t \notin [0, T].$$

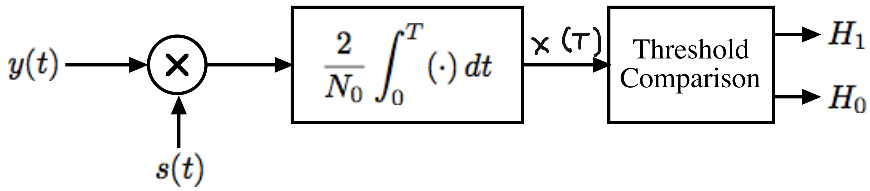
We wish to determine whether or not this signal is present in the presence of *Additive White Gaussian Noise* (AWGN).



Hypothetical Lowpass Filter:



So the processor and optimal test can be implemented using a correlator:



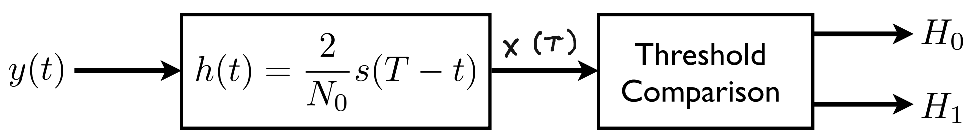
Note that this correlator implementation requires the synchronization of the reference signal $s(t)$ with the signal component in the incoming signal $y(t)$.

In a digital signal processing (DSP) implementation, this can done with a resynchronized version of $s(t)$ repeated in each sampling interval.

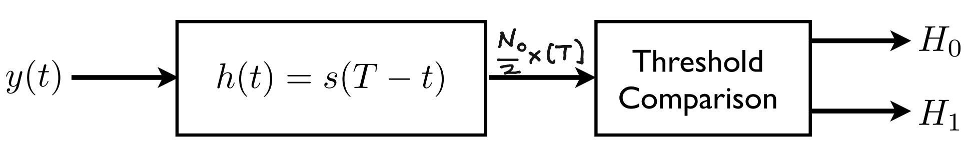
While this works, it is not the most computationally effiecient approach.

However, we can eliminate the synchronization issue altogetherby implementing the correlator using a linear time-invariant (LTI) filter.

In block diagram form, the LTI filter implementation appears as follows:



or eliminating the scale factor $2/N_0$ and adjusting the thresho9ld accordingly:



This implementation is called the <u>*matched filter*</u> implementation.

Because $h(t)$ is a LTI filter, the synchronization issue is not as critical here.

Recall...

Computing the Theshold

So we have

$$H_0 : x(T) \sim \mathcal{N}\left(0, \frac{2E_s}{N_0}\right),$$

$$H_1 : x(T) \sim \mathcal{N}\left(\frac{2E_s}{N_0}, \frac{2E_s}{N_0}\right).$$

It follows the most powerful test of size α is

$$x(T) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_1,$$

with the threshold γ_1 computed by noting that

$$\begin{aligned}\alpha &= P_{FA} \\ &= \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{4\pi E_s/N_0}} \exp\left\{-\frac{u^2}{4E_s/N_0}\right\} du \\ &= 1 - \Phi\left(\frac{\gamma_1}{\sqrt{2E_s/N_0}}\right),\end{aligned}$$

The threshold can then be found as

$$\gamma_1 = \sqrt{\frac{2E_s}{N_0}} \Phi^{-1}(1-\alpha),$$

where

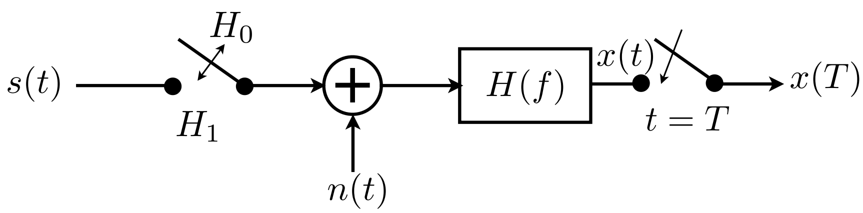
$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

The power of the test is

$$\begin{aligned}\beta &= P_D \\ &= \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{4\pi E_s/N_0}} \exp\{-\frac{(u - E_s/N_0)^2}{4E_s/N_0}\} du \\ &= 1 - \Phi\left(\frac{\gamma_1 - 2E_s/N_0}{\sqrt{2E_s/N_0}}\right) \\ &= 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{2E_s/N_0}\right).\end{aligned}$$

The Matched Filter: Signal-to-Noise Ratio Maximization

An alternative approach to deriving—and extending—the matched filter is to consider the signal-to-noise ratio at the LTI filter output at time $t = T$:



Assume $s(t)$ has duration T:

$$s(t) = 0, \qquad t \notin [0, T].$$

n.b. s(t) can be complex.

Assume $s(t)$ has energy E:

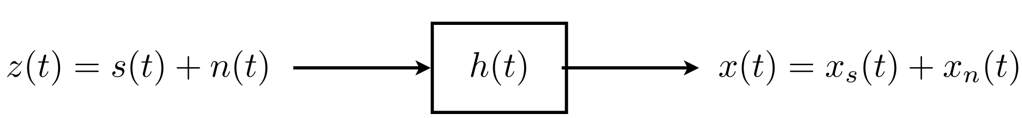
$$\int_0^T |s(t)|^2 \, dt = E.$$

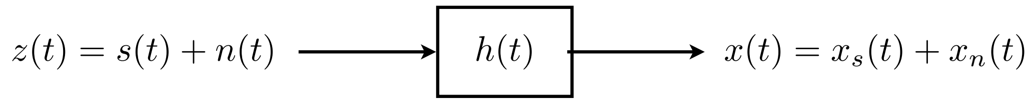
We will assume that the noise $n(t)$ is a zero-mean wide-sense stationary (WSS) random process with power spectral density $S_{nn}(f)$.

- $n(t)$ is not necessarily Gaussian.
- $n(t)$ is not necessarily white.

We wish to find the LTI filter $h(t) \overset{\mathcal{F}}{\Leftrightarrow} H(f)$ that maximizes the signal-to-noise ratio (SNR) at its output at time $t = T$.

Question: What is the SNR at the filter output at time $t = T$?





Computing the filter output $x(T)$ at time $t = T$, we have

$$\begin{aligned} x(T) &= \int_{-\infty}^{\infty} h(\tau) z(T-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\left[s(T-\tau) + n(T-\tau)\right] d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) s(T-\tau)\, d\tau + \int_{-\infty}^{\infty} h(\tau) n(T-\tau)\, d\tau \\ &= x_s(T) + x_n(T) \end{aligned}$$

The SNR of the filter output $x(T)$ at time $t = T$ is defined as

$$\text{SNR} \equiv \frac{|x_s(T)|^2}{\text{E}[|x_n(T)|^2]}.$$

Our problem becomes finding the filter $h(t)$ that maximizes the SNR at time T:

$$h_M(t) = \arg\max_{h(t)} \left[\frac{|x_s(T)|^2}{\text{E}[|x_n(T)|^2]}\right].$$

By Parseval's theorem,

$$|x_s(T)|^2 = \left|\int_{-\infty}^{\infty} h(\tau) s(T-\tau)\right|^2 = \left|\int_{-\infty}^{\infty} H(f) S(f) e^{i2\pi T f}\, df\right|^2,$$

and

$$\text{E}[|x_n(T)|^2] = R_{x_n x_n}(0) = \int_{-\infty}^{\infty} S_{x_n x_n}(f)\, df = \int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\, df.$$

Thus

$$\begin{aligned} \text{SNR} &= \frac{|\int_{-\infty}^{\infty} H(f) S(f) e^{i2\pi T f} df|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\, df} \\ &= \frac{|\int_{-\infty}^{\infty} H(f) \sqrt{S_{nn}(f)} \frac{S(f)}{\sqrt{S_{nn}(f)}} e^{i2\pi T f} df|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\, df}. \end{aligned}$$

By the Schwarz inequality, we know that any two square-integrable functions f and g satisfy

$$\left|\int f(t)g(t)\,dt\right|^2 \leq \int |f(t)|^2\,dt \cdot \int |g(t)|^2\,dt,$$

with equality if and only if $f(t) = kg^*(t)$.

Thus we have

$$\begin{aligned}
\text{SNR} &= \frac{|\int_{-\infty}^{\infty} H(f)\sqrt{S_{nn}(f)}\frac{S(f)}{\sqrt{S_{nn}(f)}}e^{i2\pi Tf}\,df|^2}{\int_{-\infty}^{\infty}|H(f)|^2 S_{nn}(f)\,df} \\
&\leq \frac{\int_{-\infty}^{\infty}|H(f)|^2 S_{nn}(f)\,df \cdot \int_{-\infty}^{\infty}\frac{|S(f)|^2}{S_{nn}(f)}df}{\int_{-\infty}^{\infty}|H(f)|^2 S_{nn}(f)\,df} \\
&= \int_{-\infty}^{\infty}\frac{|S(f)|^2}{S_{nn}(f)}\,df,
\end{aligned}$$

Apply Schwarz Inequality to numerator

with equality if and only if

$$\boxed{H(f) = \frac{kS^*(f)}{S_{nn}(f)}e^{-i2\pi Tf},}$$

<u>Matched Filter for wide-sense stationary additive colored noise</u>

where k is a complex constant.

$$\boxed{H(f) = \frac{kS^*(f)}{S_{nn}(f)}e^{-i2\pi Tf}.}$$

Note that when the noise is white, $S_{nn}(f) = N_0/2$, and this becomes

$$H(f) = \frac{2k}{N_0}S^*(f)e^{-i2\pi fT}.$$

The corresponding impulse response of is

$$h(t) = \frac{2k}{N_0}s(T-t),$$

and the resulting SNR is

$$\text{SNR} = \int_{-\infty}^{\infty}\frac{|S(f)|^2}{N_0/2}df = \frac{2E_s}{N_0}.$$

<u>*n.b.*, the exact same result we got for additive white Gaussian noise.</u>